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NONABELIAN FREE SUBGROUPS IN HOMOMORPHIC IMAGES OF VALUED QUATERNION DIVISION ALGEBRAS

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ABSTRACT. Given a quaternion division algebra D, a noncentral element $e \in D^\times$ is called *pure* if its square belongs to the center. A theorem of Rowen and Segev (2004) asserts that for any quaternion division algebra D of positive characteristic > 2 and any pure element $e \in D^\times$ the quotient $D^\times/X(e)$ of $D^\times$ by the normal subgroup $X(e)$ generated by e, is abelian-by-nilpotent-by-abelian. In this note we construct a quaternion division algebra D of characteristic zero containing a pure element $e \in D$ such that $D^\times/X(e)$ contains a nonabelian free group. This demonstrates that the situation in characteristic zero is very different.

1. INTRODUCTION

Let D be a quaternion division algebra with center K. An element $e \in D \setminus K$ is called *pure* if $e^2 \in K$. Given an element $a \in D \setminus K$, we denote by

$$X(a) \text{ and } Y(a)$$

the normal subgroups of $D^\times$ generated by $\langle a \rangle$ and $SL_1(K(a))$ respectively, where as usual $SL_1(K(a)) = K(a) \cap SL_1(D)$ and $SL_1(D)$ is the subgroup of elements having reduced norm 1.

In [7] quotients of the form $D^\times/X(e)$ were considered for pure elements $e \in D^\times$. These quotients arise in the analysis of the Whitehead group $W(G, k)$ of an absolutely simple simply connected algebraic k-group G of type ${}^{3,6}D_4$ having k-rank 1 (see Tits's Bourbaki talk [11] for the relevant terminology). It was shown that if D has positive characteristic > 2, then $D^\times/X(e)$ is abelian-by-nilpotent-by-abelian for any pure element $e \in D^\times$, which, by the explicit description of $W(G, k)$ given in [5], implies the solvability of $W(G, k)$ for G as above over a field k of characteristic > 2. Even though the solvability of $W(G, k)$ is expected to hold in any characteristic (see [6] for a general conjecture), the results of this note indicate

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that a more delicate analysis will be needed to prove this in characteristic zero. More precisely, we will prove the following.

Theorem 1.1. *There exists a quaternion division algebra D of characteristic zero and pure quaternions $e, f \in D$ such that the quotients $D^\times/X(e)$ and $D^\times/Y(f)$ contain nonabelian free groups. (In particular these quotients are nonsolvable.)*

In §3 we construct a quaternion division algebra D that supports a valuation v such that the residue algebra $\bar{D}_v$ is non-commutative of characteristic two over the residue field $\bar{K}_v$. This algebra D has the properties asserted in Theorem 1.1 by the following more general result.

Theorem 1.2. *Let D be a quaternion division algebra of characteristic zero that supports a valuation v for which the residue algebra $\bar{D}_v$ is non-commutative and has characteristic two. Then D contains pure quaternions e and f such that the quotients $D^\times/X(e)$ and $D^\times/Y(f)$ have homomorphic images containing $\bar{D}_v^\times/\bar{K}_v^\times$, and consequently $D^\times/X(e)$ and $D^\times/Y(f)$ contain nonabelian free groups.*

The proof of Theorem 1.2 proceeds as follows. Let D be a quaternion division algebra as in Theorem 1.2 and let $e, f \in D$ be pure quaternions such that $ef = -fe$. We first observe that $Y(e)$ commutes both with $X(e)$ and $Y(f)$ modulo the congruence subgroup $1+\mathfrak{m}_{D,v}$, where $\mathfrak{m}_{D,v}$ is the valuation ideal. (This follows from a stronger result that comes from [7]; see Lemma 2.1 for a short proof). We then apply the following lemma which is an easy consequence of the Cartan-Brauer-Hua Theorem (see, e.g., [3, Theorem 3.9.2, pg. 144]).

Lemma 1.3. *Let D be a finite-dimensional division algebra with center K that supports a valuation v so that the residue division algebra $\bar{D}_v$ is not commutative. Let $\mathcal{U}_v := \{x \in D^\times \mid v(x) = 0\}$ be the group of units and $\mathfrak{m}_{D,v} := \{x \in D^\times \mid v(x) > 0\}$ be the valuation ideal. Let $*\colon D^\times \to D^\times/K^\times(1+\mathfrak{m}_{D,v})$ be the canonical homomorphism. Suppose N, M are normal subgroups of $D^\times$ such that $[N, M]^* = 1^*$. Then*

1. *$\mathcal{U}_v^* = \bar{D}_v^\times/\bar{K}_v^\times$, and either $((MK^\times) \cap \mathcal{U}_v)^* = 1^*$ or $((NK^\times) \cap \mathcal{U}_v)^* = 1^*$;*
2. *for $H \in \{M, N\}$ such that $((HK^\times) \cap \mathcal{U}_v)^* = 1^*$ we have*

$$(HK^\times\mathcal{U}_v)/(HK^\times(1+\mathfrak{m}_{D,v})) \cong \bar{D}_v^\times/\bar{K}_v^\times.$$

In §3 we extend a well-known construction of valuations on fields of rational functions (cf. [2, Section 10.1, Proposition 2]) to finite-dimensional division algebras of non-commutative rational functions. We then apply this construction to obtain the quaternion division algebra D satisfying the hypotheses of Theorem 1.2 (see Proposition 3.4 and Corollary 3.5). The center of the resulting algebra D has transcendence degree 2 over $\mathbb{Q}$, so in this context we would like to mention that for a finite-dimensional division algebra $\mathcal{D}$ over a global field, any quotient of $SL_1(\mathcal{D})$ by a noncentral subgroup is finite and solvable (see [6]), implying that all quotients of $\mathcal{D}^\times$ by a noncentral subgroup are solvable.

In a preliminary version of this paper (see [9] for a report on this joint result) we proved a weaker version of Theorem 1.1 using ultra-products, however now we have a stronger result that does not require the use of ultra-products. Also, since our construction is *explicit* it is possible that the algebra D we construct could be used to demonstrate further properties of the Whitehead group.

2. The proof of Theorem 1.2

In this section we prove Theorem 1.2. We start with a lemma that comes from [7].

Lemma 2.1. *Let D be a quaternion division algebra of characteristic $\neq 2$, let S be the (normal) subgroup of $D^\times$ generated by the set $1+2SL_1(D)$, and let $\bullet\colon D^\times \to D^\bullet := D^\times/S$ be the canonical homomorphism. Then*

(1) *for any pure $e \in D$, we have $[Y(e), X(e)]^\bullet = 1^\bullet$;*
(2) *for any pure $e, f \in D$ such that $ef = -fe$, we have $[Y(e), Y(f)]^\bullet = 1^\bullet$.*

Proof. First observe that

(i) $\quad [x^\bullet, (x+s)^\bullet] = 1^\bullet, \quad$ for all $x \in D^\times$ and $s \in S$ such that $x+s \neq 0$,

because $(x+s)^\bullet = (s(s^{-1}x+1))^\bullet = (s^{-1}x+1)^\bullet$, and $(s^{-1}x+1)^\bullet$ commutes with $(s^{-1}x)^\bullet = x^\bullet$. Next, using the notation $[a,b] = a^{-1}b^{-1}ab$, $a^b = b^{-1}ab$,

$$[g, (1+x)^{-1}] = (1+x^g)(1+x)^{-1} \quad \forall x, g \in D^\times \text{ with } x \neq -1,$$

and hence

$$\left[x^g, \frac{1-x}{1+x}\right]^\bullet = [x^g, 1-2(1+x)^{-1}]^\bullet = [x^g, (1+x^g)^{-1}(1+x^g-2[g,(1+x)^{-1}])]^\bullet$$
$$= [x^g, x^g + (1-2[g,(1+x)^{-1}])]^\bullet = 1^\bullet,$$

where the last equality follows from (i) using the fact that $1-2[g,(1+x)^{-1}] \in S$. Replacing x by αe, where $\alpha \in K^\times$ is arbitrary and observing that any element $\neq -1$ in $\mathrm{SL}_1(K(e))$ has the form $\frac{1-\alpha e}{1+\alpha e}$ (for example, by Hilbert's Theorem 90), we get (1).

Part (2) follows from (1) because

$$[e, 1+\alpha f] = ((1+\alpha f)^e)^{-1}(1+\alpha f) = (1-\alpha f)^{-1}(1+\alpha f), \text{ for all } \alpha \in K^\times,$$

and as we mentioned above any element $\neq -1$ of $\mathrm{SL}_1(K(f))$ has the form $(1-\alpha f)^{-1}(1+\alpha f)$ for some $\alpha \in K^\times$. Hence $\mathrm{SL}_1(K(f)) \leq X(e)$, and so (2) follows from (1). □

The next lemma is a consequence of the Cartan-Brauer-Hua theorem and will be applied in the proof of Lemma 1.3.

Lemma 2.2. *Let D be a finite-dimensional division algebra with center K. Let A and B be two normal subgroup of $D^\times$ such that $[A,B] \leq K^\times$. Then either $A \subseteq K^\times$ or $B \subseteq K^\times$.*

Proof. First observe that

(ii) $\quad$ if $[A,B] = 1$, then either $A \subseteq K$ or $B \subseteq K$.

To prove (ii) note that the K-subalgebra $K[A]$ generated by A is a division subalgebra normalized by $D^\times$, so by the Cartan-Brauer-Hua Theorem (see [3, Theorem 3.9.2, pg. 144] for an easy proof), if A is noncentral, then $K[A] = D$, and since B centralizes $K[A]$ it follows that B is central.

Assume now that B is noncentral. Since $[B,A,A] = [A,B,A] = 1$, the three subgroup lemma ([1, (8.7)]) implies that $[[A,A],B] = 1$. Since B is noncentral, it follows from (ii) that $[A,A] \leq K^\times$. But this shows that A is nilpotent. Hence $A \leq K^\times$ since by a theorem of Scott [8], $D^\times$ contains no noncentral normal solvable subgroups. □

Proof of Lemma 1.3. (1) Let $\mathcal{O}_{D,v} := \{x \in D^\times \mid v(x) \geq 0\} \cup \{0\}$ be the valuation ring of v. Of course the canonical homomorphism $\mathcal{O}_{D,v} \to \bar{D}_v$ restricted to $\mathcal{U}_v$ induces a surjective group homomorphism $\mathcal{U}_v \to \bar{D}_v^\times$, with kernel $1 + \mathfrak{m}_{D,v}$, so $\mathcal{U}_v/(1 + \mathfrak{m}_{D,v}) = \bar{D}_v^\times$ and therefore $\mathcal{U}_v^* \cong \bar{D}_v^\times/\bar{K}_v^\times$.

Set $A := (MK^\times) \cap \mathcal{U}_v$ and $B := (NK^\times) \cap \mathcal{U}_v$. Let$^-: \mathcal{U}_v \to \mathcal{U}_v/(1+\mathfrak{m}_{D,v}) = \bar{D}_v^\times$ be the canonical homomorphism. Then $[\bar{A}, \bar{B}] \leq \bar{K}_v$. Since $\bar{D}_v$ is not commutative Lemma 2.2 implies that either $\bar{A} \leq \bar{K}_v$ or $\bar{B} \leq \bar{K}_v$, that is, either $A^* = 1^*$ or $B^* = 1^*$.

(2) Suppose $A^* = 1^*$, that is, $(MK^\times) \cap \mathcal{U}_v \leq K^\times(1 + \mathfrak{m}_{D,v})$. Then

$$\begin{aligned}(MK^\times\mathcal{U}_v)/(MK^\times(1 + \mathfrak{m}_{D,v})) &\cong \mathcal{U}_v/\mathcal{U}_v \cap (MK^\times(1 + \mathfrak{m}_{D,v})) \\ &\cong \mathcal{U}_v/(\mathcal{U}_v \cap MK^\times)(1 + \mathfrak{m}_{D,v}) = \mathcal{U}_v^*.\end{aligned}$$

□

We can now complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Let D be as in Theorem 1.2, and adopt the notation of Lemma 1.3 and its proof. Let $e, f \in D$ be pure elements such that $ef = -fe$. We claim that

(iii) $$[Y(e), X(e)]^* = 1^* = [Y(e), Y(f)]^*.$$

Recall that since D is a quaternion division algebra, $\mathrm{SL}_1(D) = [D^\times, D^\times]$. Furthermore, for any valuation v of a finite-dimensional division algebra, the value group Γ_v is commutative. Thus $\mathrm{SL}_1(D) = [D^\times, D^\times] \leq \mathcal{U}_v$, since $\mathcal{U}_v$ is the kernel of the valuation $v\colon D^\times \to \Gamma_v$. Also the fact that the characteristic of $\bar{D}_v$ is two implies that $2 \in \mathfrak{m}_{D,v}$, so $2\mathrm{SL}_1(D) \subseteq 2\mathcal{U}_v \subseteq \mathfrak{m}_{D,v}$ and therefore

$$S := \langle 1 + 2\mathrm{SL}_1(D)\rangle \leq 1 + \mathfrak{m}_{D,v}.$$

Now (iii) follows from Lemma 2.1.

Let $b, c \in \mathcal{U}_v$ such that $[\bar{b}, \bar{c}]$ is a noncentral element in $\bar{D}_v$ ($\bar{b}, \bar{c}$ exist since $\bar{D}_v$ is not commutative and hence not solvable; see, e.g., [8]). Let $a = [b, c]$ $(= b^{-1}c^{-1}bc)$ and let $e \in K(a)$ be a pure element. By the choice of a we have $a^* \neq 1^*$ and $a \in \mathrm{SL}_1(K(e)) \leq \mathcal{U}_v$. Thus $(Y(e) \cap \mathcal{U}_v)^* \neq 1^*$. Now (iii) together with Lemma 1.3 imply that $((X(e)K^\times) \cap \mathcal{U}_v)^* = 1^* = ((Y(f)K^\times) \cap \mathcal{U}_v)^*$. Now let $H \in \{X(e), Y(f)\}$. Then, by Lemma 1.3(2),

$$(HK^\times\mathcal{U}_v)/(HK^\times(1 + \mathfrak{m}_{D,v})) \cong \bar{D}_v^\times/\bar{K}_v^\times,$$

so $D^\times/HK^\times(1 + \mathfrak{m}_{D,v}))$ contains a subgroup isomorphic to $\bar{D}_v^\times/\bar{K}_v^\times$. This completes the first part of Theorem 1.2.

For the second part note first that since $\bar{D}_v^\times$ is non-commutative, it contains nonabelian free groups (see, e.g., [4, Theorem 2.1]); of course, the proof of this fact uses the celebrated Tits' alternative [10]. It follows that $\bar{D}_v^\times/\bar{K}_v^\times$ contain a nonabelian free group. Hence $D^\times/X(e)$ and $D^\times/Y(f)$ have homomorphic images containing nonabelian free groups. But then $D^\times/X(e)$ and $D^\times/Y(e)$ also contain nonabelian free groups because if we pick one preimage for each free generator of a free subgroup in a homomorphic image, the resulting elements will generate a free subgroup. □

Remarks 2.3. Note that the proof of Theorem 1.2 actually shows that if D is a quaternion division algebra that supports a valuation v such that the residue algebra $\bar{D}_v$ is non-commutative of characteristic two, and if $e, f \in D$ are pure elements such that $ef = -fe$, then one of $D^\times/Y(e)$ or $D^\times/Y(f)$ contains a nonabelian free group (and in fact one of these groups has a homomorphic image that contains a subgroup isomorphic to $\bar{D}_v^\times/\bar{K}_v^\times$).

This should be compared with the fact that for any quaternion division algebra D and any pure element $e \in D$, the quotient of $D^\times$ by the normal subgroup generated by $K(e)^\times$ is an elementary abelian 2-group (i.e. all nonidentity elements have order 2). This last fact is rather easy to prove; cf. [7].

3. Constructing valuations on division algebras of non-commutative rational functions

In this section we construct a quaternion division algebra whose center is an extension of $\mathbb{Q}$ of transcendence degree 2 that supports a valuation v such that the residue algebra $\bar{D}_v$ is non-commutative of characteristic two. In view of Theorem 1.2, this will complete the proof of Theorem 1.1. More generally, we give an explicit construction of valuations on finite-dimensional division algebras of non-commutative rational functions which enables us to control the structure of the residue algebra.

Let F be a field and let $v\colon F^\times \to \Gamma$ be a non-archimedean valuation on F. Thus $v\colon F^\times \to \Gamma$ is a nontrivial homomorphism from $F^\times$ to a totally ordered commutative group Γ (written additively) satisfying $v(a+b) \geq \min\{v(a), v(b)\}$, for all $a, b \in F^\times$, with $a + b \neq 0$. Recall that if $v(a) < v(b)$, then it follows that $v(a+b) = v(a)$.

In this section we first generalize the well-known construction of a valuation on the field of rational function $F(x)$, extending v (cf. [2, Section 10.1, Proposition 2]), to a construction of a valuation on the division algebra of fractions $F(x,\sigma)$, where $\sigma \in \operatorname{Aut}(F)$ is an automorphism of finite order satisfying $v(\sigma(a)) = v(a)$, for all $a \in F^\times$. This construction has probably been well known to experts for some time; see for example [3].

Thus let $\sigma \in \operatorname{Aut}(F)$ and let $R = F[x,\sigma]$ be the associated ring of skew polynomials in x. We recall that $F[x,\sigma]$ consists of formal expressions $a_0 + a_1x + \cdots + a_mx^m$ with $a_i \in F$ which are added in the obvious way and are multiplied according to the rule: if

$$a(x) = a_0 + a_1x + \cdots + a_mx^m \quad\text{and}\quad b(x) = b_0 + b_1x + \cdots + b_nx^n,$$

then

$$a(x)b(x) = c_0 + c_1x + \cdots + c_{m+n}x^{m+n}, \quad \text{where } c_k = \sum_{i+j=k} a_i b_j^{\sigma^i}.$$

Now suppose that $v(\sigma(a)) = v(a)$ for all $a \in F$. Define a function $w\colon R \setminus \{0\} \to \Gamma$ as follows: given a nonzero $a(x) = a_0 + a_1x + \cdots + a_mx^m$, we let

$$w(a(x)) = \min_{a_i \neq 0} v(a_i).$$

Lemma 3.1. *w is a valuation of R. In other words, for nonzero $a(x), b(x) \in R$ we have*

(1) $w(a(x)b(x)) = w(a(x)) + w(b(x))$, *and*
(2) $w(a(x) + b(x)) \geqslant \min\{w(a(x)), w(b(x))\}$ *if* $b(x) \neq -a(x)$.

Proof. The proof of (1) is identical to the usual proof of Gauss' Lemma. Namely, suppose $w(a(x)) = \alpha$, $w(b(x)) = \beta$, and set

$$i_0 = \min\{i \mid a_i \neq 0,\ v(a_i) = \alpha\} \text{ and } \ j_0 = \min\{j \mid b_j \neq 0,\ v(b_j) = \beta\}.$$

Then the coefficient $c_{i_0+j_0}$ of $x^{i_0+j_0}$ in $c(x) = a(x)b(x)$ is

$$c_{i_0+j_0} = \sum_{i+j=i_0+j_0} a_i b_j^{\sigma^i}.$$

If $i + j = i_0 + j_0$, $(i, j) \neq (i_0, j_0)$ and $a_i, b_j \neq 0$, we have either $i < i_0$ or $j < j_0$ and then respectively either $v(a_i) > \alpha$ (and $v(b_j) \geqslant \beta$) or $v(b_j) > \beta$ (and $v(a_i) \geqslant \alpha$). In all cases,

$$v(a_i b_j^{\sigma^i}) = v(a_i) + v(b_j) > \alpha + \beta.$$

It follows that

$$v(c_{i_0+j_0}) = v(a_{i_0} b_{j_0}^{\sigma^{i_0}}) = \alpha + \beta.$$

On the other hand, for any k we have

$$v(c_k) = v\left(\sum_{i+j=k} a_i b_j^{\sigma^i}\right) \geqslant \min_{a_i \neq 0 \neq b_j} v(a_i b_j^{\sigma^i}) \geqslant \alpha + \beta,$$

and (1) follows. Property (2) is obvious. $\square$

From now on, we will assume that σ has finite order d. Then the center of R is $R_0 = F^\sigma[x^d]$, where F^σ is the fixed subfield. Let $S = R_0 \smallsetminus \{0\}$. Then S is a central multiplicative subset of R, so the localization $D := R_S$ exists, and every element of D has a presentation of the form as^{-1}, where $a \in R$ and $s \in S$. Note that the localization $K := (R_0)_S$ is simply the field of fractions of R_0, and D is a finite-dimensional algebra over K without zero divisors, hence a division algebra. In fact, $\dim_K D = d^2$, as the elements $a_i x^j$, $i, j = 1, \ldots, d$, where $a_1, \ldots, a_d$ is a basis of F over F^σ, form a basis of D over K. We will denote D by $F(x, \sigma)$. If $as^{-1} = bt^{-1}$, then $at = bs$, and using Lemma 3.1(1) we immediately obtain that

$$w(a) - w(s) = w(b) - w(t),$$

so the equation

$$\tilde{w}(as^{-1}) = w(a) - w(s)$$

yields a well-defined function on $D^\times$.

Lemma 3.2. *$\tilde{w}$ is a valuation on D.*

Proof. The property that $\tilde{w}(\tilde{a}\tilde{b}) = \tilde{w}(\tilde{a}) + \tilde{w}(\tilde{b})$ for all nonzero $\tilde{a}, \tilde{b} \in D$ immediately follows from the definition. Now, suppose we have $\tilde{a} = as^{-1}, \tilde{b} = bt^{-1} \in D^\times$ such that $\tilde{b} \neq -\tilde{a}$ (i.e. $bs \neq -at$). By taking a common denominator we may assume that $t = s$. Suppose in addition that $\tilde{w}(\tilde{a}) \leqslant \tilde{w}(\tilde{b})$. Then

$$\tilde{w}(\tilde{a} + \tilde{b}) = w(a + b) - w(s) \geq w(a) - w(s) = \tilde{w}(\tilde{a}),$$

and the property $\tilde{w}(\tilde{a} + \tilde{b}) \geqslant \min\{\tilde{w}(\tilde{a}), \tilde{w}(\tilde{b})\}$ follows. $\square$

Let $\mathcal{O}_{F,v} = \{a \in F^\times \mid v(a) \geq 0\} \cup \{0\}$ and $\mathfrak{m}_{F,v} = \{a \in F^\times \mid v(a) > 0\} \cup \{0\}$ be the valuation ring and the valuation ideal of v. Note that

$$\sigma \text{ induces an automorphism } \bar{\sigma} \text{ on } \bar{F}_v,$$

because σ preserves v.

Lemma 3.3. *Let $S_0 = \{s \in S \mid w(s) = 0\}$. Then the valuation ring and the valuation ideal of $\tilde{w}$ are as follows:*

$$\text{(iv)} \qquad \mathcal{O}_{D,\tilde{w}} = \mathcal{O}_{F,v}[x,\sigma]S_0^{-1} \quad \textit{and} \quad \mathfrak{m}_{D,\tilde{w}} = \mathfrak{m}_{F,v}[x,\sigma]S_0^{-1}.$$

Furthermore, the residue algebra $\bar{D}_{\tilde{w}} = \mathcal{O}_{D,\tilde{w}}/\mathfrak{m}_{D,\tilde{w}}$ is isomorphic to $\bar{F}_v(x,\bar{\sigma})$.

Proof. Clearly, $S = (F^\sigma)^\times S_0$, from which it follows that any element in $\tilde{a} \in D^\times$ has a presentation of the form $\tilde{a} = as^{-1}$ with $s \in S_0$. Then $\tilde{w}(\tilde{a}) = w(a)$, and the descriptions in (iv) easily follow. Reducing the coefficients of polynomials in $\mathcal{O}_{F,v}[x]$ modulo $\mathfrak{m}_{F,v}$ defines a surjective ring homomorphism $\varphi\colon \mathcal{O}_{F,v}[x,\sigma] \to \bar{F}_v[x,\bar{\sigma}]$ with $\ker \varphi = \mathfrak{m}_{F,v}[x,\sigma]$. Then φ uniquely extends to a homomorphism of localizations

$$\tilde{\varphi}\colon \mathcal{O}_{F,v}[x,\sigma]S_0^{-1} \to \bar{F}_v[x,\bar{\sigma}](\varphi(S_0))^{-1}$$

with $\ker \tilde{\varphi} = \mathfrak{m}_{F,v}[x,\sigma]S_0^{-1}$. It follows that $\bar{D}_{\tilde{w}} \simeq \bar{F}_v[x,\bar{\sigma}](\varphi(S_0))^{-1}$. As the left-hand side is a division ring, so is the right-hand side, from which it follows that it in fact coincides with $\bar{F}_v(x,\bar{\sigma})$ (although in general $\varphi(S_0)$ may be smaller than the set of nonzero elements of the center of $\bar{F}_v[x,\bar{\sigma}]$). □

Now let k be a field and let u be a non-archimedean valuation on k. We use the above construction to construct a quaternion division algebra D equipped with a valuation $\tilde{w}$ extending u such that $\bar{D}_{\tilde{w}}$ is not commutative and has characteristic 2.

Consider the field of rational functions $F := k(y)$ in the variable y. Extend u to a valuation of F as above, by taking in the construction above k in place of F, u in place of v, and the identity map of k in place of σ. We thus obtain a valuation v on F extending u, and

$$\bar{F}_v = \bar{k}_u(y).$$

Now let $\sigma \in \mathrm{Aut}(F)$ be the unique automorphism (of order 2) taking $y \to \frac{1}{y}$ and fixing k pointwise. Using the definition of v one easily checks that $v(\sigma(r(y))) = v(r(y))$, for any rational function $r(y) \in k(y)$. Let $D = F(x,\sigma)$ be as above. Since the order of σ is 2, D is a quaternion division algebra. Also, $\bar{\sigma}$ is the unique map on $\bar{F}_v$ fixing $\bar{k}_u$ pointwise and taking y to $\frac{1}{y}$. Thus $\bar{\sigma}$ is nontrivial, and since $\bar{D}_{\tilde{w}} \cong \bar{F}_v(x,\bar{\sigma})$, it follows that $\bar{D}_{\tilde{w}}$ is not commutative. Thus, we have shown

Proposition 3.4. *Let k be a field that supports a non-archimedean valuation u. Then there exists a quaternion division algebra D whose center has transcendence degree 2 over k, and a valuation v on D extending u, such that the residue division algebra $\bar{D}_v$ is not commutative (and has characteristic equal to that of the residue field $\bar{k}_u$).*

Taking in Proposition 3.4 $k = \mathbb{Q}$ and u the 2-adic valuation, we get the following corollary which in conjunction with Theorem 1.2 completes the proof of Theorem 1.1.

Corollary 3.5. *There exists a quaternion division algebra D of characteristic zero and a valuation v on D such that the residue division algebra $\bar{D}_v$ is not commutative and has characteristic two.*

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