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On division algebras having the same maximal subfields

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To Gopal Prasad on his 65th birthday.

Abstract. We show that if a field K of characteristic $\neq 2$ satisfies the following property (*) for any two central quaternion division algebras D_1 and D_2 over K , the fact that D_1 and D_2 have the same maximal subfields implies that $D_1 \simeq D_2$ over K , then the field of rational functions $K(x)$ also satisfies (*). This, in particular, provides an alternative proof for the result of S. Garibaldi and D. Saltman that the fields of rational functions $k(x_1, \dots, x_r)$, where k is a number field, satisfy (*). We also show that $K = k(x_1, \dots, x_r)$, where k is either a totally complex number field with a single dyadic place (e.g. $k = \mathbb{Q}(\sqrt{-1})$) or a finite field of characteristic $\neq 2$, satisfies the analog of (*) for all central division algebras having exponent two in the Brauer group $\text{Br}(K)$.

1. Introduction

Given two (finite dimensional) central division algebras D_1 and D_2 over the same field K , we say that D_1 and D_2 have the same maximal subfields if any maximal subfield F of D_1 admits a K -embedding $F \hookrightarrow D_2$, and vice versa. In [11], 5.4, a question was raised regarding fields K having the following property:

(*) *For any two central quaternion division algebras D_1 and D_2 over K , the fact that D_1 and D_2 have the same maximal subfields implies that $D_1 \simeq D_2$ over K .*

This question was motivated by the analysis of the general problem of when weak commensurability of Zariski-dense subgroups of absolutely almost simple algebraic groups implies their commensurability, which in turn is related to the well-known open question as to whether or not two isospectral Riemann surfaces are necessarily commensurable. The fact that number fields have (*)—which follows from the Albert–Hasse–Brauer–Noether Theorem (AHBN) (cf. the proof of Corollary 4.8) and is also a consequence of the Minkowski–Hasse Theorem on quadratic forms—was used in [11] (cf. also [12]) to show that if one of the two Zariski-dense subgroups in absolutely almost simple groups of type A_1 is arithmetic and the two subgroups are weakly commensurable, then they are actually

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commensurable (in particular, the other subgroup is also arithmetic), which implies that if one of the two isospectral Riemann surfaces is arithmetically defined then the surfaces are commensurable. To extend this result to more general Zariski-dense subgroups (first and foremost, to non-arithmetic lattices in $\mathrm{SL}_2(\mathbb{R})$) using the techniques developed in [11], one needs to know what other fields have (*). It was observed by Rost, Wadsworth and others that it is possible to construct “large” (in particular, infinitely generated) fields which do not have (*) (cf. [3], Example 2.1). On the other hand, no such examples are known for finitely generated fields (and the fields that arise in the analysis of weakly commensurable finitely generated Zariski-dense subgroups are finitely generated), and the question as to what finitely generated fields have (*) remains wide open. In fact, until recently no fields other than global fields were known to have (*). In [3], Garibaldi and Saltman answered in the affirmative one of the questions posed in earlier versions of [11] by showing that a purely transcendental extension $K = k(x_1, \dots, x_r)$ of a number field k has (*) (more generally, it was shown in [3] that any *transparent* field of characteristic $\neq 2$ has (*)).

The goal of this note is to present two further results on (*) and related issues. All fields below will be of characteristic $\neq 2$. First, we show that the property to have (*) is stable under purely transcendental extensions, which gives an alternative proof of the fact that $K = k(x_1, \dots, x_r)$, with k a number field, has (*).

Theorem A. *Let K be a field of characteristic $\neq 2$. If (*) holds for K then it also holds for the field of rational functions $K(x)$.*

Second, we give examples of fields for which a property similar to (*) holds for algebras that are more general than quaternions. We notice that given any central division algebra D over K , the opposite algebra D^{op} has the same maximal subfields as D (cf. Lemma 3.6 for a more general statement), so (the natural analog of) (*) definitely fails unless $D \simeq D^{\mathrm{op}}$, i.e. the class $[D]$ has exponent two in the Brauer group $\mathrm{Br}(K)$. On the other hand, there are division algebras of exponent two that are more general than quaternion algebras, and whether or not (*) holds for them is a meaningful question (as the following theorem demonstrates).

Theorem B. *Let $K = k(x_1, \dots, x_r)$ be a purely transcendental extension of a field k which is either a totally imaginary number field with a single dyadic place (e.g. $k = \mathbb{Q}(\sqrt{-1})$), or is a finite field of characteristic > 2 , and let D_1 and D_2 be two finite dimensional central division algebras over K such that the classes $[D_1], [D_2] \in \mathrm{Br}(K)$ have exponent two. If D_1 and D_2 have the same maximal subfields then $D_1 \simeq D_2$.*

One can view Theorem B as giving some indication that an analog of (*) may hold over some fields more general than number fields for some other absolutely almost simple algebraic groups associated with algebras of exponent two. More precisely, following [11], 5.4, we call two K -forms G_1 and G_2 of an absolutely almost simple algebraic group G *weakly K -isomorphic* if they have the same isomorphism classes of maximal K -tori, and the question is in what situations weakly K -isomorphic groups are necessarily K -isomorphic. Based on Theorem B, one

can hope that the affirmative answer is possible in certain cases where G is of type B_n , C_n and G_2 (and maybe E_7 and F_4) - see the end of Sect. 3 for a more detailed discussion, but none of these types has been investigated so far.

The proofs of Theorems A (Sect. 4) and B (Sect. 3), just as the argument in [3], are based on an analysis of ramification.

Notations and conventions. All fields in this note will be of characteristic $\neq 2$. For a central simple algebra A over a field K , $[A]$ will denote the corresponding class in the Brauer group $\text{Br}(K)$. For $a, b \in K^\times$, we let $\left(\frac{a, b}{K}\right)$ denote the corresponding quaternion algebra. Given a valuation v of a field K (and all valuations in this note will be discrete), we let $\mathcal{O}_{K,v}$, K_v and $\bar{K}^{(v)}$ denote the corresponding valuation ring, the completion and the residue field, respectively.

2. Preliminaries

Let K be a field endowed with a discrete valuation v . For a finite dimensional K -algebra A , we set $A_v = A \otimes_K K_v$ endowed with the topology of a vector space over K_v . We recall that an étale K -algebra is defined to be a finite direct product of finite separable extensions of K . Then the notion for two simple algebras to have the same maximal étale subalgebras is defined in the obvious way (clearly, algebras with the same maximal étale subalgebras have the same dimension).

Lemma 2.1 ([3], Lemma 3.1). *Let A_1 and A_2 be two central simple algebras over K , and let v be a discrete valuation of K . If A_1 and A_2 have the same maximal étale subalgebras then the algebras A_{1v} and A_{2v} also have the same maximal étale subalgebras.*

Proof. We give an argument based on a construction described in [10], proof of Theorem 3(ii) and in [11], Lemma 3.4. Let E_1 be a maximal étale subalgebra of A_{1v} . Let $G_1 = \text{GL}_{1,A_1}$ be the algebraic K -group associated to A_1 , and let $T_1 = \text{R}_{E_1/K_v}(\text{GL}_1)$ be the maximal K_v -torus of G_1 corresponding to E_1 . Consider the Zariski-open subset $T_1^{\text{reg}} \subset T_1$ of regular elements and the regular map

$$\varphi: G_1 \times T_1^{\text{reg}} \longrightarrow G, \quad (g, t) \mapsto gtg^{-1}.$$

It is easy to check that the differential $d_{(g,t)}\varphi$ is surjective for any $(g, t) \in G_1 \times T_1^{\text{reg}}$, so it follows from the Implicit Function Theorem that the map

$$\varphi_v: G_1(K_v) \times T_1^{\text{reg}}(K_v) \longrightarrow G_1(K_v), \quad (g, t) \mapsto gtg^{-1},$$

is open in the topology defined by v . In particular, $U_1 = \text{Im } \varphi_v$ is open in $G_1(K_v) = A_{1v}^\times$. On the other hand, by weak approximation for K , we have that A_1 is dense in A_{1v} . So, pick $a \in A_1 \cap U_1$ and set $E_1^0 = K[a]$. Then E_1^0 is a maximal étale subalgebra of A_1 and there exists $g \in A_{1v}^\times$ such that $E_1^0 \otimes_K K_v = gE_1g^{-1}$. By our assumption, there exists an embedding $\iota^0: E_1^0 \hookrightarrow A_2$. Then

$$\iota = (\iota^0 \otimes \text{id}_{K_v}) \circ \text{Int } g$$

is a required embedding of E_1 into A_{2v} . By symmetry, A_{1v} and A_{2v} have the same maximal étale subalgebras.

Remark 2.2. The above argument uses the property of weak approximation for G_1 which may fail for some algebraic groups. Nevertheless, the following analog of Lemma 2.1 is true in general: Let G_1 and G_2 be two reductive algebraic groups over a field K , and let v be a discrete valuation of K . If G_1 and G_2 have the same isomorphism classes of maximal tori over K then they have the same isomorphism classes of maximal tori over K_v . To prove this, one needs to invoke weak approximation in the variety of maximal tori - cf. the proof of Theorem 1(ii) in [9].

Lemma 2.3. *Let $A_i = M_{d_i}(D_i)$ for some $d_i \geq 1$ and some central division algebra D_i over K , where $i = 1, 2$. If A_1 and A_2 have the same maximal étale subalgebras then $d_1 = d_2$, and D_1 and D_2 have the same maximal subfields that are separable over K .*

Proof. Let P_1 be a maximal separable subfield of D_1 . Then $L_1 = P_1 \oplus \cdots \oplus P_1$ (d_1 times) is a maximal étale subalgebra of A_1 . Let $e_1 = (1, 0, \dots, 0), \dots, e_{d_1} = (0, \dots, 0, 1)$ be the orthogonal idempotents in L_1 . By our assumption, L_1 admits an embedding into A_2 , and we identify the former with its image in the latter. Consider the right D_2 -vector space $V_2 = (D_2)^{d_2}$ as a left A_2 -module. Then the relations $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$ imply that the sum of the nonzero D_2 -subspaces $W_i = e_i V_2$, where $i = 1, \dots, d_1$, is direct, hence $d_1 \leq d_2$. By symmetry, $d_1 = d_2$, and then D_1 and D_2 have the same dimension. In the above notations, it follows that each W_i is 1-dimensional over D_2 and $V_2 = \bigoplus W_i$. Clearly, $P_1 \simeq L_1 e_1$ embeds in $\text{End}_{D_2} W_1 \simeq D_2$, and the required fact follows. \square

Corollary 2.4. *Let A_1 and A_2 be two central simple algebras over K such that $\dim_K A_1 = \dim_K A_2$ is relatively prime to $\text{char } K$, and let v be a discrete valuation of K . Write $A_{i_v} = M_{d_i}(D_i)$ with $d_i \geq 1$ and D_i a central division K_v -algebra. If A_1 and A_2 have the same maximal étale subalgebras then $d_1 = d_2$ and D_1 and D_2 have the same maximal subfields.*

Proof. By Lemma 2.1, the K_v -algebras A_{1_v} and A_{2_v} have the same maximal étale subalgebras. So, by Lemma 2.3, we have $d_1 = d_2$ and D_1 and D_2 have the same maximal separable subfields. However, since $\dim_K A_1 = \dim_K A_2$ is prime to $\text{char } K$, all maximal subfields of D_1 and D_2 are separable over K_v .

To formulate our next lemma, we need to introduce one condition on a field k :

(LD) Given finite separable extensions $E \subset F$ of k , any central division algebra Δ over E contains a maximal separable subfield P that is linearly disjoint from F over E .

(It is worthwhile to observe that the fields of real and p -adic numbers do not have (LD), while global fields do have it—cf. Proposition 2.7 as (trivially) do finite and algebraically closed fields.)

Lemma 2.5. *Let \mathcal{K} be a field complete with respect to a discrete valuation v , with residue field $k = \bar{\mathcal{K}}^{(v)}$, and let $\mathcal{D}_1, \mathcal{D}_2$ be two central division algebras over \mathcal{K} . Assume that the center $\mathcal{E}_i := Z(\bar{\mathcal{D}}_i)$ is a separable extension of k for $i = 1, 2$. If \mathcal{D}_1 and \mathcal{D}_2 have the same maximal subfields then*

- (i) $[\mathcal{E}_1 : k] = [\mathcal{E}_2 : k]$;
 (ii) in each of the following situations: (a) \mathcal{D}_1 and \mathcal{D}_2 are of prime degree, (b) k satisfies (LD), we have $\mathcal{E}_1 = \mathcal{E}_2$.

Proof. (i): Let \tilde{v}_i be the extension of v to \mathcal{D}_i . It is well-known that since \mathcal{E}_i/k is separable, \mathcal{D}_i is inertially split (this, for example, easily follows from the fact that discrete valuation are defectless, see [17], Sect. 2). Then $[\mathcal{E}_i : k]$ coincides with the ramification index $e(\tilde{v}_i|v) = [\tilde{v}_i(\mathcal{D}_i^\times) : v(\mathcal{K}^\times)]$ (cf. [15], Chap. XII, Sect. 2, Example 3, or [17], Theorem 3.4), so it is enough to show that $e(\tilde{v}_1|v) = e(\tilde{v}_2|v)$. Let $a \in \mathcal{D}_1$ be such that $\tilde{v}_1(a)$ generates the value group $\tilde{v}_1(\mathcal{D}_1^\times)$, and let \mathcal{L} be a maximal subfield of \mathcal{D}_1 containing a . Then $e(\tilde{v}_1|v) = [\tilde{v}_1(\mathcal{L}^\times) : v(\mathcal{K}^\times)]$. On the other hand, by our assumption \mathcal{L} can be embedded in \mathcal{D}_2 , and if we identify the former with its image in the latter, then $\tilde{v}_2|_{\mathcal{L}} = \tilde{v}_1|_{\mathcal{L}}$ because v has a unique extension to \mathcal{L} as \mathcal{K} is complete (cf. [15], Chap. II, Sect. 2, Corollary 2). It follows that $e(\tilde{v}_1|v) \leq e(\tilde{v}_2|v)$. By symmetry, $e(\tilde{v}_1|v) = e(\tilde{v}_2|v)$, as required.

- (ii): First, suppose \mathcal{D}_1 and \mathcal{D}_2 have prime degree p . If \mathcal{D}_1 is unramified then $\mathcal{E}_1 = k$, so we obtain from (i) that $\mathcal{E}_2 = k$, and there is nothing to prove. If \mathcal{D}_1 is ramified then

$$e(\tilde{v}_1|v) = p = [\mathcal{E}_1 : k],$$

so it follows from the formula in [15], *loc. cit.*, that $[\bar{\mathcal{D}}_1 : \mathcal{E}_1]$ divides p , and therefore in fact $\bar{\mathcal{D}}_1 = \mathcal{E}_1$. Similarly, $\bar{\mathcal{D}}_2 = \mathcal{E}_2$. Pick a in the valuation ring $\mathcal{O}_{\mathcal{D}_1, \tilde{v}_1}$ so that for its residue $\bar{a} \in \bar{\mathcal{D}}_1$ we have $\bar{\mathcal{D}}_1 = k(\bar{a})$. Then $\mathcal{L} := \mathcal{K}(a)$ is a maximal unramified subfield of \mathcal{D}_1 , with residue field $\tilde{\mathcal{L}}^{(\tilde{v}_1)} = \mathcal{E}_1$. As in the proof of (i), \mathcal{L} embeds into \mathcal{D}_2 and $\tilde{v}_1|_{\mathcal{L}} = \tilde{v}_2|_{\mathcal{L}}$. So, $[\tilde{\mathcal{L}}^{(\tilde{v}_2)} : k] = p$, hence $\tilde{\mathcal{L}}^{(\tilde{v}_2)} = \bar{\mathcal{D}}_2 = \mathcal{E}_2$. Finally,

$$\mathcal{E}_1 = \tilde{\mathcal{L}}^{(\tilde{v}_1)} = \tilde{\mathcal{L}}^{(\tilde{v}_2)} = \mathcal{E}_2,$$

as required.

Now, assume that k possesses property (LD), and let n denote the common degree of \mathcal{D}_1 and \mathcal{D}_2 . We will prove that $\mathcal{E}_2 \subset \mathcal{E}_1$; then (i) will yield $\mathcal{E}_1 = \mathcal{E}_2$. Set $\mathcal{F} = \mathcal{E}_1\mathcal{E}_2$ (in a fixed algebraic closure of k). Since \mathcal{E}_1 is separable over k , by (LD), there exists a maximal separable subfield \mathcal{P} of $\bar{\mathcal{D}}_1$ that is linearly disjoint from \mathcal{F} over \mathcal{E}_1 . Write $\mathcal{P} = k(\bar{a})$, and set $\mathcal{L} = \mathcal{K}(a)$. As \mathcal{D}_1 is inertially split, we conclude, using ([17], Theorem 3.4(iii)), that

$$[\mathcal{P} : k]^2 = [\mathcal{P} : \mathcal{E}_1]^2 [\mathcal{E}_1 : k]^2 = [\bar{\mathcal{D}}_1 : k] [\mathcal{E}_1 : k] = [\mathcal{D}_1 : \mathcal{K}],$$

i.e. $[\mathcal{P} : k] = n$. It follows that \mathcal{L} is a maximal subfield of \mathcal{D}_1 with residue field $\tilde{\mathcal{L}}^{(\tilde{v}_1)} = \mathcal{P}$. As above, \mathcal{L} embeds into \mathcal{D}_2 , and therefore $\mathcal{P} = \tilde{\mathcal{L}}^{(\tilde{v}_1)} = \tilde{\mathcal{L}}^{(\tilde{v}_2)}$ embeds into $\bar{\mathcal{D}}_2$. It follows that $\mathcal{P} \supset \mathcal{E}_2$. (Indeed, otherwise $\mathcal{E}_2\mathcal{P}$ would be a separable extension of k contained in $\bar{\mathcal{D}}_2$ and having degree $> n$. Writing $\mathcal{E}_2\mathcal{P} = k(\bar{b})$ for some $b \in \mathcal{O}_{\mathcal{D}_2, \tilde{v}_2}$, we would find that $\mathcal{K}(b)$ would be an extension of \mathcal{K} contained in \mathcal{D}_2 and of degree $> n$, which is impossible.) Since \mathcal{P} was chosen to be linearly disjoint from $\mathcal{F} = \mathcal{E}_1\mathcal{E}_2$ over \mathcal{E}_1 , we conclude that $\mathcal{F} = \mathcal{E}_1$, i.e. $\mathcal{E}_2 \subset \mathcal{E}_1$. \square

Remark 2.6. We would like to point out that the assertion of Lemma 2.5 (ii) that $\mathcal{E}_1 = \mathcal{E}_2$ also holds if \mathcal{D}_1 and \mathcal{D}_2 are of degree 4 and k is of characteristic $\neq 2$ such that the quaternion algebra $\left(\frac{-1, -1}{k}\right)$ is not a division algebra (in particular, if $\sqrt{-1} \in k$). Indeed, the argument in the cases where $[\mathcal{E}_1 : k] = [\mathcal{E}_2 : k]$ equals 1 or 4 is identical to the one given in Case (a) of Lemma 2.5 (ii), so we only need to consider the situation where $[\mathcal{E}_1 : k] = [\mathcal{E}_2 : k] = 2$, hence $\bar{\mathcal{D}}_i$ is a central quaternion division algebra over \mathcal{E}_i for $i = 1, 2$. To mimic the argument used in Case (b), we need to show that $\bar{\mathcal{D}}_1$ contains a maximal subfield \mathcal{P} which is linearly disjoint from $\mathcal{E}_1\mathcal{E}_2$ over \mathcal{E}_1 , and vice versa. Since $[\mathcal{E}_1\mathcal{E}_2 : \mathcal{E}_1] \leq 2$, this will obviously hold automatically if not all maximal subfields of $\bar{\mathcal{D}}_1$ are \mathcal{E}_1 -isomorphic. Now, according to ([7], Example 4 in Sect. 13.6), if all maximal subfields of a quaternion division algebra D over a field F of characteristic $\neq 2$ are isomorphic then F is formally real, pythagorean and $D \simeq \left(\frac{-1, -1}{F}\right)$. However, our assumption implies that $\left(\frac{-1, -1}{\mathcal{E}_1}\right)$ is not a division algebra, and the required fact follows.

We will now describe a class of fields having property (LD).

Proposition 2.7. *Let k be finitely generated over its prime subfield. Then k has property (LD).*

Proof. We will assume (as we may) that k is infinite. Let $E \subset F$ be finite separable extensions of k , and let Δ be a central division algebra over E . Enlarging F , we can assume that $\Delta \otimes_E F \simeq M_n(F)$. It is enough to construct a discrete valuation w of E so that the completion E_w is locally compact and coincides with $F_{\tilde{w}}$ for some extension $\tilde{w}|w$. Indeed, we can then pick a separable extension \mathcal{P} of E_w of degree n and embed it into $\Delta_w = \Delta \otimes_E E_w \simeq M_n(E_w)$. The argument given in the proof of Lemma 2.1 (or the standard Krasner's Lemma, cf. [6], Lemma 8.1.6) enables us to construct a maximal subfield $P \subset \Delta$ such that $P \otimes_E E_w \simeq \mathcal{P}$. Then

$$[PF : F] \geq [\mathcal{P}F_{\tilde{w}} : F_{\tilde{w}}] = [\mathcal{P} : E_w] = n.$$

So, $[PF : F] = n$, and therefore P and F are linearly disjoint over E .

In characteristic zero, according to Proposition 1 in [10], for infinitely many primes p there exists an embedding $F \hookrightarrow \mathbb{Q}_p$, and then the valuations w and \tilde{w} of E and F respectively, obtained as pull-backs of the p -adic valuation, are as required. If $\text{char } k = p > 0$ then we need to use a suitable modification of the proof of Proposition 1 in [10]. Let \mathbb{F}_p be the field with p elements. There exist algebraically independent $t_1, \dots, t_r \in E$ with $r > 0$ such that E is a finite separable extension of $\ell := \mathbb{F}_p(t_1, \dots, t_r)$, and then F is also a finite separable extension of ℓ . Furthermore, we can pick a primitive element $a \in F$ over ℓ having minimal polynomial f of the form

$$f(s, t_1, \dots, t_r) = s^d + c_{d-1}(t_1, \dots, t_r)s^{d-1} + \dots + c_0(t_1, \dots, t_r)$$

where $c_i(t_1, \dots, t_r) \in \mathbb{F}_p[t_1, \dots, t_r]$. Since f is prime to its derivative f_s , there exist polynomials $g(s, t_1, \dots, t_r)$, $h(s, t_1, \dots, t_r)$ and $m(t_1, \dots, t_r)$ with coefficients in \mathbb{F}_p such that

$$gf + hf_s = m \neq 0. \quad (1)$$

One can find polynomials $\alpha_1(t), \dots, \alpha_r(t) \in \mathbb{F}_p[t]$ such that $\alpha_1(t) = t$ and $m(\alpha_1(t), \dots, \alpha_r(t)) \neq 0$. Set $\varphi(s) = f(s, \alpha_1(t), \dots, \alpha_r(t))$. Let β be a root of $\varphi(s)$ (in a fixed algebraic closure of $\mathbb{F}_p(t)$), and let $R = \mathbb{F}_p(t)(\beta)$. By Tchebotarev's Density Theorem (cf. [1], Chap. VII, 2.4), one can find a valuation v of $\mathbb{F}_p(t)$ associated to some irreducible polynomial in $\mathbb{F}_p[t]$ such that

$$v(m(\alpha_1(t), \dots, \alpha_r(t))) = 0, \quad (2)$$

and $R_{\tilde{v}} = \mathbb{F}_p(t)_v$ for some extension $\tilde{v}|v$. Let $\mathbb{F}_q = \mathcal{O}_{R, \tilde{v}}/\mathfrak{P}_{\tilde{v}} = \mathcal{O}_{\mathbb{F}_p(t), v}/\mathfrak{p}_v$ be the residue field, and let $\beta^0, \alpha_1^0, \dots, \alpha_r^0$ be the images of $\beta, \alpha_1(t), \dots, \alpha_r(t)$ in \mathbb{F}_q . Since $\alpha_1(t) = t$, we see that $\mathbb{F}_q = \mathbb{F}_p(\alpha_1^0)$. It follows from (1) and (2) that

$$f(\beta^0, \alpha_1^0, \dots, \alpha_r^0) = 0, \quad \text{but} \quad f_s(\beta^0, \alpha_1^0, \dots, \alpha_r^0) \neq 0. \quad (3)$$

Let $\mathcal{F} = \mathbb{F}_q((T))$. Pick r elements $\tilde{t}_1, \dots, \tilde{t}_r \in \mathbb{F}_q[[T]]$ that are algebraically independent over \mathbb{F}_q and satisfy

$$\tilde{t}_1 = \alpha_1^0 + T, \quad \tilde{t}_i = \alpha_i^0 \pmod{T} \quad \text{for } i > 1, \quad (4)$$

and then consider the embedding $\ell \hookrightarrow \mathcal{F}$ sending t_i to \tilde{t}_i for all $i = 1, \dots, r$. We claim that

$$\overline{\iota(\ell)} = \mathcal{F}, \quad (5)$$

where here, and in the rest of the proof of the proposition, the bar denotes the closure in the T -adic topology and not the residue field like elsewhere in the paper. Indeed, it follows from (4) that the image of $\iota(t_1)$ in $\mathbb{F}_q[[T]]/T\mathbb{F}_q[[T]] = \mathbb{F}_q$ generates \mathbb{F}_q , which implies (e.g. by Hensel's Lemma) that $\mathbb{F}_q \subset \overline{\iota(\ell)}$. We then see from (4) that $T \in \overline{\iota(\ell)}$, and (5) follows. To complete the argument, we will now show that ι can be extended to an embedding $\tilde{\iota}: F \hookrightarrow \mathcal{F}$ as then the pullbacks of the natural valuation on \mathcal{F} will give us the required valuations w and \tilde{w} on E and F respectively. Using Hensel's Lemma, one obtains from (3) and (4) that \mathcal{F} contains a root to $f(s, \tilde{t}_1, \dots, \tilde{t}_r) = 0$, and the existence of $\tilde{\iota}$ follows. \square

Remark 2.8. The following was communicated to us by Skip Garibaldi. One can consider the following property of a field E :

(+) For every finite Galois extension F/E there exists a discrete valuation on E that splits completely in F .

The proof of Proposition 2.7 is actually the combination of two statements: (i) If every finite separable extension E/k has (+) then k has (LD), and (ii) Infinite fields that are finitely generated over the prime field have (+). Regarding (ii), one can prove the following: *Let K be an arbitrary field, and let E/K be a separable finitely generated extension of transcendence degree ≥ 1 . Then E has property (+).* This statement, which is contained in [8] in a different language, gives Proposition 2.7 because global fields have (+) by Tchebotarev's Density Theorem.

3. Proof of Theorem B

First, we will single out some conditions on the field K that imply the assertion of Theorem B (see Theorem 3.2). We will then verify these conditions for the fields considered in Theorem B (see Proposition 3.4).

For a field \mathcal{K} complete with respect to a discrete valuation v , we let $\mathrm{Br}^0(\mathcal{K})$, or $\mathrm{Br}_v^0(\mathcal{K})$, denote the subgroup of $\mathrm{Br}(\mathcal{K})$ consisting of elements that split over an unramified extension of \mathcal{K} (in other words, $\mathrm{Br}^0(\mathcal{K}) = \mathrm{Br}(\mathcal{K}_{\mathrm{nr}}/\mathcal{K})$ where $\mathcal{K}_{\mathrm{nr}}$ is the maximal unramified extension of \mathcal{K})¹. We recall that for a central division algebra \mathcal{D} over \mathcal{K} , we have $[\mathcal{D}] \in \mathrm{Br}^0(\mathcal{K})$ if and only if the center $Z(\bar{\mathcal{D}}^{(v)})$ of the residue algebra is a separable extension of $\bar{\mathcal{K}}^{(v)}$ ([17], Theorem 3.4); the elements of $\mathrm{Br}^0(\mathcal{K})$ are called “inertially split” (cf. [14], [17]). It follows that if n is relatively prime to $\mathrm{char} \bar{\mathcal{K}}^{(v)}$ then the n -torsion subgroup $\mathrm{Br}(\mathcal{K})_n$ is contained in $\mathrm{Br}^0(\mathcal{K})$.

Furthermore, we let ρ or ρ_v denote the reduction map $\mathrm{Br}_v^0(\mathcal{K}) \rightarrow \mathrm{Hom}(G(v), \mathbb{Q}/\mathbb{Z})$ where $G(v)$ is the absolute Galois group of the residue field $\bar{\mathcal{K}}^{(v)}$ (see, for example, [15], Chap. XII, Sect. 3, or [17], (3.9)). Then $\mathrm{Br}'_v(\mathcal{K}) := \mathrm{Ker} \rho_v$ is known to consist precisely of the classes of all unramified (or “inertial”) division algebras (equivalently, those division algebras that arise from the Azumaya algebras over the valuation ring $\mathcal{O}_{\mathcal{K},v}$, cf. [17], Theorem 3.2). We recall that $\mathrm{Br}'_v(\mathcal{K})$ can be naturally identified with the Brauer group of the residue field of \mathcal{K} . More generally, given a discrete valuation v of a field K , we let $\mathrm{Br}'_v(K)$ denote the subgroup of classes $[A] \in \mathrm{Br}(K)$ for which $[A \otimes_K K_v] \in \mathrm{Br}'_v(K_v)$.

Definition 3.1. Let K be an infinite field of characteristic $\neq 2$. We say that K is *2-balanced* if there exists a set V of discrete valuations of K such that

- (a) for each $v \in V$, the residue field $\bar{K}^{(v)}$ satisfies (LD) and is of characteristic $\neq 2$;
- (b) $\bigcap_{v \in V} \mathrm{Br}'_v(K)_2 = \{e\}$ (in other words, the 2-component of the unramified Brauer group of K with respect to V is trivial).

Theorem 3.2. Let K be a 2-balanced field, and let D_1, D_2 be two central division algebras over K such that $[D_1], [D_2] \in \mathrm{Br}(K)_2$. If D_1 and D_2 have the same maximal subfields then $D_1 \simeq D_2$.

Proof. For $v \in V$, we let ρ_v denote the reduction map $\mathrm{Br}^0(K_v) \rightarrow \mathrm{Hom}(G(v), \mathbb{Q}/\mathbb{Z})$. The assumption that $\mathrm{char} \bar{K}^{(v)} \neq 2$ implies that $[D_1 \otimes_K K_v], [D_2 \otimes_K K_v] \in \mathrm{Br}^0(K_v)$, and then due to condition (b) in the above definition, it is enough to show that

$$\rho_v([D_1 \otimes_K K_v]) = \rho_v([D_2 \otimes_K K_v]), \quad (6)$$

for all $v \in V$. Fix $v \in V$, and set $\mathcal{K} = K_v$. According to Lemma 2.1, the algebras $D_1 \otimes_K \mathcal{K}$ and $D_2 \otimes_K \mathcal{K}$ have the same maximal étale subalgebras. Using Corollary 2.4, we see that

$$D_i \otimes_K \mathcal{K} = M_{d_i}(\mathcal{D}_i), \quad i = 1, 2,$$

¹ As usual, the definition of an unramified extension \mathcal{L}/\mathcal{K} includes the requirement that the corresponding extension of the residue fields $\bar{\mathcal{L}}^{(v)}/\bar{\mathcal{K}}^{(v)}$ be separable.

where $d_1 = d_2$ and the division algebras \mathcal{D}_1 and \mathcal{D}_2 have the same maximal subfields. To prove (6), it suffices to show that

$$\rho_v([\mathcal{D}_1]) = \rho_v([\mathcal{D}_2]). \quad (7)$$

Let $\chi_i = \rho_v([\mathcal{D}_i]) \in \text{Hom}(G(v), \mathbb{Q}/\mathbb{Z})$. Since \mathcal{D}_1 and \mathcal{D}_2 have the same maximal subfields and $\bar{K}^{(v)} = \bar{K}^{(v)}$ satisfies (LD), we conclude from Lemma 2.5 (ii), case (b), that $Z(\bar{\mathcal{D}}_1) = Z(\bar{\mathcal{D}}_2)$. It is known, however, that the extension $Z(\bar{\mathcal{D}}_i)/\bar{K}$ corresponds to $\text{Ker } \chi_i$ (cf. [17], Theorem 3.5). So, we obtain that $\text{Ker } \chi_1 = \text{Ker } \chi_2$, and since χ_1 and χ_2 both have order two, we conclude that $\chi_1 = \chi_2$, yielding (7).

Remark 3.3. Using Remark 2.6 in place of Lemma 2.5 (ii), we obtain that if a field K with the property that the quaternion algebra $\left(\frac{-1, -1}{K}\right)$ is not a division algebra, has a set V of discrete valuations such that $\text{char } \bar{K}^{(v)} \neq 2$ for each $v \in V$ and $\bigcap_{v \in V} \text{Br}'_v(K) = \{e\}$ then for any central division algebras D_1 and D_2 of degree four over K such that $[D_1], [D_2] \in \text{Br}(K)_2$, the fact that D_1 and D_2 have the same maximal subfields implies that $D_1 \simeq D_2$ (we only need to observe that $\left(\frac{-1, -1}{\bar{K}^{(v)}}\right)$ is not a division algebra for all $v \in V$).

The following proposition establishes that the fields in the statement of Theorem B are 2-balanced, completing thereby the proof of the latter.

Proposition 3.4. *Let $K = k(x_1, \dots, x_r)$ be a purely transcendental extension of a field k which is either a totally imaginary number field with a single dyadic place (e.g. $k = \mathbb{Q}(\sqrt{-1})$) or a finite field of characteristic > 2 . Then K is 2-balanced.*

Proof. Let $K_i = k(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r)$, and let V_i be the set of all valuations of $K = K_i(x_i)$ that are trivial on K_i . Then for $v \in V_i$ the residue field $\bar{K}^{(v)}$ is a finite extension of K_i , hence a finitely generated field of characteristic $\neq 2$. Invoking Proposition 2.7, we see that $\bar{K}^{(v)}$ satisfies (LD), and therefore

$$V_0 := \bigcup_{i=1}^r V_i$$

satisfies condition (a) of Definition 3.1. Henceforth, we will identify $\text{Br}(k)$ with a subgroup of $\text{Br}(K)$ using the natural embedding. We claim that

$$\bigcap_{v \in V_0} \text{Br}'_v(K)_2 = \text{Br}(k)_2. \quad (8)$$

This is proved by induction on r using the following consequence of Faddeev's exact sequence (cf. [4], Corollary 6.4.6, or [7], Sect. 19.5; for the case of a nonperfect field of constants, see [2], Example 9.21 on p. 26, and [13]): Let F be a field of characteristic $\neq 2$, and let V^F be the set of valuations of the field of rational functions $F(x)$ that are trivial on F ; then

$$\bigcap_{v \in V^F} \text{Br}'_v(F(x))_2 = \text{Br}(F)_2. \quad (9)$$

For $r = 1$, (8) is identical to (9), and there is nothing to prove. For $r > 1$, set $k' = k(x_r)$ and $V'_0 = \bigcup_{i=1}^{r-1} V_i$. By induction hypothesis,

$$\bigcap_{v \in V'_0} \text{Br}'_v(K)_2 = \text{Br}(k')_2. \quad (10)$$

On the other hand, there is a natural bijection between $V^{k'}$ and V_r , and for a given $v \in V^{k'}$ we let \hat{v} denote the corresponding valuation in V_r . Clearly, if a central division algebra Δ over k' (of exponent two) is ramified at $v \in V^{k'}$, i.e. $[\Delta] \notin \text{Br}'_v(k')_2$, then $\hat{\Delta} = \Delta \otimes_{k'} K$ is ramified at \hat{v} , i.e. $[\hat{\Delta}] \notin \text{Br}'_{\hat{v}}(K)$. Thus, if a central division algebra D over K represents an element from the left-hand side of (8), then by (10) we can write $D = \Delta \otimes_{k'} K$ for some central division algebra Δ over k' . Furthermore, it follows from our previous remark that $[\Delta] \in \bigcap_{v \in V^{k'}} \text{Br}'_v(k')$. So, using (9) for $F = k$, we see that $[\Delta] \in \text{Br}(k)$, proving (8). It immediately follows that $V = V_0$ is as required if k is finite.

Let now k be a totally imaginary number field with a single dyadic place. It follows from the Albert-Hasse-Brauer-Noether Theorem (cf. [7], Sect. 18.4) that for the set W of all non-dyadic nonarchimedean places of k the natural map

$$\text{Br}(k) \longrightarrow \bigoplus_{w \in W} \text{Br}(k_w)$$

is injective. Since $\text{Br}'_w(k_w) = \{e\}$, this can be restated as

$$\bigcap_{w \in W} \text{Br}'_w(k) = \{e\}. \quad (11)$$

For $w \in W$, we let \tilde{w} denote its natural extension to K given by

$$\tilde{w} \left(\sum_{i_1, \dots, i_r} a_{i_1 \dots i_r} x_1^{i_1} \cdots x_r^{i_r} \right) = \inf w(a_{i_1 \dots i_r}).$$

Then the residue field $\bar{K}^{(\tilde{w})} = \bar{k}^{(w)}(x_1, \dots, x_r)$ is a finitely generated field of characteristic $\neq 2$, hence satisfies (LD) (see Proposition 2.7). It follows that $V := V_0 \cup \tilde{W}$, where $\tilde{W} = \{\tilde{w} \mid w \in W\}$, satisfies condition (a) of Definition 3.1. At the same time, using (8) and (11) and arguing as above, we see that

$$\bigcap_{v \in V} \text{Br}'_v(K)_2 = \{e\},$$

which is condition (b) of Definition 3.1. Thus, V is as required. \square

Remark 3.5. (Continuation of Remark 2.8 by Skip Garibaldi) Using the statement from Remark 2.8, one can prove the following: *If F is 2-balanced and has (LD) then $F(t)$ is also 2-balanced and has (LD).* This provides another perspective on the proof of Proposition 3.4.

As we already mentioned in the introduction, for any central division K -algebra D , the opposite algebra D^{op} has the same maximal subfields, but $D \not\cong D^{\text{op}}$ unless $[D] \in \text{Br}(K)_2$. It should be noted, however, that the associated norm one groups $\text{SL}_{1,D}$ and $\text{SL}_{1,D^{\text{op}}}$ are always K -isomorphic. So, we would like to point out the following general construction of division algebras D_1 and D_2 that have the same maximal subfields, but for which $\text{SL}_{1,D_1} \not\cong \text{SL}_{1,D_2}$. Let Δ_1 and Δ_2 be two central division algebras over K of relatively prime degrees $n_1, n_2 > 2$. Then $D_1 = \Delta_1 \otimes_K \Delta_2$ and $D_2 = \Delta_1 \otimes_K \Delta_2^{\text{op}}$ are division algebras of degree $n = n_1 n_2$, which are neither isomorphic nor anti-isomorphic, so the corresponding norm one groups are not K -isomorphic. At the same time, if P is a maximal subfield of D_1 then it splits Δ_1 and Δ_2 , hence also Δ_2^{op} . It follows that P splits D_2 , and therefore is isomorphic to its maximal subfield. A more general perspective on this construction can be derived from the following (known) statement (to see the connection, one observes that for the algebras D_1 and D_2 as above, the classes $[D_1], [D_2] \in \text{Br}(K)$ generate the same subgroup, which coincides with the subgroup generated by $[\Delta_1]$ and $[\Delta_2]$, hence $[D_2] = m[D_1]$ for some m relatively prime to n).

Lemma 3.6. *Let D be a central division algebra of degree n over a field K . Then for any $m \geq 1$ which is relatively prime to n , the class $m[D] \in \text{Br}(K)$ is represented by a central division algebra D_m of the same degree n and having the same maximal subfields as D .*

Proof. First, we observe that if A_1 and A_2 are two central simple algebras over K of the same degree d containing a field extension P/K of degree d then $A := A_1 \otimes_K A_2$ is isomorphic to $M_d(A')$ where A' is a central simple algebra of degree d that also contains P . Indeed, we have $A_1 \otimes_K P \simeq M_d(P)$ so A contains $B := M_d(K)$. Using the Double Centralizer Theorem, we conclude that $A \simeq B \otimes_K C_A(B)$, i.e. $A \simeq M_d(A')$ where $A' = C_A(B)$ is a central simple algebra of degree d . Since

$$A \otimes_K P \simeq (A_1 \otimes_K P) \otimes_P (A_2 \otimes_K P) \simeq M_{d^2}(P),$$

we see that P splits A' and therefore is isomorphic to a maximal subfield of the latter (see [7], Sect. 13.3). This remark combined with simple induction shows that for any $m \geq 1$ we have

$$D^{\otimes m} \simeq M_{n^{m-1}}(D_m) \quad (12)$$

where D_m is a central simple algebra of degree n such that every maximal subfield of D embeds in D_m . Let now m be relatively prime to n , and pick $\ell \geq 1$ so that $\ell m \equiv 1 \pmod{n}$, hence $\ell[D_m] = [D]$. If $D_m = M_s(\Delta)$ where Δ is a division algebra of degree t , $st = n$, then $\ell[D_m] = \ell[\Delta]$. Applying (12) to Δ and ℓ in place of D and m , we see that $\ell[\Delta]$ is represented by a central simple algebra Δ_ℓ of degree t . Then $[\Delta_\ell] = [D]$ is possible only if D_m is isomorphic to Δ , hence a division algebra, and $\Delta_\ell \simeq D$. Furthermore, as we have seen, every maximal subfield of D embeds in D_m , and every maximal subfield of $D_m \simeq \Delta$ embeds in $\Delta_\ell \simeq D$, i.e. D and D_m have the same maximal subfields.

Lemma 3.6 seems to suggest that as a potential generalization of (*) to arbitrary division algebras one should consider the question of whether for two central division algebras D_1 and D_2 over a field K , the classes $[D_1]$ and $[D_2]$ generate the same subgroup of $\text{Br}(K)$. Unfortunately, the answer to this question is negative already over number fields for algebras of any degree (exponent) $n > 2$. To see this, one can pick four nonarchimedean places v_1, \dots, v_4 of a given number field K , and then consider, for any $n > 2$, the division algebras D_1 and D_2 over K having local invariants $1/n, 1/n, -1/n, -1/n$ and $1/n, -1/n, 1/n, -1/n$ respectively at v_1, \dots, v_4 , and 0 everywhere else (cf. [11], Example 6.5). It follows from ([7], Corollary b in Sect. 18.4) that D_1 and D_2 have the same maximal subfields; at the same time, $[D_1]$ and $[D_2]$ generate different subgroups of $\text{Br}(K)$. Thus, there appears to be no sensible analog of Theorem B for algebras of exponent > 2 . On the other hand, in addition to generalizing Theorem B to other fields, one may consider similar questions for algebras with involution. More precisely, let (A_1, τ_1) and (A_2, τ_2) be two central simple algebras over K with involutions of the first kind (i.e., acting trivially on K) and of the same type (symplectic or orthogonal) - then of course $[A_1], [A_2] \in \text{Br}(K)_2$. Assume that A_1 and A_2 have the same isomorphism classes of maximal étale subalgebras invariant under the involutions². In what situations can one guarantee that $A_1 \simeq A_2$ as K -algebras? $(A_1, \tau_1) \simeq (A_2, \tau_2)$ as K -algebras with involutions? (Affirmative) results in this direction may lead to some progress on the problem, mentioned in the introduction, of when two weakly isomorphic forms of an absolutely almost simple algebraic group are necessarily isomorphic, particularly for types B_n and C_n .

4. Proof of Theorem A

For a quaternion algebra $D = \left(\frac{a, b}{K} \right)$ corresponding to $a, b \in K^\times$ we let q_D denote the quadratic form

$$q_D(s, t, u) = as^2 + bt^2 - abu^2.$$

Then D is a division algebra if and only if q_D does not represent nonzero squares, in which case the maximal subfields of D are isomorphic to the quadratic extensions of the form $K(\sqrt{d})$ where $d \in K^\times$ is represented by q_D . Furthermore, it is known ([7], Sect. 1.7) that two central quaternion algebras D_1 and D_2 over K are isomorphic if and only if the corresponding quadratic forms q_{D_1} and q_{D_2} are equivalent. Thus, (*) reduces to the statement that the quadratic forms q_{D_1} and q_{D_2} associated to two quaternion division algebras D_1 and D_2 are equivalent given that they represent the same elements of K .

The proof of Theorem A is based on Faddeev's exact sequence (see [4], Corollary 6.4.6, [7], Sect. 19.5, [2], Example 9.21 on p. 26, or [13]). Let K^{sep} be a

² This assumption has two possible interpretations: for any τ_1 -invariant maximal étale subalgebra $E_1 \subset A_1$ there exist a τ_2 -invariant maximal étale subalgebra $E_2 \subset A_2$ such that $E_1 \simeq E_2$ as K -algebras, or such that $(E_1, \tau_1|_{E_1}) \simeq (E_2, \tau_2|_{E_2})$ as K -algebras with involutions (and vice versa).

separable closure of K , and $G = \text{Gal}(K^{\text{sep}}/K)$ be its absolute Galois group. Furthermore, let V be the set of valuations of $K(x)$ corresponding to all irreducible polynomials $p(x) \in K[x]$. For each $v \in V$, we fix its extension \tilde{v} to $K^{\text{sep}}(x)$, and let $G(v) = G(\tilde{v}|_v)$ be the corresponding decomposition group; we observe that $G(v)$ is naturally identified with the absolute Galois group of the residue field $\overline{K(x)}^{(v)}$. Then we have the following exact sequence:

$$0 \rightarrow \text{Br}(K) \xrightarrow{\iota} \text{Br}(K^{\text{sep}}(x)/K(x)) \xrightarrow{\phi} \bigoplus_{v \in V} \text{Hom}(G(v), \mathbb{Q}/\mathbb{Z}). \quad (13)$$

in which ι is the natural embedding $[A] \mapsto [A \otimes_K K(x)]$, and $\phi = (\phi_v)$, where the local components ϕ_v are related to the reduction maps $\rho_v: \text{Br}_v^0(K(x)_v) \rightarrow \text{Hom}(G(v), \mathbb{Q}/\mathbb{Z})$ by $\phi_v = \rho_v \circ \lambda_v$ with $\lambda_v: \text{Br}(K(x)) \rightarrow \text{Br}(K(x)_v)$ being the natural map (notice that $\lambda_v(\text{Br}(K^{\text{sep}}(x)/K(x))) \subset \text{Br}_v^0(K(x)_v)$). Since $\text{char } K \neq 2$, we have the inclusion $\text{Br}(K(x))_2 \subset \text{Br}(K^{\text{sep}}(x)/K(x))$, so (13) gives rise to the following exact sequence

$$0 \rightarrow \text{Br}(K)_2 \xrightarrow{\iota} \text{Br}(K(x))_2 \xrightarrow{\phi} \bigoplus_{v \in V} \text{Hom}(G(v), \mathbb{Z}/2). \quad (14)$$

Lemma 4.1 (Cf. [3], Proposition 3.5). *Let D_1 and D_2 be two central quaternion division algebras over $K(x)$ having the same maximal subfields. Then $\phi([D_1]) = \phi([D_2])$.*

Proof. (Cf. the proof of Theorem 3.2) Fix $v \in V$. By Corollary 2.4, we have

$$D_i \otimes_{K(x)} K(x)_v = M_{d_i}(\mathcal{D}_i), \quad i = 1, 2,$$

where $d_1 = d_2$ and the division algebras \mathcal{D}_1 and \mathcal{D}_2 have the same maximal subfields. Using Lemma 2.5 (ii), case (a), we see that $Z(\tilde{\mathcal{D}}_1) = Z(\tilde{\mathcal{D}}_2)$. Since the extension $Z(\mathcal{D}_i)/\overline{K(x)}^{(v)}$ corresponds to the subgroup $\text{Ker } \chi_i \subset G(v)$, where $\chi_i = \rho_v([\mathcal{D}_i])$, we conclude that $\text{Ker } \chi_i = \text{Ker } \chi_2$, and eventually $\chi_1 = \chi_2$, implying that $\phi_v([D_1]) = \phi_v([D_2])$. \square

Corollary 4.2. *Let D_1 and D_2 be as in Lemma 4.1. Then there exists a central quaternion algebra D over K such that $D_1 \otimes_{K(x)} D_2 \simeq M_2(D \otimes_K K(x))$.*

Proof. It follows from Lemma 4.1 that there exists a division algebra \mathcal{D} over K such that

$$[D_1 \otimes_{K(x)} D_2] = [\mathcal{D} \otimes_K K(x)].$$

Notice that $\mathcal{D} \otimes_K K(x)$ is also a division algebra. On the other hand, since D_1 and D_2 possess a common subfield, we have $D_1 \otimes_{K(x)} D_2 \simeq M_2(\Delta)$ for some central quaternion algebra Δ over $K(x)$ (see [4], Lemma 1.5.2). From the uniqueness in Wedderburn's Theorem we conclude that either $\mathcal{D} = K$ or \mathcal{D} is a quaternion algebra over K . Set $D = M_2(K)$ in the former case, and $D = \mathcal{D}$ in the latter. Then our construction implies that D is as required. \square

Now, to complete the proof of Theorem A, it remains to show that in the notations of Corollary 4.2, the class $[D]$ in $\text{Br}(K)$ is trivial. For this, we need to make a couple of preliminary observations.

Lemma 4.3. *Let \mathcal{K} be a field with a discrete valuation v such that $\text{char } \bar{\mathcal{K}}^{(v)} \neq 2$. For $i = 1, 2$, let $a_i, b_i \in \mathcal{K}^\times$ be such that $v(a_i) = v(b_i) = 0$, and set*

$$\mathcal{D}_i = \left(\frac{a_i, b_i}{\mathcal{K}} \right), \quad \check{\mathcal{D}}_i = \left(\frac{\bar{a}_i, \bar{b}_i}{\bar{\mathcal{K}}^{(v)}} \right),$$

where \bar{a}_i, \bar{b}_i are the images of a_i, b_i in $\bar{\mathcal{K}}^{(v)}$. Assume that \mathcal{D}_1 and \mathcal{D}_2 are division algebras having the same maximal subfields. Then if one of the $\check{\mathcal{D}}_i$'s is a division algebra then both of them are, in which case they have the same maximal subfields.

Proof. Let $q_{\mathcal{D}_i}$ and $q_{\check{\mathcal{D}}_i}$ be the corresponding quadratic forms. First, we will show that if a nonzero $\bar{d} \in \bar{\mathcal{K}}^{(v)}$ is represented by $q_{\check{\mathcal{D}}_1}$ then it is also represented by $q_{\check{\mathcal{D}}_2}$. Indeed, let $\bar{s}_1, \bar{t}_1, \bar{u}_1 \in \bar{\mathcal{K}}^{(v)}$ be such that

$$q_{\check{\mathcal{D}}_1}(\bar{s}_1, \bar{t}_1, \bar{u}_1) = \bar{d}.$$

Pick arbitrary lifts $s_1, t_1, u_1 \in \mathcal{O}_{\mathcal{K},v}$ and set $d = q_{\mathcal{D}_1}(s_1, t_1, u_1)$. Since \mathcal{D}_1 and \mathcal{D}_2 have the same maximal subfields, there exist $s_2, t_2, u_2 \in \mathcal{K}$ such that $q_{\mathcal{D}_2}(s_2, t_2, u_2) = d$. If

$$\alpha := \min\{v(s_2), v(t_2), v(u_2)\} \geq 0$$

then taking reductions we obtain

$$q_{\check{\mathcal{D}}_2}(\bar{s}_2, \bar{t}_2, \bar{u}_2) = \bar{d},$$

as required. On the other hand, if $\alpha < 0$ then for

$$s'_2 = \pi^{-\alpha} s_2, \quad t'_2 = \pi^{-\alpha} t_2 \quad \text{and} \quad u'_2 = \pi^{-\alpha} u_2,$$

where $\pi \in \mathcal{K}^\times$ is a uniformizer, we will have $(\bar{s}'_2, \bar{t}'_2, \bar{u}'_2) \neq (\bar{0}, \bar{0}, \bar{0})$ and

$$q_{\check{\mathcal{D}}_2}(\bar{s}'_2, \bar{t}'_2, \bar{u}'_2) = \bar{0},$$

i.e. $q_{\check{\mathcal{D}}_2}$ represents zero. But then, being nondegenerate, it represents all elements of $\bar{\mathcal{K}}^{(v)}$.

Now, if $\check{\mathcal{D}}_1$ is not a division algebra, then $q_{\check{\mathcal{D}}_1}$ represents a nonzero square in $\bar{\mathcal{K}}^{(v)}$. By the above remark, the same is true for $q_{\check{\mathcal{D}}_2}$, hence $\check{\mathcal{D}}_2$ is not a division algebra, proving our first assertion. On the other hand, if both $\check{\mathcal{D}}_1$ and $\check{\mathcal{D}}_2$ are division algebras then by symmetry the above remark implies that $q_{\check{\mathcal{D}}_1}$ and $q_{\check{\mathcal{D}}_2}$ represent the same elements, and therefore $\check{\mathcal{D}}_1$ and $\check{\mathcal{D}}_2$ have the same maximal subfields. \square

Another ingredient we need is a consequence of the existence of the specialization map in Milnor's K -theory. For a field F , we let $K_2(F)$ denote its second Milnor K -group, and for $a, b \in F^\times$ let $\{a, b\} \in K_2(F)$ denote the corresponding symbol. According to the Merkurjev-Suslin Theorem ([4], Chap. 8), for any field of characteristic $\neq 2$ there is an isomorphism $K_2(F)/2K_2(F) \simeq \text{Br}(F)_2$ sending $\{a, b\}$ to the class of the quaternion algebra $\left(\frac{a, b}{F}\right)$. Let now \mathcal{K} be a field complete with respect to a discrete valuation v . Then there exists a homomorphism $s_v: K_2(\mathcal{K}) \rightarrow K_2(\bar{\mathcal{K}}^{(v)})$ such that for any $a, b \in \mathcal{K}^\times$ with $v(a) = v(b) = 0$ we have

$$s_v(\{a, b\}) = \{\bar{a}, \bar{b}\}$$

where \bar{a}, \bar{b} are the images of a, b in $\bar{\mathcal{K}}^{(v)}$ (see [4], Proposition 7.1.4); notice that s_v depends on the choice of a uniformizer in \mathcal{K} . Combining s_v with the Merkurjev-Suslin isomorphism, we obtain the following.

Lemma 4.4. *Let \mathcal{K} be a field complete with respect to a discrete valuation v . Assume that $\text{char } \bar{\mathcal{K}}^{(v)} \neq 2$. Then there exists a homomorphism, called the specialization homomorphism, $\sigma_v: \text{Br}(\mathcal{K})_2 \rightarrow \text{Br}(\bar{\mathcal{K}}^{(v)})_2$ such that for any $a, b \in \mathcal{K}^\times$ with $v(a) = v(b) = 0$ we have*

$$\sigma_v\left(\left[\left(\frac{a, b}{\mathcal{K}}\right)\right]\right) = \left[\left(\frac{\bar{a}, \bar{b}}{\bar{\mathcal{K}}^{(v)}}\right)\right].$$

We can extend the notion of the specialization homomorphism σ_v to any field F with a discrete valuation v such that $\text{char } \bar{F}^{(v)} \neq 2$ by defining it to be the composition

$$\text{Br}(F)_2 \longrightarrow \text{Br}(F_v)_2 \longrightarrow \text{Br}(\bar{F}^{(v)})_2$$

of the extension of scalars with the specialization homomorphism described in Lemma 4.4. We then have

Corollary 4.5. *Let D be a quaternion algebra over K . Then for any valuation v of $K(x)$ we have $\sigma_v([D \otimes_K K(x)]) = [D \otimes_K \bar{K}(x)^{(v)}]$.*

The conclusion of the proof of Theorem A. It follows from (AHBN) that (*) holds for global fields (cf. the proof of Corollary 4.8), so we may assume that K is infinite. Write the given central quaternion division algebras D_i over $K(x)$ in the form

$$D_i = \left(\frac{a_i(x), b_i(x)}{K(x)}\right) \quad \text{where } a_i(x), b_i(x) \in K[x] \text{ for } i = 1, 2.$$

Since K is infinite, we can replace x by $x - \alpha$ to ensure that $a_i(0), b_i(0) \neq 0$ for $i = 1, 2$. We then set

$$\check{D}_i = \left(\frac{a_i(0), b_i(0)}{K}\right).$$

By Corollary 4.2, we have

$$[D_1][D_2] = [D \otimes_K K(x)] \quad (15)$$

for some central quaternion algebra D over K , and all we need to show is that the class $[D] \in \text{Br}(K)$ is trivial. Let v be the valuation of $K(x)$ associated with x ; then $\overline{K(x)}^{(v)} = K$. It follows from Lemma 4.4 and Corollary 4.5 that for the corresponding specialization homomorphism $\sigma_v: \text{Br}(K(x))_2 \rightarrow \text{Br}(K)_2$ we have

$$\sigma_v([D_i]) = [\check{D}_i] \quad \text{for } i = 1, 2, \text{ and } \sigma_v([D \otimes_K K(x)]) = [D].$$

Thus, it follows from (15) that

$$[D] = [\check{D}_1][\check{D}_2]. \quad (16)$$

However,

$$[\check{D}_1] = [\check{D}_2]. \quad (17)$$

Indeed, if one of the classes $[\check{D}_i]$ is trivial then, by Lemma 4.3, so is the other, and (17) is obvious. On the other hand, if both classes $[\check{D}_1]$ and $[\check{D}_2]$ are nontrivial, then by Lemma 4.3 the division algebras \check{D}_1 and \check{D}_2 have the same maximal subfields, and (17) follows from our assumption that (*) holds for K . Now, (16) and (17) imply that $[D]$ is trivial, as required. \square

The specialization technique based on Lemmas 4.3 and 4.4 can be used to analyze (*) for fields of rational functions on other curves, which will be done elsewhere. In fact, it can also be used to obtain some finiteness results pertaining to (*). More precisely, let us consider the following property of a field K :

- (Φ) There exists $n = n(K)$ such that for any central quaternion division algebra D over K , the set of isomorphism classes of central quaternion division algebras over K having the same maximal subfields as D contains $\leq n$ elements.

Theorem 4.6. *Let X be an absolutely irreducible smooth projective curve over a field K of characteristic $\neq 2$. Assume that the quotient $\text{Br}_{ur}(\mathcal{K})_2 / \iota(\text{Br}(K)_2)$, where $\text{Br}_{ur}(\mathcal{K})$ is the unramified Brauer group of $\mathcal{K} = K(X)$ (over K) and $\iota: \text{Br}(K) \rightarrow \text{Br}(\mathcal{K})$ is the natural map, is finite of order m , and there exists a family $\mathcal{L} = \{L\}$ of odd degree extensions L/K such that*

- (α) if $L \in \mathcal{L}$ and $K \subset P \subset L$ then $P \in \mathcal{L}$;
- (β) $\bigcup_{L \in \mathcal{L}} X(L)$ is infinite;
- (γ) each $L \in \mathcal{L}$ has property (Φ) and $\sup_{L \in \mathcal{L}} n(L) =: n_0 < \infty$ where $n(L)$ is the number from the definition of (Φ).

Then \mathcal{K} has property (Φ) with $n(\mathcal{K}) = m \cdot n_0$. In particular, if $m < \infty$, K has property (Φ) and X has infinitely many K -rational points then \mathcal{K} has property (Φ) with $n(\mathcal{K}) = m \cdot n(K)$.

Proof. Let $V^{\mathcal{K}}$ be the set of discrete valuations of \mathcal{K} trivial on K , and for $v \in V^{\mathcal{K}}$ let $\phi_v: \text{Br}(\mathcal{K})_2 \rightarrow \text{Hom}(G(v), \mathbb{Z}/2)$, where $G(v)$ is the absolute Galois group of the residue field $\bar{\mathcal{K}}^{(v)}$, denote the composition $\rho_v \circ \lambda_v$ of the natural map $\lambda_v: \text{Br}(\mathcal{K}) \rightarrow \text{Br}(\mathcal{K}_v)$ with the reduction map $\rho_v: \text{Br}(\mathcal{K}_v)_2 \rightarrow \text{Hom}(G(v), \mathbb{Z}/2)$. Then by definition $\text{Br}_{ur}(\mathcal{K}) = \bigcap_{v \in V^{\mathcal{K}}} \text{Ker } \phi_v$.

Now, fix a central quaternion division algebra D over \mathcal{K} , and let $\mathcal{I}(D)$ denote the collection of classes $[D'] \in \text{Br}(\mathcal{K})$ for D' a central quaternion division algebra over \mathcal{K} having the same maximal subfields as D . It follows from Lemmas 2.1 and 2.5 that given $[D'] \in \mathcal{I}(D)$, for any $v \in V^{\mathcal{K}}$ we have

$$\rho_v([D' \otimes_{\mathcal{K}} \mathcal{K}_v]) = \rho_v([D \otimes_{\mathcal{K}} \mathcal{K}_v]), \quad (18)$$

i.e. $\phi_v([D']) = \phi_v([D])$ (cf. the proof of Lemma 4.1). Thus, $\mathcal{I}(D) \subset [D] \cdot \text{Br}_{ur}(\mathcal{K})_2$. By our assumption, $\text{Br}_{ur}(\mathcal{K})_2$ is the union of m cosets $\mathcal{C}_1, \dots, \mathcal{C}_m$ modulo $\iota(\text{Br}(K)_2)$. So, to prove that $n(\mathcal{K}) = m \cdot n_0$ satisfies the definition of property (Φ) , it is enough to show that

$$|\mathcal{I}(D) \cap \mathcal{C}_i| \leq n_0 \quad \text{for all } i = 1, \dots, m. \quad (19)$$

If $\mathcal{I}(D) \cap \mathcal{C}_i = \emptyset$ then there is nothing to prove; otherwise, all central quaternion algebras D' with $[D'] \in \mathcal{I}(D) \cap \mathcal{C}_i$ have the same maximal subfields as D . So, it is enough to show that for any central quaternion division algebra D over \mathcal{K} , the number of classes $[D']$, where D' is a central quaternion division algebra having the same maximal subfields as D and such that $[D'] \in [D] \cdot \iota(\text{Br}(K)_2)$, is $\leq n_0$.

Lemma 4.7. *For a central division algebra Δ over K of degree $\ell = 2^d$, the algebra $\Delta \otimes_K \mathcal{K}$ is also a division algebra. Consequently, (1) if for such Δ the algebra $\Delta \otimes_K \mathcal{K}$ is Brauer-equivalent to a quaternion algebra then Δ is itself a quaternion algebra, and (2) the natural map $\text{Br}(K)_2 \rightarrow \text{Br}(\mathcal{K})_2$ is injective.*

Proof. By (β) , there is an odd degree extension L/K and a rational point $p_0 \in X(L)$. Let $v_0 \in V^{\mathcal{K}}$ be the valuation of \mathcal{K} obtained as the restriction of the valuation of $L(X)$ associated with p_0 . Then the residue field $P = \bar{\mathcal{K}}^{(v_0)}$ is an odd degree extension of K . Let $f(x_1, \dots, x_{\ell^2})$ be the homogeneous polynomial of degree ℓ representing the reduced norm $\text{Nrd}_{\Delta/K}$. If $\Delta \otimes_K \mathcal{K}$ is not a division algebra then f represents zero over \mathcal{K} . Then f also represents zero over P , i.e. $\Delta \otimes_K P$ is not a division algebra. This, however, cannot happen as $\ell = 2^d$ and $[P : K]$ is odd (see [7], Sect. 13.4, part (vi) of the proposition). A contradiction, proving our first claim. The remaining assertions easily follow. \square

For central quaternion division algebras D and D' over \mathcal{K} having the same maximal subfields, we have $D \otimes D' \simeq M_2(D'')$ for some central quaternion algebra D'' ([4], Lemma 1.5.2). If in addition $[D'] = [D][\Delta \otimes_K \mathcal{K}]$ for a central division algebra Δ over K with $[\Delta] \in \text{Br}(K)_2$ then it follows from the lemma that either $\Delta = K$ or Δ is a quaternion division algebra. So, if we let $\mathcal{J}(D)$ denote the set of classes $[A] \in \text{Br}(K)_2$ where A is a central quaternion algebra over K such that the class $[D][\Delta \otimes_K \mathcal{K}] \in \text{Br}(\mathcal{K})$ is represented by a central quaternion division algebra D' over \mathcal{K} having the same maximal subfields as D , then to prove (19) it

is enough to show that $|\mathcal{J}(D)| \leq n_0$ for any D . Finally, since for an odd degree extension P/K the natural map $\lambda_P: \text{Br}(K)_2 \rightarrow \text{Br}(P)_2$ is injective, it is enough to find, for a fixed D , such an extension for which $|\lambda_P(\mathcal{J}(D))| \leq n_0$, and this is what we are going to do now.

Fix a central quaternion division algebra $D = \left(\frac{a, b}{\mathcal{K}} \right)$ over \mathcal{K} . It follows from assumption (β) in the statement of the theorem that there exists $L \in \mathcal{L}$ and a point $p_0 \in X(L)$ which is neither a zero nor a pole of a or b . As in the proof of Lemma 4.7, we let v_0 be the restriction to \mathcal{K} of the valuation of $L(X)$ associated with p_0 . Then

$$v_0(a) = v_0(b) = 0, \quad (20)$$

and the residue field $P = \bar{\mathcal{K}}^{(v_0)} \subset L$ belongs to \mathcal{L} (by (α)); in particular $[P : K]$ is odd. Any other central quaternion division algebra D' over \mathcal{K} having the same maximal subfields as D can be written in the form $D' = \left(\frac{a, b'}{\mathcal{K}} \right)$ for some $b' \in \mathcal{K}^\times$. Furthermore, it follows from (20) that D is unramified at v_0 , and then due to (18), D' is also unramified at v_0 . Then we can choose b' so that $v_0(b') = 0$ (see [3], 3.4).

Consider the quaternion algebras $\check{D} = \left(\frac{\bar{a}, \bar{b}}{P} \right)$ and $\check{D}' = \left(\frac{\bar{a}, \bar{b}'}{P} \right)$ where \bar{a}, \bar{b} and \bar{b}' are the images in $P = \bar{\mathcal{K}}^{(v_0)}$ of a, b and b' , respectively. Set $\check{\mathcal{I}}(\check{D}) = \{e\}$ if $\check{D} \simeq M_2(P)$, and let $\check{\mathcal{I}}(\check{D})$ denote the set of classes $[A] \in \text{Br}(P)$ where A is a central quaternion division algebra over P having the same maximal subfields as \check{D} if the latter is a division algebra. Since $P \in \mathcal{L}$, it follows from (γ) that $|\check{\mathcal{I}}(\check{D})| \leq n_0$. On the other hand, according to Lemma 4.3 we have $[\check{D}'] \in \check{\mathcal{I}}(\check{D})$. Now, if $[D'] = [D][\Delta \otimes_{\mathcal{K}} \mathcal{K}]$ where $[\Delta] \in \mathcal{J}(D)$ then it follows from Lemma 4.4 that

$$[\check{D}'] = [\check{D}][\Delta \otimes_{\mathcal{K}} P] \quad \text{in } \text{Br}(P).$$

This shows that $\lambda_P(\mathcal{J}(D)) \subset [\check{D}]^{-1} \check{\mathcal{I}}(\check{D})$, and consequently,

$$|\lambda_P(\mathcal{J}(D))| \leq |\check{\mathcal{I}}(\check{D})| \leq n_0,$$

as required. \square

Corollary 4.8. *Let \mathcal{H} be a finitely generated subgroup of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, and let $K = \overline{\mathbb{Q}}^{\mathcal{H}}$ be the corresponding fixed field. Furthermore, let $F(x, y)$ be an absolutely irreducible polynomial over K such that at least one of the numbers $\deg F$, $\deg_x F$ or $\deg_y F$ is odd. Then the field $\mathcal{K} = K(X_0)$ of K -rational functions on the affine curve X_0 given by $F(x, y) = 0$ has property (Φ) .*

Proof. We first note the following elementary statement.

Lemma 4.9. *Let $F(x, y)$ be an absolutely irreducible polynomial over an arbitrary field K such that one of the numbers $\deg F$, $\deg_x F$ or $\deg_y F$ is odd. Let X be a smooth projective K -defined model for the affine curve X_0 given by $F(x, y) = 0$. Then $\bigcup_{L \in \mathcal{L}} X(L)$, where \mathcal{L} is the family of all finite extensions L/K of odd degree, is infinite.*

Proof. It is enough to show that $\mathfrak{X}_{\mathcal{L}} = \bigcup_{L \in \mathcal{L}} X_0(L)$ is infinite. Assume the contrary, i.e. $\mathfrak{X}_{\mathcal{L}} = \{(x_1, y_1), \dots, (x_r, y_r)\}$. We will first consider the case where $d := \deg_x F$ is odd. We have $F(x, y) = f_d(y)x^d + f_{d-1}(y)x^{d-1} + \dots$ with $f_d \neq 0$, so one can find an odd degree extension K'/K containing an element $y_0 \notin \{y_1, \dots, y_r\}$ such that $f_d(y_0) \neq 0$ (if K is infinite then such an element y_0 can already be found in $K' = K$). Then the degree of $\varphi(x) = F(x, y_0) \in K'[x]$ is d , hence odd, and therefore $\varphi(x)$ has an irreducible factor $\psi(x)$ of odd degree. Let x_0 be a root of $\psi(x)$ (in a fixed algebraic closure of K), and set $L = K'(x_0)$. Then L is of odd degree over K' , hence over K , i.e. $L \in \mathcal{L}$. So, $(x_0, y_0) \in X_0(L) \subset \mathfrak{X}_{\mathcal{L}}$, contradicting our construction. The case where $\deg_y F$ is odd is reduced to the case just considered by switching x and y . Finally, if $\deg F$ is odd then one can find an odd degree extension K''/K and $a \in K''$ so that for $\Phi(x, y) = F(x, y + ax)$ we have $\deg_x \Phi = \deg F$, hence odd (again, if K is infinite one can find such an a already in $K'' = K$). Then our claim holds for the K'' -defined curve given by $\Phi(x, y) = 0$, which implies its truth for X_0 as any odd degree extension L/K'' is of odd degree over K .

Next, we recall that as follows from (AHBN) ([7], Sect. 18.5) any algebraic extension L/\mathbb{Q} satisfies (*), i.e. it satisfies (Φ) with $n(L) = 1$. Indeed, let D_1 and D_2 be central quaternion division algebras over L having the same maximal subfields. Assume that $D_1 \not\cong D_2$; then $D := D_1 \otimes_L D_2$ represents a nontrivial class in $\text{Br}(L)$. We can find a finite extension L^0/\mathbb{Q} contained in L and central quaternion division algebras D_i^0 over L^0 such that $D_i = D_i^0 \otimes_{L^0} L$ for $i = 1, 2$. Set $D^0 = D_1^0 \otimes_{L^0} D_2^0$. Then for any finite extension P/L^0 , contained in L , the algebra $D_P^0 = D^0 \otimes_{L^0} P$ represents a nontrivial class in $\text{Br}(P)$, and therefore by (AHBN) there exists a valuation w of P (which can be archimedean) such that the class $[D_P^0 \otimes_P P_w] \in \text{Br}(P_w)$ is nontrivial. The standard argument using the nonemptiness of the inverse limit of an inverse system of nonempty finite sets shows that there exists a valuation \tilde{v} of L such that for any finite subextension $L^0 \subset P \subset L$ and $v_P := \tilde{v}|_P$, the class $[D_P^0 \otimes_P P_{v_P}] \in \text{Br}(P_{v_P})$ is nontrivial. Let $v_0 = \tilde{v}|_{L^0}$. Since $\text{Br}(L_{v_0}^0)_2$ is of order ≤ 2 and the class $[D^0 \otimes_{L^0} L_{v_0}^0]$ is nontrivial, one of the classes $[D_i^0 \otimes_{L^0} L_{v_0}^0]$, where $i = 1, 2$, is trivial and the other is nontrivial. Suppose that $[D_1^0 \otimes_{L^0} L_{v_0}^0]$ is trivial. Then for any finite subextension $L^0 \subset P \subset L$, the class $[D_2^0 \otimes_{L^0} P_{v_P}] \in \text{Br}(P_{v_P})$ is nontrivial. Let V be the (finite) set of ramification places of D_1^0 . Then $v_0 \notin V$, so by the weak approximation theorem there exists $t \in (L^0)^\times$ such that $t \notin (L_v^0)^\times$ for all $v \in V$ and $t \in (L_{v_0}^0)^\times$. It follows from (AHBN) that $L^0(\sqrt{t})$ is isomorphic to a maximal subfield of D_1^0 ([7], Corollary b in Sect. 18.4), hence $L(\sqrt{t})$ is isomorphic to a maximal subfield of D_1 . Since D_1 and D_2 have the same maximal subfields, there exists a finite subextension $L^0 \subset P \subset L$ such that $P(\sqrt{t})$ is isomorphic to a maximal subfield of $D_2^0 \otimes_{L^0} P$. However by our construction $t \in P_{v_P}^{\times 2}$, so the latter is impossible as $D_2^0 \otimes_{L^0} P_{v_P}$ is a division algebra.

Finally, since $\text{Gal}(\overline{K}/K) = \mathcal{H}$ is finitely generated, the field K is of type (F) as defined by Serre ([16], Chap. III, Sect. 4.2), and therefore $H^1(K, C)$ is finite for any finite $\text{Gal}(\overline{K}/K)$ -module C (*loc. cit.*, Theorem 4). Then it follows from

exact sequence (9.25) in [2], p. 27, that for $\mathcal{K} = K(X_0) = K(X)$, where X is the K -defined smooth projective model for X_0 , the quotient $\mathrm{Br}_{ur}(\mathcal{K})_2/\iota(\mathrm{Br}(K)_2)$ is finite. Thus, our claim follows from Theorem 4.6 applied to the family $\mathcal{L} = \{L\}$ of all odd degree extensions L/K .

Remarks 4.10. 1. Lemma 4.9 (and hence Corollary 4.8) applies to any elliptic curve as well as to any hyperelliptic curve given by $y^2 = f(x)$ where f is a polynomial of odd degree without multiple roots.

2. We observe that (*) for a field K is equivalent to (Φ) with $n(K) = 1$. For $X = \mathbb{P}_K^1$, Faddeev's exact sequence yields $m = 1$, so we obtain from Theorem 4.6 (assuming, as we may, K to be infinite) that $n(K) = 1$ implies $n(K(x)) = 1$, which is precisely Theorem A. Thus, Theorem 4.6 contains Theorem A as a particular case. For clarity of exposition, however, we decided to give first a streamlined proof of Theorem A which is not loaded with extra technical details.

In the recent preprint [5], some finiteness results in the spirit of property (Φ) were obtained for division algebras of any prime degree over a field of rational functions.

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