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The multinorm principle for linearly disjoint Galois extensions

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ABSTRACT

Text. Let L_1 and L_2 be finite separable extensions of a global field K, and let E_i be the Galois closure of L_i over K for $i = 1, 2$. We establish a local-global principle for the product of norms from L_1 and L_2 (so-called *multinorm principle*) provided that the extensions E_1 and E_2 are linearly disjoint over K.

Video. For a video summary of this paper, please click here or visit http://www.youtube.com/watch?v=6fwHt_k5dVQ.

1. Introduction

Let K be a global field. Given a finite extension L/K, we let J_K and J_L denote the groups of ideles of K and L respectively, and let $N_{L/K} : J_L \to J_K$ denote the natural extension of the norm map associated with L/K (cf. [2, pp. 73–75]). Then the extension L/K is said to satisfy the *Hasse norm principle* if

$$K^\times \cap N_{L/K}(J_L) = N_{L/K}\left(L^\times\right).$$

The classical result of Hasse states that this is always the case if L/K is a cyclic Galois extension. For general extensions (even Galois extensions), the Hasse principle does not necessarily hold, and its

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investigation has received a lot of attention. The obstruction to the Hasse principle is given by the quotient

$$Ш(L/K) = \frac{K^\times \cap N_{L/K}(J_L)}{N_{L/K}(L^\times)}$$

which is a finite group called the *Tate–Shafarevich group* of the extension L/K. (We note that it coincides with the Tate–Shafarevich group of the corresponding norm torus $\mathrm{R}^{(1)}_{L/K}(\mathrm{GL}_1)$, cf. [18, Section 11].)

In [2, p. 198], Tate gave the following cohomological computation of $Ш(L/K)$ for a Galois extension L/K: *Let $G = \mathrm{Gal}(L/K)$, and for a valuation v of K, let G^v be the decomposition group of* (*a fixed extension of*) v. *Then $Ш(L/K)$ is the dual of* (*hence is isomorphic to*) *the kernel of the map $H^3(G,\mathbb{Z}) \to \prod_v H^3(G^v,\mathbb{Z})$ induced by restriction.* Various aspects of the Hasse principle were investigated in [7, 12,13], and a computation of $Ш(L/K)$ for an arbitrary finite extension L/K in terms of so-called representation groups of the relevant Galois groups was given by Drakokhrust [5].

In [9], Hürlimann considered the tori of norm type associated with a pair of finite extensions L_1, L_2 of a global field K. The triviality of the Tate–Shafarevich group for this torus is equivalent to the fact that

$$K^\times \cap N_{L_1/K}(J_{L_1})N_{L_2/K}(J_{L_2}) = N_{L_1/K}\left(L_1^\times\right)N_{L_2/K}\left(L_2^\times\right). \tag{M}$$

Following [14], we say that the pair L_1, L_2 satisfies the *multinorm principle* if (M) holds. It was shown in [9] that this is indeed the case if L_1 is a cyclic Galois extension of K and L_2 is an arbitrary Galois extension (a similar result was independently obtained by Colliot-Thélène and Sansuc [4]). A more general sufficient condition for the multinorm principle was given in [14, Proposition 6.11]. This result was used to give a simplified proof of the Hasse principle for Galois cohomology of simply connected outer forms of type A_n over number fields (cf. [14, Chapter VI]) and in the analysis of the Margulis–Platonov conjecture for anisotropic inner forms of type A_n [14, Section 9.2]; it was also employed in [16] in the computation of the metaplectic kernel. More recently, another sufficient condition for the multinorm principle was given in [17] (cf. Proposition 4.2) in order to study the local–global principle for embedding fields with an involutive automorphism into simple algebras with involution; some further applications of this result can be found in [6].

It should be emphasized that in *all* of these results it was assumed that one of the extensions satisfies the Hasse principle. In this light, the main result of this note looks quite surprising: we show that no assumption of this nature is actually needed.

Theorem. *Let L_1 and L_2 be two finite separable extensions of a global field K, and let E_i be the Galois closure of L_i over K for $i = 1, 2$. If $E_1 \cap E_2 = K$ (i.e., E_1 and E_2 are linearly disjoint over K) then the pair L_1, L_2 satisfies the multinorm principle.*

We notice that the conclusion of the theorem can be false for non-linearly disjoint extensions. For example, if $L_1 = L_2 =: L$, then the multinorm principle is equivalent to the norm principle for L/K, hence may fail. See Section 4 for more sophisticated examples and a discussion of a more general conjecture.

The proof of the theorem is based on the following sufficient condition for the multinorm principle.

Proposition 1. *Let L_1 and L_2 be two finite separable extensions of K such that their Galois closures E_1 and E_2 satisfy $E_1 \cap E_2 = K$. Set $L = L_1L_2$. If the map*

$$\phi : Ш(L/K) \to Ш(L_1/K) \times Ш(L_2/K)$$

induced by the diagonal embedding $K^\times \hookrightarrow K^\times \times K^\times$ is surjective, then the pair L_1, L_2 satisfies the multinorm principle.

In Section 2, we prove the proposition and also reduce the proof of the theorem to the case where both L_1 and L_2 are Galois extensions of K. Then, to complete the proof of the theorem, we verify that the map ϕ is in fact surjective for any two linearly disjoint Galois extensions – cf. Proposition 3 in Section 3. In Section 4 we give some additional results and examples related to the multinorm principle. Finally, Section 5 which describes connections with the theory of algebraic tori was written following the advice of J.-L. Colliot-Thélène while making revisions in the original manuscript (arXiv:1203.1458).

2. Proof of Proposition 1

The following statement will enable us to prove Proposition 1, but is also of independent interest.

Proposition 2. *Let L_1 and L_2 be finite extensions of K such that their Galois closures E_1 and E_2 satisfy $E_1 \cap E_2 = K$. Let $L = L_1L_2$, and let*

$$S = K^\times \cap N_{L/K}(J_L) \quad \textit{and} \quad T = N_{L_1/K}\big(L_1^\times\big)N_{L_2/K}\big(L_2^\times\big).$$

Then the following conditions are equivalent:

(1) *The pair L_1, L_2 satisfies the multinorm principle;*
(2) $K^\times \cap N_{L_i/K}(J_{L_i}) \subset T$ *for $i = 1$ and 2;*
(3) $K^\times \cap N_{L_i/K}(J_{L_i}) \subset T$ *for at least one index $i \in \{1, 2\}$;*
(4) $S \subset T$.

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious, while the nontrivial implication (4) ⇒ (1) is a consequence of the following statement which is extracted from the proof of Proposition 6.11 in [14].

Lemma 3. *Let L_1 and L_2 be as in Proposition 2. Then in the above notations we have*

$$K^\times \cap N_{L_1/K}(J_{L_1})N_{L_2/K}(J_{L_2}) = ST.$$

Proof. For completeness, we (succinctly) reproduce the argument given in [14]. Let M_i be the maximal abelian extension of K contained in L_i for $i = 1, 2$, and let M be the maximal abelian extension of K contained in L. Then by Galois theory the fact that $E_1 \cap E_2 = K$ implies that

- $M = M_1M_2$ and $\mathrm{Gal}(M/K)$ is naturally isomorphic to $\mathrm{Gal}(M/M_1) \times \mathrm{Gal}(M/M_2)$;
- the maximal abelian extension of L_i contained in L is L_iM_{3-i} for $i = 1, 2$.

The crucial observation is that the map

$$\varphi : J_{L_1}/L_1^\times N_{L/L_1}(J_L) \times J_{L_2}/L_2^\times N_{L/L_2}(J_L) \to J_K/K^\times N_{L/K}(J_L),$$

induced by the product of the norm maps $N_{L_1/K}$ and $N_{L_2/K}$, is an isomorphism, which is proved by showing that φ is surjective and that its domain and target have the same order. To this end, we consider the following commutative diagram

$$\begin{array}{ccc} J_{M_1}/M_1^\times N_{M/M_1}(J_M) \times J_{M_2}/M_2^\times N_{M/M_2}(J_M) & \xrightarrow{\psi} & J_K/K^\times N_{M/K}(J_M) \\ \downarrow{\scriptstyle \theta_1 \times \theta_2} & & \downarrow{\scriptstyle \theta} \\ \mathrm{Gal}(M/M_1) \times \mathrm{Gal}(M/M_2) & \xrightarrow{\iota} & \mathrm{Gal}(M/K), \end{array} \tag{1}$$

where ψ is constructed analogously to φ,

$$\theta_i : J_{M_i}/M_i^{\times} N_{M/M_i}(J_M) \to \mathrm{Gal}(M/M_i) \quad \text{and} \quad \theta : J_K/K^{\times} N_{M/K}(J_M) \to \mathrm{Gal}(M/K)$$

are the isomorphisms given by the corresponding Artin maps (cf. [2, Chapter VII]), and ι is induced by the canonical embeddings $\mathrm{Gal}(M/M_i) \to \mathrm{Gal}(M/K)$; the commutativity of (1) follows from Proposition 4.3 in [2]. In our situation, ι is an isomorphism, so ψ is also an isomorphism, implying that

$$J_K = K^{\times} N_{M_1/K}(J_{M_1}) N_{M_2/K}(J_{M_2}). \tag{2}$$

We now recall the fact that for any finite separable extension P/F of global fields we have

$$F^{\times} N_{P/F}(J_P) = F^{\times} N_{R/F}(J_R),$$

where R is the maximal abelian extension of F contained in P (cf. [2, Exercise 8]). Thus,

$$K^{\times} N_{L_i/K}(J_{L_i}) = K^{\times} N_{M_i/K}(J_{M_i}) \quad \text{for } i = 1, 2$$

which in conjunction with (2) yields that

$$J_K = K^{\times} N_{L_1/K}(J_{L_1}) N_{L_2/K}(J_{L_2}),$$

proving that φ is surjective. On the other hand, since L_1M_2 is the maximal abelian extension of L_1 contained in L, using the fundamental isomorphism of global class field theory we obtain

$$\begin{aligned} \left| J_{L_1}/L_1^{\times} N_{L/L_1}(J_L) \right| &= \left| J_{L_1}/L_1^{\times} N_{L_1M_2/L_1}(J_{L_1M_2}) \right| = [L_1M_2 : L_1] \\ &= [M_2 : K] = [M : M_1] = \left| J_{M_1}/M_1^{\times} N_{M/M_1}(J_M) \right|, \end{aligned}$$

and similarly

$$\left| J_{L_2}/L_2^{\times} N_{L/L_2}(J_L) \right| = \left| J_{M_2}/M_2^{\times} N_{M/M_2}(J_M) \right| \quad \text{and} \quad \left| J_K/K^{\times} N_{L/K}(J_L) \right| = \left| J_K/K^{\times} N_{M/K}(J_M) \right|.$$

Since ψ is an isomorphism, these equations imply that the domain and the target of φ have the same order, proving that φ is in fact an isomorphism.

Now, take any $a \in K^{\times} \cap N_{L_1/K}(J_{L_1}) N_{L_2/K}(J_{L_2})$, and write it in the form

$$a = N_{L_1/K}(x_1) N_{L_2/K}(x_2) \quad \text{with } x_i \in J_{L_i}.$$

Then $(x_1 L_1^{\times} N_{L/L_1}(J_L), x_2 L_2^{\times} N_{L/L_2}(J_L)) \in \operatorname{Ker} \varphi$. Using the injectivity of φ established above, we see that we can write

$$x_i = y_i N_{L/L_i}(z_i) \quad \text{with } y_i \in L_i^{\times},\ z_i \in J_L \text{ for } i = 1, 2.$$

Then

$$a = \big(N_{L_1/K}(y_1) N_{L_2/K}(y_2)\big) N_{L/K}(z_1 z_2) \in TS.$$

This proves the inclusion

$$K^\times \cap N_{L_1/K}(J_{L_1})N_{L_2/K}(J_{L_2}) \subset ST,$$

while the reverse inclusion is obvious. □

Remark. If one of the L_i's satisfies the usual Hasse norm principle then condition (3) of Proposition 2 obviously holds for this i. This yields the multinorm principle in this situation, which is precisely the assertion of Proposition 6.11 in [14]. Thus, the latter is a particular case of our Proposition 2.

Before proceeding with the proof of Proposition 1, we will now use Lemma 3 to give

Reduction of the theorem to the Galois case. Let L_1, L_2 be as in the theorem, and let us assume that we already know that their Galois closures E_1, E_2 satisfy the multinorm principle. We will now show that the pair L_1, L_2 satisfies the multinorm principle as well. Generalizing the notions introduced in the proof of Proposition 2, for a pair of finite extensions P_1 and P_2 of K, we set

$$S_{P_1,P_2} = K^\times \cap N_{P_1P_2/K}(J_{P_1P_2}) \quad \text{and} \quad T_{P_1,P_2} = N_{P_1/K}\bigl(P_1^\times\bigr)N_{P_2/K}\bigl(P_2^\times\bigr).$$

We also set

$$R_{P_1,P_2} = K^\times \cap N_{P_1/K}(J_{P_1})N_{P_2/K}(J_{P_2}).$$

We note that for any other finite extensions P_1' and P_2' of K we have the inclusions

$$S_{P_1,P_2} \subset R_{P_1',P_2} \quad \text{and} \quad S_{P_1,P_2} \subset R_{P_1,P_2'}. \tag{3}$$

Now, applying Lemma 3 twice in conjunction with (3), we obtain

$$R_{L_1,L_2} = T_{L_1,L_2}S_{L_1,L_2} \subset T_{L_1,L_2}R_{E_1,L_2} = T_{L_1,L_2}T_{E_1,L_2}S_{E_1,L_2} \subset T_{L_1,L_2}T_{E_1,L_2}R_{E_1,E_2}. \tag{4}$$

Since by our assumption the multinorm principle holds for the pair E_1, E_2, we have $R_{E_1,E_2} = T_{E_1,E_2}$, so (4) becomes

$$R_{L_1,L_2} \subset T_{L_1,L_2}T_{E_1,L_2}T_{E_1,E_2} = T_{L_1,L_2},$$

which means that the multinorm principle holds for the pair L_1, L_2. □

To complete the proof of Proposition 1, we need the following elementary group-theoretic lemma.

Lemma 4. *Let $\mathcal{A}$ be an abelian group with subgroups $\mathcal{B}$ and $\mathcal{C}$. Then the sequence*

$$\mathcal{A} \xrightarrow{f} \frac{\mathcal{A}}{\mathcal{B}} \times \frac{\mathcal{A}}{\mathcal{C}} \xrightarrow{g} \frac{\mathcal{A}}{\mathcal{B}\mathcal{C}} \to 1,$$

where f and g are defined by

$$f(x) = (x\mathcal{B}, x\mathcal{C}) \quad \textit{and} \quad g(x\mathcal{B}, y\mathcal{C}) = xy^{-1}\mathcal{B}\mathcal{C},$$

is exact.

Proof of Proposition 1. Applying Lemma 4 to the group $\mathcal{A} = K^\times \cap N_{L/K}(J_L)$ and its subgroups

$$\mathcal{B} = N_{L_1/K}\big(L_1^\times\big) \cap N_{L/K}(J_L) \quad \text{and} \quad \mathcal{C} = N_{L_2/K}\big(L_2^\times\big) \cap N_{L/K}(J_L),$$

we obtain the following exact sequence

$$K^\times \cap N_{L/K}(J_L) \xrightarrow{f} \frac{K^\times \cap N_{L/K}(J_L)}{N_{L_1/K}(L_1^\times) \cap N_{L/K}(J_L)} \times \frac{K^\times \cap N_{L/K}(J_L)}{N_{L_2/K}(L_2^\times) \cap N_{L/K}(J_L)}$$
$$\xrightarrow{g} \frac{K^\times \cap N_{L/K}(J_L)}{(N_{L_1/K}(L_1^\times) \cap N_{L/K}(J_L))(N_{L_2/K}(L_2^\times) \cap N_{L/K}(J_L))} \to 1. \tag{5}$$

By our assumption, the composite homomorphism

$$Ш(L/K) = \frac{K^\times \cap N_{L/K}(J_L)}{N_{L/K}(L^\times)} \xrightarrow{\bar{f}} \frac{K^\times \cap N_{L/K}(J_L)}{N_{L_1/K}(L_1^\times) \cap N_{L/K}(J_L)} \times \frac{K^\times \cap N_{L/K}(J_L)}{N_{L_2/K}(L_2^\times) \cap N_{L/K}(J_L)}$$
$$\xrightarrow{h} \frac{K^\times \cap N_{L_1/K}(J_{L_1})}{N_{L_1/K}(L_1^\times)} \times \frac{K^\times \cap N_{L_2/K}(J_{L_2})}{N_{L_2/K}(L_2^\times)} = Ш(L_1/K) \times Ш(L_2/K),$$

where $\bar{f}$ is induced by f and h by the inclusions $K^\times \cap N_{L/K}(J_L) \subset K^\times \cap N_{L_i/K}(J_{L_i})$ for $i = 1, 2$, is surjective. Since h is obviously injective, we conclude that $\bar{f}$, hence f, is surjective. So, the exact sequence (5) yields that its third term is trivial, i.e.

$$S = K^\times \cap N_{L/K}(J_L) = \big(N_{L_1/K}\big(L_1^\times\big) \cap N_{L/K}(J_L)\big)\big(N_{L_2/K}\big(L_2^\times\big) \cap N_{L/K}(J_L)\big)$$
$$\subset N_{L_1/K}\big(L_1^\times\big)N_{L_2/K}\big(L_2^\times\big) = T.$$

This verifies condition (4) of Proposition 2, thereby yielding the validity of the multinorm principle for the pair L_1, L_2. □

3. Proof of the Main Theorem

As we have seen in Section 2, it is enough to prove the Main Theorem assuming that both L_1 and L_2 are Galois extensions of K. In this case, the claim is a consequence of Proposition 1 combined with the following statement.

Proposition 5. *Let L_1 and L_2 be Galois extensions of K with $L_1 \cap L_2 = K$, and let $L = L_1 L_2$. Then the map*

$$\phi : Ш(L/K) \to Ш(L_1/K) \times Ш(L_2/K)$$

induced by the diagonal embedding $K^\times \hookrightarrow K^\times \times K^\times$ is surjective.

Our proof relies on properties of the deflation and residuation maps for the Tate cohomology groups, introduced in [20] and [8], and their interaction with the fundamental isomorphisms of class field theory. Since these maps are rarely used, we briefly recall in Appendix A their construction, which is needed to prove the key Lemma 8.

Given a finite group G and a G-module A, we let $\hat{H}^i(G, A)$ denote the ith Tate cohomology group (cf., for example, [2, Chapter IV, Section 6]). For a normal subgroup H of G and any $i \geqslant 0$, one can define the *deflation map*

$$\mathrm{Def}^G_{G/H} : \hat{H}^{-i}(G, A) \to \hat{H}^{-i}\big(G/H, A^H\big).$$

The deflation map is natural; in particular, it has the following properties.

Lemma 6. *For any G-module homomorphism $f : A \to B$ and any $i \geqslant 0$, the diagram*

$$\begin{array}{ccc} \hat{H}^{-i}(G, A) & \longrightarrow & \hat{H}^{-i}(G, B) \\ \downarrow{\scriptstyle \mathrm{Def}^G_{G/H}} & & \downarrow{\scriptstyle \mathrm{Def}^G_{G/H}} \\ \hat{H}^{-i}(G/H, A^H) & \longrightarrow & \hat{H}^{-i}(G/H, B^H) \end{array}$$

in which the horizontal maps are induced by f, is commutative.

Proof. This is Proposition 8 in [20]. □

Lemma 7. *Let*

$$0 \to A \to B \to C \to 0 \tag{6}$$

be an exact sequence of G-modules, and assume that the induced sequence of G/H-modules

$$0 \to A^H \to B^H \to C^H \to 0 \tag{7}$$

is also exact. Then for any $i \geqslant 1$ the diagram

$$\begin{array}{ccc} \hat{H}^{-i}(G, C) & \longrightarrow & \hat{H}^{-i+1}(G, A) \\ \downarrow{\scriptstyle \mathrm{Def}^G_{G/H}} & & \downarrow{\scriptstyle \mathrm{Def}^G_{G/H}} \\ \hat{H}^{-i}(G/H, C^H) & \longrightarrow & \hat{H}^{-i+1}(G/H, A^H) \end{array}$$

in which the horizontal maps are the coboundary maps arising from the exact sequences (6) and (7), is commutative.

Proof. This is Proposition 4 in [20]. □

Our proof also makes use of the *residuation map* $\mathrm{Rsd}^G_{G/H}$ – see Appendix A. The key property that we need is that in the case of interest to us, the residuation map is the dual of the usual inflation map. More precisely, we have the following.

Lemma 8. *Let $G = H \times K$ and identify G/K with H. Then for $i \geqslant 2$ the residuation and inflation maps in the following diagram*

$$\begin{array}{ccccc} \hat{H}^{-i}(G, \mathbb{Z}) & \times & \hat{H}^{i}(G, \mathbb{Z}) & \xrightarrow{\cup} & \hat{H}^{0}(G, \mathbb{Z}) \\ \downarrow{\scriptstyle \mathrm{Rsd}^G_H} & & {\scriptstyle \mathrm{Inf}^G_H}\uparrow & & {\scriptstyle \mathrm{Cor}^G_H}\uparrow \\ \hat{H}^{-i}(H, \mathbb{Z}) & \times & \hat{H}^{i}(H, \mathbb{Z}) & \xrightarrow{\cup} & \hat{H}^{0}(H, \mathbb{Z}) \end{array}$$

are adjoint with respect to the pairings given by the $\cup$*-products. That is,*

$$f \cup \mathrm{Inf}_H^G(\psi) = \mathrm{Cor}_H^G\big(\mathrm{Rsd}_H^G(f) \cup \psi\big)$$

for every $f \in \hat{H}^{-i}(G, \mathbb{Z})$ *and* $\psi \in \hat{H}^i(H, \mathbb{Z})$.

Proof. This uses an explicit construction of the residuation map and will be given in Appendix A. □

Another critical ingredient of the proof of Proposition 5 is the following result of K. Horie and M. Horie [8] that shows how the deflation and residuation maps interact with the isomorphisms from class field theory. For a global field K, we let $C_K = J_K/K^\times$ denote the idele class group. Furthermore, given a Galois extension F/K of global fields, for any $\mathrm{Gal}(F/K)$-module A we write $\hat{H}^i(F/K, A)$ instead of $\hat{H}^i(\mathrm{Gal}(F/K), A)$, and then for any $i \in \mathbb{Z}$ there is a canonical isomorphism $\Phi_F : \hat{H}^{i-2}(F/K, \mathbb{Z}) \to \hat{H}^i(F/K, C_F)$ called the *Tate isomorphism* (cf. [2, Chapter VII]).

Lemma 9. *(See [8, Theorem 1].) Let* $E \subset F$ *be Galois extensions of a global field* K*. Then for any* $i \geqslant 0$*, the following diagram*

$$\begin{array}{ccc} \hat{H}^{-i-2}(F/K, \mathbb{Z}) & \xrightarrow{\Phi_F} & \hat{H}^{-i}(F/K, C_F) \\ \Big\downarrow \mathrm{Rsd}_{\mathrm{Gal}(E/K)}^{\mathrm{Gal}(F/K)} & & \Big\downarrow \mathrm{Def}_{\mathrm{Gal}(E/K)}^{\mathrm{Gal}(F/K)} \\ \hat{H}^{-i-2}(E/K, \mathbb{Z}) & \xrightarrow{\Phi_E} & \hat{H}^{-i}(E/K, C_E) \end{array} \qquad (8)$$

commutes.

(We will only use this lemma for $i = 1$.)

Proof of Proposition 5. For a finite Galois extension F/K, we let

$$\kappa_F : \hat{H}^0\big(F/K, F^\times\big) \to \hat{H}^0(F/K, J_F)$$

denote the map induced by the inclusion $F^\times \to J_F$. Then clearly $Ш(F/K) = \mathrm{Ker}\,\kappa_F$. Now, let $G_j = \mathrm{Gal}(L_j/K)$ for $j = 1, 2$. Since L_1 and L_2 are assumed to be linearly disjoint, for $L = L_1L_2$ and $G = \mathrm{Gal}(L/K)$ there is a natural isomorphism

$$G = G_1 \times G_2,$$

which in particular allows us to identify G/G_{3-j} with G_j for $j = 1, 2$. Considering the inclusion $L^\times \to J_L$ as part of the exact sequence of G-modules $1 \to L^\times \to J_L \to C_L \to 1$ and applying Lemmas 6 and 7 to $H = G_{3-j}$ with $i = 1$ we obtain (observing that the corresponding sequence (7) is $1 \to L_j^\times \to J_{L_j} \to C_{L_j} \to 1$, cf. [2, Chapter VII, Proposition 8.1]) the following commutative diagram with exact rows:

$$\begin{array}{ccccc} \hat{H}^{-1}(G, C_L) & \longrightarrow & \hat{H}^0(G, L^\times) & \xrightarrow{\kappa_L} & \hat{H}^0(G, J_L) \\ \Big\downarrow \mathrm{Def}_{G_j}^G & & \Big\downarrow \mathrm{Def}_{G_j}^G & & \Big\downarrow \mathrm{Def}_{G_j}^G \\ \hat{H}^{-1}(G_j, C_{L_j}) & \longrightarrow & \hat{H}^0(G_j, L_j^\times) & \xrightarrow{\kappa_{L_j}} & \hat{H}^0(G_j, J_{L_j}) \end{array} \qquad (9)$$

for each $j = 1, 2$. Since the deflation map in dimension 0 is induced by the identity map (cf. Appendix A), we see that the map ϕ in Proposition 5 is the map $\mathrm{Ker}(\kappa_L) \to \mathrm{Ker}(\kappa_{L_1}) \times \mathrm{Ker}(\kappa_{L_2})$ induced by $\mathrm{Def}^G_{G_1} \times \mathrm{Def}^G_{G_2}$. So, it follows from (9) that ϕ is surjective if

$$\mathrm{Def}^G_{G_1} \times \mathrm{Def}^G_{G_2} : \hat{H}^{-1}(G, C_L) \to \hat{H}^{-1}(G_1, C_{L_1}) \times \hat{H}^{-1}(G_2, C_{L_2}) \tag{10}$$

is such. Now, using Lemma 9 with $i = 1$, we obtain the following commutative diagram

$$\begin{array}{ccc}
\hat{H}^{-3}(G, \mathbb{Z}) & \xrightarrow{\Phi_L} & \hat{H}^{-1}(G, C_L) \\
\downarrow{\scriptstyle \mathrm{Rsd}^G_{G_j}} & & \downarrow{\scriptstyle \mathrm{Def}^G_{G_j}} \\
\hat{H}^{-3}(G_j, \mathbb{Z}) & \xrightarrow{\Phi_{L_j}} & \hat{H}^{-1}(G_j, C_{L_j})
\end{array}$$

for each $j = 1, 2$. So, the surjectivity of (10) is equivalent to that of

$$\mathrm{Rsd}^G_{G_1} \times \mathrm{Rsd}^G_{G_2} : \hat{H}^{-3}(G, \mathbb{Z}) \to \hat{H}^{-3}(G_1, \mathbb{Z}) \times \hat{H}^{-3}(G_2, \mathbb{Z}). \tag{11}$$

For this, we will use the duality between the residuation and inflation maps provided by Lemma 8. More precisely, it is well-known (cf., for example, [1, Theorem 6.6, p. 250]) that for any finite group H and any $i \in \mathbb{Z}$, the $\cup$-product defines a perfect pairing

$$\alpha_H : \hat{H}^{-i}(H, \mathbb{Z}) \times \hat{H}^{i}(H, \mathbb{Z}) \to \hat{H}^{0}(H, \mathbb{Z}) = \mathbb{Z}/|H|\mathbb{Z}.$$

On the other hand, in our situation, $\mathrm{Cor}^G_{G_j}$ identifies $\hat{H}^0(G_j, \mathbb{Z}) = \mathbb{Z}/|G_j|\mathbb{Z}$ with

$$|G_{3-j}|\mathbb{Z}/|G|\mathbb{Z} \subset \mathbb{Z}/|G|\mathbb{Z} = \hat{H}^0(G, \mathbb{Z}).$$

It follows that $\alpha = \mathrm{Cor}^G_{G_1} \circ \alpha_{G_1} + \mathrm{Cor}^G_{G_2} \circ \alpha_{G_2}$ defines a perfect pairing

$$\big(\hat{H}^{-i}(G_1, \mathbb{Z}) \times \hat{H}^{-i}(G_2, \mathbb{Z})\big) \times \big(\hat{H}^{i}(G_1, \mathbb{Z}) \times \hat{H}^{i}(G_2, \mathbb{Z})\big) \to \hat{H}^0(G, \mathbb{Z}).$$

Furthermore, by Lemma 8, we have the following commutative diagram

$$\begin{array}{ccccc}
\hat{H}^{-3}(G, \mathbb{Z}) & \times & \hat{H}^{3}(G, \mathbb{Z}) & \overset{\cup}{\searrow} & \\
\downarrow{\scriptstyle \mathrm{Rsd}^G_{G_1} \times \mathrm{Rsd}^G_{G_2}} & & \uparrow{\scriptstyle \mathrm{Inf}^G_{G_1} + \mathrm{Inf}^G_{G_2}} & & \hat{H}^0(G, \mathbb{Z}). \\
(\hat{H}^{-3}(G_1, \mathbb{Z}) \times \hat{H}^{-3}(G_2, \mathbb{Z})) & \times & (\hat{H}^{3}(G_1, \mathbb{Z}) \times \hat{H}^{3}(G_2, \mathbb{Z})) & \overset{\alpha}{\nearrow} &
\end{array}$$

Thus, the surjectivity of (17) is equivalent to the injectivity of $\mathrm{Inf}^G_{G_1} + \mathrm{Inf}^G_{G_2}$, and the proof of the proposition is completed by the following statement.

Lemma 10. *For any finite group G of the form $G = G_1 \times G_2$ and any $i \geqslant 1$, the map*

$$\mathrm{Inf}_{G_1}^{G} + \mathrm{Inf}_{G_2}^{G} : \hat{H}^i(G_1, \mathbb{Z}) \times \hat{H}^i(G_2, \mathbb{Z}) \to \hat{H}^i(G, \mathbb{Z})$$

is injective.

Proof. For a subgroup $H \subset G$, we let $\mathrm{Res}_H^G : \hat{H}^i(G, \mathbb{Z}) \to \hat{H}^i(H, \mathbb{Z})$ denote the corresponding restriction map. Identifying G/G_{3-j} with G_j as above, it is easy to see that the composition

$$\mathrm{Res}_{G_j}^{G} \circ \mathrm{Inf}_{G_j}^{G} : \hat{H}^i(G_j, \mathbb{Z}) \to \hat{H}^i(G_j, \mathbb{Z})$$

is the identity map, while the composition $\mathrm{Res}_{G_{3-j}}^{G} \circ \mathrm{Inf}_{G_j}^{G}$ is zero, and our assertion follows. □

Remark. We note that the deflation map in the context of Tate–Shafarevich groups and its connection with the inflation map was used in [11, p. 97] for a different purpose.

4. Examples and extensions

In this section we give examples where the multinorm principle fails and prove some results that compliment and extend the Main Theorem.

Example 1. *For non-Galois extensions, the condition $L_1 \cap L_2 = K$ may not imply the multinorm principle for the pair L_1, L_2.* Indeed, let F/K be a Galois extension with Galois group $G = \mathrm{Gal}(F/K)$ isomorphic to A_6 as in Lemma 2 of [12], and let H be a subgroup of G of index 10 (see [12] or [14, p. 311]). Since A_6 is simple, we can choose $\sigma \in G$ such that $\sigma H \sigma^{-1} \neq H$. Set

$$L_1 = F^H \quad \text{and} \quad L_2 = F^{\sigma H \sigma^{-1}} = \sigma(L_1).$$

Clearly, A_6 does not have any subgroups of index 2 or 5, so $\langle H, \sigma H \sigma^{-1} \rangle = G$ and therefore

$$L_1 \cap L_2 = K. \tag{12}$$

On the other hand, since L_1 and L_2 are Galois-conjugate over K, we have

$$N_{L_1/K}\left(L_1^\times\right) = N_{L_2/K}\left(L_2^\times\right) \quad \text{and} \quad N_{L_1/K}(J_{L_1}) = N_{L_2/K}(J_{L_2}).$$

This means that the multinorm principle for the pair L_1, L_2 is equivalent to the Hasse norm principle for L_1/K. However, according to Theorem 1 of [12], the latter actually fails for L_1/K. Thus, the pair L_1, L_2 does not satisfy the Hasse norm principle despite (12). □

We note that the extensions L_1 and L_2 in Example 1 are not linearly disjoint. However, even for linearly disjoint extensions L_1, L_2 their Galois closures E_1 and E_2 need not satisfy $E_1 \cap E_2 = K$ (e.g. for the linearly disjoint extensions $L_1 = \mathbb{Q}(\sqrt[3]{5})$ and $L_2 = \mathbb{Q}(\sqrt[3]{7})$ of $\mathbb{Q}$, we have $E_1 \cap E_2 = \mathbb{Q}(\zeta_3)$ where ζ_3 is a primitive 3rd root of unity), which is required to apply our Main Theorem. So, the question of whether any pair L_1, L_2 of linearly disjoint extensions of K satisfies the multinorm principle remains open.

On the other hand, it would be interesting to analyze the multinorm principle for at least pairs of Galois extensions L_1, L_2 such that $L_1 \cap L_2 \neq K$. This case is not well-understood as of now, but the following proposition clarifies the nature of additional conditions one needs to impose to avoid obvious counter-examples.

Proposition 11. *Let L_1 and L_2 be finite Galois extensions of K satisfying $L_1 \cap L_2 = K$, and let L_3 be any finite extension of L_1. If L_1/K fails to satisfy the norm principle, then the pair L_1L_2, L_3 fails to satisfy the multinorm principle.*

Proof. It follows from Proposition 5 that the natural homomorphism

$$\text{Ш}(L_1L_2/K) \to \text{Ш}(L_1/K)$$

is surjective. Since $\text{Ш}(L_1/K)$ is nontrivial, this means that there exists $x \in K^\times \cap N_{L_1L_2/K}(J_{L_1L_2})$ that is not in $N_{L_1/K}(L_1^\times)$. Then x lies in $K^\times \cap N_{L_1L_2/K}(J_{L_1L_2})N_{L_3/K}(J_{L_3})$, but cannot be contained in $N_{L_1L_2/K}((L_1L_2)^\times)N_{L_3/K}(L_3^\times) \subseteq N_{L_1/K}(L_1^\times)$. □

Based on the (negative) result of the proposition, we would like to propose the following.

Conjecture. *Let L_1 and L_2 be finite Galois extensions of K. If every extension P of K contained in $L_1 \cap L_2$ satisfies the norm principle then the pair L_1, L_2 satisfies the multinorm principle.* (*It may be enough to require that only the intersection $L_1 \cap L_2$ satisfies the norm principle.*)

We note that, if proved, this conjecture would imply that a pair L_1, L_2 of finite Galois extensions of K satisfies the multinorm whenever the intersection $L_1 \cap L_2$ is a cyclic extension of K.

Next, we would like to point out that in some simple cases the Main Theorem can be proved without any use of group cohomology. The first such instance is when both extensions are biquadratic.

Proposition 12. *Let L_1 and L_2 be biquadratic extensions of K satisfying $L_1 \cap L_2 = K$. Then the pair L_1, L_2 satisfies the multinorm principle.*

Proof. Write $L_1 = K(\sqrt{a}, \sqrt{b})$ and $L_2 = K(\sqrt{c}, \sqrt{d})$. If at least one of the extensions satisfies the norm principle then the result follows from Proposition 2 (see the remark after the proposition). So, we only need to consider the case were both extensions fail to satisfy the norm principle. Using Tate's computation of the Tate–Shafarevich group for a Galois extension mentioned in the introduction, one readily sees that all local degrees of L_i over K are either 1 or 2, and then $\text{Ш}(L_i/K)$ is of order 2 for both $i = 1, 2$. We let S and T denote the sets of places of K that split in $K(\sqrt{a})$ and $K(\sqrt{c})$ respectively. Following [2, Exercise 5], consider the following homomorphisms of $K^\times$ to $\{\pm 1\}$:

$$\varphi_1(x) = \prod_{v \in S} (x, b)_v \quad \text{and} \quad \varphi_2(x) = \prod_{v \in T} (x, d)_v,$$

where $(x, y)_v$ denotes the Hilbert symbol at v. Clearly $\ker \varphi_i$ is an index two subgroup in $K^\times$ that according to [2] admits the following description

$$\ker \varphi_i = \{x \in K^\times \mid x^2 \in N_{L_i/K}(L_i^\times)\} \tag{13}$$

for $i = 1, 2$. Since b and d define different cosets modulo $K^{\times 2}$, it follows from properties of the Hilbert symbol (cf. [2, Exercise 2.6]) that the homomorphisms φ_1 and φ_2 are distinct, hence $(\ker \varphi_1)(\ker \varphi_2) = K^\times$. Using (13), we obtain the inclusion

$$K^{\times 2} \subset N_{L_1/K}(L_1^\times) N_{L_2/K}(L_2^\times). \tag{14}$$

Now, let $x_i \in K^\times$ be such that $\varphi_i(x_i) = -1$. Then $x_i^2 \notin N_{L_i/K}(L_i^\times)$. On the other hand, since all the local degrees of L_i over K are either 1 or 2, we see that $x_i^2 \in K^\times \cap N_{L_i/K}(J_{L_i})$. This means that the coset $x_i^2 N_{L_i/K}(L_i^\times)$ is a generator of $\text{Ш}(L_i/K) \simeq \mathbb{Z}/2\mathbb{Z}$, hence

$$K^\times \cap N_{L_i/K}(J_{L_i}) = \{1, x_i^2\} N_{L_i/K}(L_i^\times).$$

Now, taking into account (14), we see that

$$K^\times \cap N_{L_i/K}(L_i^\times) \subset N_{L_1/K}(L_1^\times) N_{L_2/K}(L_2^\times),$$

verifying thereby condition (2) of Proposition 2 and completing the proof of the multinorm principle for the pair L_1, L_2. □

Another instance is when both extensions are of a prime degree p. We recall that any extension L/K of degree p satisfies the norm principle (cf. [14, Proposition 6.10]). The following proposition provides an analog of this fact for the multinorm principle.

Proposition 13. *Let L_1 and L_2 be two separable extensions of K of a prime degree p. Then the pair L_1, L_2 satisfies the multinorm principle.*

(Note that in this proposition we don't need to assume that our extensions or their Galois closures are linearly disjoint.)

Lemma 14. *Let L_1 and L_2 be finite extensions of K. For any finite extension P of K of degree relatively prime to both $[L_1 : K]$ and $[L_2 : K]$, the validity of the multinorm principle for the pair L_1P, L_2P of extensions of P implies its validity for the pair L_1, L_2.*

Proof. For $i = 1, 2$, since $[L_i : K]$ is coprime to $[P : K]$, the extensions L_i and P are linearly disjoint over K, which implies that the norm map $N_{L_i/K}$ coincides (on J_{L_i} and $L_i^\times$) with the restriction of the norm map $N_{L_iP/P}$. Now, suppose that the multinorm principle holds for the pair L_1P, L_2P over P, and let

$$x \in K^\times \cap N_{L_1/K}(J_{L_1}) N_{L_2/K}(J_{L_2}).$$

Then it follows from the above remark that $x \in P^\times \cap N_{L_1P/P}(J_{L_1P}) N_{L_2P/P}(J_{L_2P})$, and hence

$$x = N_{L_1P/P}(y_1) N_{L_2P/P}(y_2) \quad \text{for some } y_i \in (L_iP)^\times,\ i = 1, 2.$$

Applying $N_{P/K}$, we obtain

$$x^{[P:K]} = N_{L_1/K}\big(N_{L_1P/L_1}(y_1)\big) N_{L_2/K}\big(N_{L_2P/L_2}(y_2)\big) \in N_{L_1/K}(L_1^\times) N_{L_2/K}(L_2^\times).$$

Since $x^{[L_1:K]} \in N_{L_1/K}(L_1^\times)$ and the degrees $[L_1 : K]$ and $[P : K]$ are relatively prime, we conclude that

$$x \in N_{L_1/K}(L_1^\times) N_{L_2/K}(L_2^\times),$$

proving the multinorm principle for L_1, L_2. □

Proof of Proposition 13. We first reduce the proof to the case where both L_1 and L_2 are Galois extensions of K. Let E_1 be the Galois closure of L_1 and let $G = \mathrm{Gal}(E_1/K)$. Then G is isomorphic to a subgroup of the symmetric group S_p, so its Sylow p-subgroup G_p is a cyclic group of order p. Set $P = E_1^{G_p}$; then $E_1 = L_1P$. Since the degree $[P : K]$ is coprime to p, according to Lemma 14, it suffices to prove the multinorm principle for the pair L_1P, L_2P of extensions of P. This enables us to assume without any loss of generality that one of the extensions is Galois. Repeating the argument for the other extension, we can assume that both extensions are Galois.

Now, let us consider the case where L_1 and L_2 are cyclic Galois extensions of K of degree p. By the Hasse theorem, L_i/K satisfies the norm principle for $i = 1, 2$. So, if $L_1 \cap L_2 = K$ then the multinorm principle for L_1, L_2 follows from Proposition 2 as condition (2) therein obviously holds. In the remaining case $L_1 = L_2$, the multinorm principle reduces to the norm principle for L_i, and therefore holds as well. □

Remark. If L_1 and L_2 are two separable extensions of K of a prime degree p, and E_1 and E_2 are their Galois closures, then one of the following occurs: either the degree of $E := E_1 \cap E_2$ is prime to p, or $E_1 = E_2$. To see this, one first proves the following elementary lemma from group theory: *Let G be a transitive subgroup of S_p. If $N \neq \{1\}$ is a normal subgroup of G then the order $|N|$ is divisible by p.* Then, if $E_1 \neq E_2$, for at least one $i \in \{1, 2\}$, the group $\mathrm{Gal}(E_i/E)$ is a nontrivial normal subgroup of $\mathrm{Gal}(E_i/K) \subset S_p$, hence has order divisible by p. Since the order of S_p is not divisible by p^2, we obtain that $[E:K]$ is prime to p, as claimed.

Now, if $[E:K]$ is prime to p then by Lemma 14 it is enough to prove the multinorm principle for the pair of extensions $L'_1 := L_1E, L'_2 := L_2E$ of E. But the Galois closures of L'_1 and L'_2 coincide with E_1 and E_2 respectively, hence are linearly disjoint over E. So, the multinorm principle for L'_1, L'_2 immediately follows from Proposition 2 as L'_1/E and L'_2/E satisfy the norm principle.

An obvious way to construct distinct degree $p > 2$ extensions L_1 and L_2 of K such that $E_1 = E_2$ is to pick an arbitrary non-Galois degree p extension L_1 and take for L_2 its suitable Galois conjugate. We note, however, that the group-theoretic constructions in [10] allow one to produce *non-conjugate* extensions with this property. In any case, letting P denote the fixed field of a Sylow p-subgroup of $\mathrm{Gal}(E/K)$, we will have $L_1P = L_2P = E$. Then arguing as in Lemma 14 one shows that

$$N_{L_1/K}\big(L_1^\times\big) = N_{L_2/K}\big(L_2^\times\big) \quad \text{and} \quad N_{L_1/K}(J_{L_1}) = N_{L_2/K}(J_{L_2})$$

(even when L_1 and L_2 are not Galois conjugate!). Thus, in this case the multinorm principle for L_1, L_2 reduces to the norm principle for L_i/K. This provides a somewhat more detailed perspective on the result of Proposition 13.

Finally, we observe that the multinorm can be considered not only for pairs but for any finite families of finite extensions of K. More precisely, we say that a family $L_1, \ldots, L_m$ ($m \geqslant 2$) satisfies the multinorm principle if

$$K^\times \cap N_{L_1/K}(J_{L_1}) \cdots N_{L_m/K}(J_{L_m}) = N_{L_1/K}\big(L_1^\times\big) \cdots N_{L_m/K}\big(L_m^\times\big).$$

Example 2. *The multinorm principle may fail for a triple L_1, L_2, L_3 of finite Galois extensions of K even when the fields L_i and L_j are pairwise linearly disjoint over K.* Indeed, set $K = \mathbb{Q}$ and

$$L_1 = \mathbb{Q}(\sqrt{13}), \qquad L_2 = \mathbb{Q}(\sqrt{17}), \quad \text{and} \quad L_3 = \mathbb{Q}(\sqrt{13 \cdot 17}).$$

Then

$$K^\times \cap N_{L_1/K}(J_{L_1}) N_{L_2/K}(J_{L_2}) N_{L_3/K}(J_{L_3}) = K^\times,$$

but $N_{L_1/K}(L_1^\times) N_{L_2/K}(L_2^\times) N_{L_3/K}(L_3^\times)$ is a subgroup of $K^\times$ of index 2 (cf. [2, Exercise 5] and [17, Lemma 4.8]), hence the multinorm principle fails (see also [9, Section 2]). □

Generalizing the Main Theorem of this note, one can show that if $L_1, \ldots, L_m$ are finite Galois extensions of K such that

$$\mathrm{Gal}(L_1 \cdots L_m/K) \simeq \mathrm{Gal}(L_1/K) \times \cdots \times \mathrm{Gal}(L_m/K)$$

(in other words, the whole family $L_1, \ldots, L_m$ is linearly disjoint over K) then the multinorm principle still holds for $L_1, \ldots, L_m$. This, however, requires some new considerations which will be described in [15].

5. Post factum

Let L_1 and L_2 be finite separable extensions of a field K, and let T be the corresponding *multinorm torus* (cf. [9]), i.e. the kernel of the map

$$\mathrm{R}_{L_1/K}(\mathrm{GL}_1) \times \mathrm{R}_{L_2/K}(\mathrm{GL}_1) \to \mathrm{GL}_1$$

given by the product of the norm maps for our extensions. Our Main Theorem on the multinorm principle is equivalent to the statement that if K is a global field and the normal closures E_1 and E_2 of L_1, L_2 are linearly disjoint over K, then the Tate–Shafarevich group $Ш(T)$ is trivial. J.-L. Colliot-Thélène pointed out to us that it is natural to consider the question about the Hasse principle in conjunction with the question about the weak approximation for the corresponding torus. Indeed, according to Voskresenskiĭ [18, Section 11.6], for a K-torus T there is an exact sequence

$$0 \to A(T) \to H^1\big(K, \mathrm{Pic}(\overline{X})\big)^{\vee} \to Ш(T) \to 0,$$

where $A(T)$ is the defect of weak approximation, X is a smooth projective model of T over K and $\overline{X} = X \otimes_K \overline{K}$ (with $\overline{K}$ being a separable closure of K), and $M^{\vee} = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is the Pontrjagin dual of a finite abelian group M. Furthermore, it was shown by Colliot-Thélène and Sansuc [3, Proposition 9.5] that the group $H^1(K, \mathrm{Pic}(\overline{X}))$ is isomorphic to

$$Ш^2_{\omega}(\widehat{T}) = \mathrm{Ker}\Big(H^2(G, \widehat{T}) \to \prod_{g \in G} H^2\big(\langle g \rangle, \widehat{T}\big)\Big)$$

where $\widehat{T}$ is the group of characters of T considered as a module over the Galois group $G = \mathrm{Gal}(L/K)$ of a finite Galois extension L/K that splits T. So, it is natural to try to prove the Hasse principle AND the weak approximation for a given class of tori T by proving that $Ш^2_{\omega}(\widehat{T})$ vanishes. In fact, a couple of weeks before we posted the original version of this note (arXiv:1203.1458), Dasheng Wei had posted his paper [19] in which the vanishing of $Ш^2_{\omega}(\widehat{T})$ for a multinorm torus T was derived from his computation of the Brauer–Manin obstruction whenever E_1 and E_2 are linearly disjoint in our notations (and even is a sightly more general situation). So, J.-L. Colliot-Thélène asked us if one can show the vanishing of $Ш^2_{\omega}(\widehat{T})$ by a direct computation – this would clarify our result as well as Wei's. The goal of this section is to provide this computation.

Proposition 15. *Let T be a multinorm torus associated with a pair L_1, L_2 of finite separable extensions of a field K. If their normal closures E_1 and E_2 are linearly disjoint over K, then $Ш^2_{\omega}(\widehat{T}) = \{0\}$.*

Proof. Let $G_i = \mathrm{Gal}(E_i/K)$ and $H_i = \mathrm{Gal}(E_i/L_i)$. The torus T splits over $E := E_1E_2$, and the fact that E_1 and E_2 are linearly disjoint implies that $G = \mathrm{Gal}(E/K)$ can be canonically identified with $G_1 \times G_2$. Then the character module $\widehat{T}$ is obtained from the following exact sequence of G-modules

$$0 \to \mathbb{Z} \to \mathbb{Z}[G_1/H_1] \oplus \mathbb{Z}[G_2/H_2] \to \widehat{T} \to 0, \tag{15}$$

where the first nontrivial map is given by

$$1 \mapsto (\epsilon_1, \epsilon_2),$$

where $\epsilon_i = \sum_{g_i H_i \in G_i/H_i} g_i H_i$. For $i = 1, 2$, pick a $\mathbb{Z}$-submodule $M_i \subset \mathbb{Z}[G_i/H_i]$ complementing $\langle \epsilon_i \rangle$, and consider it as a $\mathbb{Z}[G_{3-i}]$-module with trivial G_{3-i}-action. Then we have the following decomposition of $\mathbb{Z}[G_{3-i}]$-modules:

$$\mathbb{Z}[G_1/H_1] \oplus \mathbb{Z}[G_2/H_2] = M_i \oplus \big\langle (\epsilon_1, \epsilon_2) \big\rangle \oplus \mathbb{Z}[G_{3-i}/H_{3-i}]. \tag{16}$$

It follows that (15) splits as a sequence of $\mathbb{Z}[G_i]$-modules for both $i = 1, 2$.

Lemma 16. *The product*

$$\rho : H^2(G, \widehat{T}) \to H^2(G_1, \widehat{T}) \times H^2(G_2, \widehat{T})$$

of the restriction maps $\mathrm{Res}^G_{G_j}$ *for* $j = 1, 2$ *is injective.*

Proof. First, we claim that $H^1(G_1, \widehat{T}) = 0$. Using Shapiro's lemma and the fact that $\mathbb{Z}[G_2/H_2] \simeq \mathbb{Z}^{[G_2:H_2]}$ as G_1-modules, we obtain

$$H^1\big(G_1, \mathbb{Z}[G_1/H_1] \oplus \mathbb{Z}[G_2/H_2]\big) \simeq H^1(H_1, \mathbb{Z}) \oplus H^1(G_1, \mathbb{Z})^{[G_2:H_2]} = 0.$$

Since (15) splits as a sequence of $\mathbb{Z}[G_1]$-modules, it follows that $H^1(G_1, \widehat{T}) = 0$. This fact enables us to write the following inflation–restriction sequence (where G/G_1 is identified with G_2):

$$0 \to H^2\big(G_2, \widehat{T}^{G_1}\big) \xrightarrow{\iota} H^2(G, \widehat{T}) \to H^2(G_1, \widehat{T}).$$

To prove the assertion of the lemma, it is now enough to show that $\iota(H^2(G_2, \widehat{T}^{G_1}))$ intersects the kernel of $\mathrm{Res}^G_{G_2}$ trivially. But the composite map $\mathrm{Res}^G_{G_2} \circ \iota$ coincides with the natural map

$$\nu : H^2\big(G_2, \widehat{T}^{G_1}\big) \to H^2(G_2, \widehat{T}),$$

so it remains to show that it is injective. Note that if $z = (z_1, z_2)$ and $g_1 \in G_1$ are such that $g_1 z - z = n(\epsilon_1, \epsilon_2)$ with $n \in \mathbb{Z}$, then $n = 0$. This means that $\widehat{T}^{G_1}$ coincides with the image in $\widehat{T}$ of $\langle \epsilon_1 \rangle \oplus \mathbb{Z}[G_2/H_2] = \langle (\epsilon_1, \epsilon_2) \rangle \oplus \mathbb{Z}[G_2/H_2]$. So, it follows from (16) that $\widehat{T} = \overline{M}_1 \oplus \widehat{T}^{G_1}$, where $\overline{M}_1$ is the (isomorphic) image of M_1 in $\widehat{T}$, as $\mathbb{Z}[G_2]$-modules, hence the injectivity of ν. □

By Lemma 16, it is enough to prove that for $i = 1, 2$, the map

$$H^2(G_i, \hat{T}) \to \prod_{g \in G_i} H^2\big(\langle g \rangle, \hat{T}\big)$$

is injective. We give the argument for $i = 1$. Since (15) splits as a sequence of $\mathbb{Z}[G_1]$-modules, it suffices to prove that

$$\beta : H^2\big(G_1, \mathbb{Z}[G_1/H_1] \oplus \mathbb{Z}[G_2/H_2]\big) \to \prod_{g \in G_1} H^2\big(\langle g \rangle, \mathbb{Z}[G_1/H_1] \oplus \mathbb{Z}[G_2/H_2]\big)$$

is injective. We have $\beta = \beta_1 \oplus \beta_2$ where

$$\beta_i : H^2\big(G_1, \mathbb{Z}[G_i/H_i]\big) \to \prod_{g \in G_1} H^2\big(\langle g \rangle, \mathbb{Z}[G_i/H_i]\big),$$

and we need to prove that each β_i is injective. To prove this for $i = 2$, we observe that $\mathbb{Z}[G_2/H_2] \simeq \mathbb{Z}^{[G_2:H_2]}$ as G_1-modules, and therefore β_2 is the sum of $[G_2 : H_2]$ copies of

$$\gamma : H^2(G_1, \mathbb{Z}) \to \prod_{g \in G_1} H^2\big(\langle g \rangle, \mathbb{Z}\big).$$

Via dimension shifting, γ can be identified with the map

$$\mathrm{Hom}(G_1, \mathbb{Q}/\mathbb{Z}) = H^1(G_1, \mathbb{Q}/\mathbb{Z}) \to \prod_{g \in G_1} H^1\big(\langle g \rangle, \mathbb{Q}/\mathbb{Z}\big) = \prod_{g \in G_1} \mathrm{Hom}\big(\langle g \rangle, \mathbb{Q}/\mathbb{Z}\big),$$

which is clearly injective, and the injectivity of β_2 follows.

To establish the injectivity of β_1, we will actually prove a stronger statement that the map

$$\delta : H^2\big(G_1, \mathbb{Z}[G_1/H_1]\big) \to \prod_{h \in H_1} H^2\big(\langle h \rangle, \mathbb{Z}[G_1/H_1]\big)$$

is injective. For this we will need an explicit description of the isomorphism provided by Shapiro's lemma. As in [21, p. 130], we let $\mathbb{Z}^*$ denote the co-induced module $M_{H_1}^{G_1}(\mathbb{Z})$, i.e.

$$\mathbb{Z}^* = \big\{ f : G_1 \to \mathbb{Z} \;\big|\; f(ht) = f(t) \text{ for all } t \in G_1,\ h \in H_1 \big\}$$

with the G_1-action given by

$$(gf)(t) = f(tg).$$

As discussed in [21, p. 131], Shapiro's lemma yields an isomorphism

$$\mathrm{sh} : H^2\big(G_1, \mathbb{Z}^*\big) \to H^2(H_1, \mathbb{Z})$$

which explicitly can be described as restriction to H_1 followed by the map induced by evaluation at 1. On the other hand, the map $\phi : \mathbb{Z}[G_1/H_1] \to \mathbb{Z}^*$ defined by

$$\phi(\Sigma)(t) := \text{the coefficient of } t^{-1}H_1 \text{ in } \Sigma$$

is an isomorphism of G_1-modules, so the map

$$\varphi : H^2\big(G_1, \mathbb{Z}[G_1/H_1]\big) \to H^2(H_1, \mathbb{Z})$$

obtained by composing the isomorphism sh with the map induced by ϕ is an isomorphism as well. Explicitly, φ can be described as restriction to H_1 followed by the map induced by the H_1-module homomorphism $\mathbb{Z}[G_1/H_1] \to \mathbb{Z}$ that records the coefficient of the trivial coset $1 \cdot H_1$. From this description, we obtain the following commutative diagram

$$\begin{array}{ccc} H^2(G_1, \mathbb{Z}[G_1/H_1]) & \xrightarrow{\varphi} & H^2(H_1, \mathbb{Z}) \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\lambda} \\ \prod_{h \in H_1} H^2(\langle h \rangle, \mathbb{Z}[G_1/H_1]) & \xrightarrow{\psi} & \prod_{h \in H_1} H^2(\langle h \rangle, \mathbb{Z}) \end{array}$$

where λ is given by restriction maps, and ψ is induced by the above map $\mathbb{Z}[G_1/H_1] \to \mathbb{Z}$. Using dimension shifting as in the proof of the injectivity of γ, we conclude that λ is injective. This implies that δ, hence also β_1, is injective, completing the proof of the injectivity of β. □

Even though Proposition 15, combined with the results of Voskresenskiĭ and Colliot-Thélène and Sansuc, gives an alternative proof of our Main Theorem as well as of weak approximation for the corresponding multinorm torus, we decided to keep Sections 1–4 intact. The reason is that our explicit arithmetic argument shows how things work and relies only on standard facts from class field theory. At the same time, it turns out to be applicable in other situations, e.g. it enables one to compute the obstruction to the multinorm principle when L_1 and L_2 are abelian but not necessarily disjoint extensions, to prove a version of the multinorm principle for more than two extensions, etc. (see [15]).

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Appendix A. Deflation and residuation maps and their properties

In this appendix, we briefly sketch the construction of the deflation and residuation maps and prove Lemma 8 (note that our account, unlike that in [20] and [8], is based on homogeneous cochains).

Given a finite group G, we let $X = \{X_i\}_{i \in \mathbb{Z}}$ denote the standard complex used to define the Tate cohomology groups (cf. [2, Chapter IV, Section 6]). More precisely, for $i \geqslant 0$, $X_i = \mathbb{Z}[G^{i+1}]$ with the G-action $s(g_0, \dots, g_i) = (sg_0, \dots, sg_i)$, and the differential $d\colon X_{i+1} \to X_i$ given by

$$d(g_0, \dots, g_{i+1}) = \sum_{j=0}^{i+1} (-1)^j (g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_{i+1}).$$

Furthermore, for $i \geqslant 1$, we set $X_{-i} = \mathrm{Hom}_{\mathbb{Z}}(X_{i-1}, \mathbb{Z})$, which is a free $\mathbb{Z}$-module with a basis $(s_1^*, \dots, s_i^*)$, where all $s_j \in G$, defined by

$$\left(s_1^*, \dots, s_i^*\right)(g_0, \dots, g_{i-1}) = \begin{cases} 1 & \text{if } s_j = g_{j-1} \text{ for all } j, \\ 0 & \text{otherwise,} \end{cases}$$

and the G-action $g(s_1^*, \dots, s_i^*) = ((gs_1)^*, \dots, (gs_i)^*)$. The differential $d\colon X_{-i} \to X_{-i-1}$ is given by

$$d\left(s_1^*, \dots, s_i^*\right) = \sum_{j=1}^{i+1} \sum_{g \in G} (-1)^j \left(s_1^*, \dots, s_{j-1}^*, g^*, s_j^*, \dots, s_i^*\right).$$

Finally, the "special" differential $d : X_0 \to X_{-1}$ is defined by

$$d(g_0) = \sum_{s \in G} s^*.$$

Then for any G-module A and all $i \in \mathbb{Z}$ we have

$$\hat{H}^i(G, A) = H^i\big(\mathrm{Hom}_G(X, A)\big).$$

Deflation map. Given any normal subgroup H of G, we let Y denote the standard complex for G/H. Then for any G-module A and each $i \geqslant 1$ there is a map $\delta_{-i} : \mathrm{Hom}_G(X_{-i}, A) \to \mathrm{Hom}_G(Y_{-i}, A)$ given by

$$(\delta_{-i} f)\big(\alpha_1^*, \dots, \alpha_i^*\big) = \sum_{g_i H = \alpha_i} f\big(g_1^*, \dots, g_i^*\big)$$

for $f \in \mathrm{Hom}_G(X_{-i}, A)$ and $\alpha_1, \dots, \alpha_i \in G/H$. One can check that the image of δ_{-i} lies in $\mathrm{Hom}_{G/H}(Y_{-i}, A^H)$, hence δ_{-i} induces a map

$$\mathrm{Def}^G_{G/H} : \hat{H}^{-i}(G, A) \to \hat{H}^{-i}\big(G/H, A^H\big)$$

called the *deflation map*. For $i = 0$ one gives an ad hoc definition of the deflation map. Namely, for any group G and any G-module A we have $\hat{H}^0(G, A) \simeq A^G / N_G(A)$, where N_G is the norm map, $N_G(a) = \sum_{g \in G} ga$. Then

$$\mathrm{Def}^G_{G/H} : \hat{H}^0(G, A) \to \hat{H}^0\big(G/H, A^H\big)$$

is induced by the identification $A^G \to (A^H)^{G/H}$ and the inclusion $N_G(A) \hookrightarrow N_{G/H}(A^H)$. (In terms of homogeneous cochains, every element of $\hat{H}^0(G, A)$ is represented by a function $f \in \mathrm{Hom}_G(\mathbb{Z}[G], A)$ with values in A^G. Then $\mathrm{Def}^G_{G/H}$ is induced by the map $\delta : \mathrm{Hom}_G(\mathbb{Z}[G], A^G) \to \mathrm{Hom}_{G/H}(\mathbb{Z}[G/H], A^H)$ given by $\delta(f)(g_0 H) = f(g_0)$.)

Residuation map. Let G, H, X, Y, and A be as above. We let I_H denote the augmentation ideal of $\mathbb{Z}[H]$, and set $A_H = A/I_H A$. For each $i \geqslant 1$ there is a map $\delta'_{-i} : \mathrm{Hom}_G(X_{-i}, A) \to \mathrm{Hom}_{G/H}(Y_{-i}, A_H)$ given by

$$\big(\delta'_i f\big)\big(\alpha^*, \alpha_2^*, \dots, \alpha_i^*\big) = \sum_{g_i H = \alpha_i} f\big(g^*, g_2^*, \dots, g_i^*\big) + I_H,$$

where g is an arbitrary (single) element such that $gH = \alpha$; since f is a G-map, this definition does not depend on the choice of g. Then for $i \geqslant 2$, δ'_{-i} induces a map on cohomology

$$\mathrm{Rsd}^G_{G/H} : \hat{H}^{-i}(G, A) \to \hat{H}^{-i}(G/H, A_H),$$

called the *residuation map*. We note that in the special case where A is a trivial G-module, we have $A = A^H = A_H$, and

$$|H| \cdot \mathrm{Rsd}^G_{G/H} = \mathrm{Def}^G_{G/H}. \tag{17}$$

We will make use of this fact below for $A = \mathbb{Z}$.

Proof of Lemma 8. Fix $i \geqslant 2$, and to simplify notation we will write Inf, Def, ... instead of Inf_H^G, Def_H^G, etc. Let $\bar{f} \in \hat{H}^{-i}(G,\mathbb{Z})$ and $\bar{\psi} \in \hat{H}^i(H,\mathbb{Z})$ be represented by the homogeneous cocycles $f \in \mathrm{Hom}_G(\mathbb{Z}[(G^*)^i],\mathbb{Z})$, where $(G^*)^i = \{(s_1^*,\ldots,s_i^*) \mid s_j \in G\}$, and $\psi \in \mathrm{Hom}_H(\mathbb{Z}[H^{i+1}],\mathbb{Z})$. Furthermore, $\mathrm{Def}(\bar{f})$ and $\mathrm{Rsd}(\bar{f})$ are represented respectively by $\tilde{f}_1$ and $\tilde{f}_2 \in \mathrm{Hom}_H(\mathbb{Z}[(H^*)^i],\mathbb{Z})$ defined by

$$\tilde{f}_1\big(h_1^*,\ldots,h_i^*\big) = \sum_{k_j \in K} f\big((h_1k_1)^*,\ldots,(h_ik_i)^*\big) \quad \text{and}$$

$$\tilde{f}_2\big(h_1^*,h_2^*,\ldots,h_i^*\big) = \sum_{k_j \in K} f\big(h_1^*,(h_2k_2)^*,\ldots,(h_ik_i)^*\big),$$

and $\mathrm{Inf}(\bar{\psi})$ is represented by $\tilde{\psi} \in \mathrm{Hom}_G(\mathbb{Z}[G^{i+1}],\mathbb{Z})$ given by

$$\tilde{\psi}(h_0k_0,\ldots,h_ik_i) = \psi(h_0,\ldots,h_i).$$

Next, as shown in [2, pp. 105–108], the cup-product $\bar{a} \cup \bar{b}$ of classes $\bar{a} \in \hat{H}^{-i}(G,\mathbb{Z})$ and $\bar{b} \in \hat{H}^i(G,\mathbb{Z})$ that are represented by the cocycles a and b, is represented by the function

$$g_0 \mapsto \sum_{s_1,\ldots,s_i \in G} a\big(s_1^*,\ldots,s_i^*\big) b(s_i,\ldots,s_1,g_0),$$

and the cup-product of classes in $\hat{H}^{-i}(H,\mathbb{Z})$ and $\hat{H}^i(H,\mathbb{Z})$ is described similarly. Finally, the corestriction map from $\hat{H}^0(H,\mathbb{Z}) = \mathbb{Z}/|H|\mathbb{Z}$ to $\hat{H}^0(G,\mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z}$ is given by multiplication by $[G:H] = |K|$.

Putting this information together, we obtain that $\mathrm{Cor}(\mathrm{Rsd}(\bar{f}) \cup \bar{\psi})$ is represented by the function

$$h_0k_0 \mapsto |K| \sum_{h_1,\ldots,h_i \in H} \tilde{f}_2\big(h_1^*,\ldots,h_i^*\big)\psi(h_i,\ldots,h_1,h_0),$$

and therefore in view of (17) by the function

$$\begin{aligned} h_0k_0 \mapsto & \sum_{h_1,\ldots,h_i \in H} \tilde{f}_1\big(h_1^*,\ldots,h_i^*\big)\psi(h_i,\ldots,h_1,h_0) \\ &= \sum_{h_j \in H}\sum_{k_j \in K} f\big((h_1k_1)^*,\ldots,(h_ik_i)^*\big)\psi(h_i,\ldots,h_1,h_0) \\ &= \sum_{h_j \in H, k_j \in K} f\big((h_1k_1)^*,\ldots,(h_ik_i)^*\big)\tilde{\psi}(h_1k_1,\ldots,h_ik_i,h_0k_0) \\ &= \sum_{s_j \in G} f\big(s_1^*,\ldots,s_i^*\big)\tilde{\psi}(s_i,\ldots,s_1,h_0k_0). \end{aligned}$$

But the function

$$h_0k_0 \mapsto \sum_{s_j \in G} f\big(s_1^*,\ldots,s_i^*\big)\tilde{\psi}(s_i,\ldots,s_1,h_0k_0)$$

also represents $\bar{f} \cup \mathrm{Inf}(\bar{\psi})$, yielding our claim. □

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