ARITHMETIC AND ZARISKI-DENSE SUBGROUPS: weak commensurability, eigenvalue rigidity, and applications to locally symmetric spaces

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1. Algebraic groups and their arithmetic and Zariski-dense subgroups
   - Basic definitions
   - Field of definition
   - Algebraic groups: important classes and structure theory
   - Basic results about arithmetic groups
   - Arithmetic lattices in simple Lie groups
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such that

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Thus, algebraic groups = Zariski-closed subgroups of $\mathrm{GL}_n(\mathbb{C})$.

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such that $\varphi(X) = (f_{k\ell}(X))$ for all $X = (x_{ij}) \in G$.

Example of algebraic groups

1. $G = \mathrm{SL}_n(\mathbb{C})$ is given by $\det(X) - 1 = 0$ where $X = (x_{ij})$.

2. $G = \mathrm{Sp}_{2m}(\mathbb{C})$ is given by ${}^tXEX = E$, where

$$E = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix}$$

3. $G = \mathrm{O}_n(q)$, q a nondegenerate quadratic form.

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For $G = \mathrm{GL}_n(\mathbb{C})$, it is simply defined by conjugation action on $M_n(\mathbb{C})$:

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Analytically: G can be considered as a complex Lie group, and then we take its Lie algebra in this (analytic) context.

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(2) A morphism of F-groups $\varphi \colon G \to H$ is F-defined **if** it can be given by functions from

$$F\left[x_{11}, \ldots, x_{nn}, \frac{1}{\det(x_{ij})}\right].$$

Examples

1. SL_n and Sp_{2m} are defined over $\mathbb{Q}$.

2. $\mathrm{O}_n(q)$ and $\mathrm{SO}_n(q)$ are defined over any $F \subset \mathbb{C}$ that contains coefficients of q.

3. Determinant morphism $\mathrm{GL}_n \to \mathrm{GL}_1$, $g \mapsto \det(g)$, is defined over $\mathbb{Q}$.

4. $G = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \right\}$ is defined over $\mathbb{Q}$.

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   Determinant restricts to character

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mapsto x^2 + y^2$$

Examples, cont'd

Another character $\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mapsto x + iy$
is **not** defined over $\mathbb{Q}$ (or $\mathbb{R}$) but over $\mathbb{Q}(i)$.

5. If G is F-defined, then Ad_G is F-defined (in a suitable basis of $\mathfrak{g}$)

6. Let q_1 and q_2 be two nondegenerate n-dimensional quadratic forms over $\mathbb{Q}$.

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However, there may or may not be $\mathbb{Q}$-isomorphism!

Modification:

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(The group of integral points depends on basis in $\mathbb{Q}^n$.)

On the other hand, $g\,\mathrm{GL}_n(\mathbb{Z})\,g^{-1}$ is commensurable to $\mathrm{GL}_n(\mathbb{Z})$, for any $g \in \mathrm{GL}_n(\mathbb{Q})$.

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Note: conclusion remains valid for any *surjective* φ, but proof requires *reduction theory*.

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- In order to formulate basic results about (usual) arithmetic groups, we need to review important classes of algebraic groups.

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(equivalently, all eigenvalues of u are equal to 1).

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In general, for T of rank m all characters form an abelian group $X(T)$ isomorphic to $\mathbb{Z}^m$.

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• Any F-torus T has F-subtori T_s (split) and T_a (anisotropic) such that

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- $\dim T_s = \mathrm{rk}_F\, T$ (F-rank of T).

Example

$G = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \right\}$ is defined over $F = \mathbb{R}$ (even $\mathbb{Q}$), and is a torus.

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Indeed, for $g = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$ we have

$$g^{-1} \begin{pmatrix} x & -y \\ y & x \end{pmatrix} g = \begin{pmatrix} x+iy & 0 \\ 0 & x-iy \end{pmatrix} \quad \Rightarrow \quad g^{-1}\, G\, g = \mathbb{D}_2.$$

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Using characters: We have following characters

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(which actually form a basis of $X(T) \simeq \mathbb{Z}^2$).

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In fact, $G_s = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \right\}$ and $G_a = G \cap \mathrm{SL}_2(\mathbb{C})$

(note that $G_s \bigcap G_a = \{\pm 1\}$).

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On the other hand, $\chi = \chi_1 + \chi_2$ is *fixed* by complex conjugation (note that $\chi(x) = \det(x)$, $x \in G$) $\Rightarrow$ G is **not** anisotropic either.

In fact, $G_s = \left\{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \right\}$ and $G_a = G \cap \mathrm{SL}_2(\mathbb{C})$

(note that $G_s \bigcap G_a = \{\pm 1\}$).

- **So,** $\mathrm{rk}\, G = 2$ (absolute rank) and $\mathrm{rk}_{\mathbb{R}}\, G = 1$ ($\mathbb{R}$-rank).

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1. Algebraic groups and their arithmetic and Zariski-dense subgroups
   - Basic definitions
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   - Algebraic groups: important classes and structure theory
   - **Basic results about arithmetic groups**
   - Arithmetic lattices in simple Lie groups
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a subgroup $\Gamma \subset \mathcal{G}$ *is* arithmetic *if there is* $\mathbb{R}/\mathbb{Q}$*-form* G' *of* G **and** $\mathbb{R}$*-isomorphism* $\varphi \colon G' \to G$ *such that* Γ *is commensurable with* $\varphi(G'(\mathbb{Z}))$.

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Then each of the *rational* quadratic forms

$$q_1 = x^2 + y^2 - 3z^2 \quad \text{and} \quad q_2 = x^2 + y^2 - 7z^2,$$

being equivalent to q over $\mathbb{R}$, defines a family of arithmetic subgroups of $\mathcal{G}$.

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So, $\Gamma = \mathrm{SO}_3(q_3)(\mathbb{Z}[\sqrt{2}])$ embeds as a *discrete* subgroup in

$$\mathcal{H} = \mathcal{G}_3 \times \mathcal{G}_3'$$

where $\mathcal{G}_3 = \mathrm{SO}_3(q_3)(\mathbb{R})$, $\mathcal{G}_3' = \mathrm{SO}_3(q_3')(\mathbb{R})$, $q_3' = x^2 + y^2 + \sqrt{2}z^2$.

By *restriction of scalars*, one constructs a semi-simple $\mathbb{Q}$-group H such that

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So, a "reasonable definition" of an arithmetic group/lattice must include groups that arise from rings of algebraic integers other than $\mathbb{Z}$.

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If G is not adjoint and $\pi \colon G \to \overline{G}$ is F-isogeny onto adjoint group, then $\Gamma \subset G(F)$ is $(K, \mathcal{G})$-arithmetic if $\pi(\Gamma) \subset \overline{G}(F)$ is such.

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Γ_1 and Γ_2 are commensurable up to F-isomorphism between G_1 and G_2 if there exists F-isomorphism

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such that $\varphi(\Gamma_1)$ and Γ_2 are commensurable in usual sense.

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$$i^2 = a, \ \ j^2 = b, \ \ k = ij = -ji \quad \text{etc.}$$

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Then $\mathrm{N}(z'z'') = \mathrm{N}(z')\mathrm{N}(z')$ (also for $z', z'' \in D \otimes_K F$).

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(For computations, one still uses realization of G as hypersurface.)

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- May assume that $a, b \in \mathcal{O}$ (ring of integers in K), then
  $$\mathcal{A} = \mathcal{O} + \mathcal{O}i + \mathcal{O}j + \mathcal{O}k$$
  is a subring of D (called an $\mathcal{O}$-order in D).

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  - $D \otimes_{K,\epsilon} \mathbb{R}$ is a division algebra.

1. Algebraic groups and their arithmetic and Zariski-dense subgroups
   - Basic definitions
   - Field of definition
   - Algebraic groups: important classes and structure theory
   - Basic results about arithmetic groups
   - Arithmetic lattices in simple Lie groups
   - Zariski-dense subgroups

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Then:

- $u^+(1)$ and $u^-(1)$ generate $\mathrm{SL}_2(\mathbb{Z})$ which is arithmetic;
- $u^+(2)$ and $u^-(2)$ generate a subgroup of index 12 $\mathrm{SL}_2(\mathbb{Z})$, which is again arithmetic;
- for $m \geqslant 3$, $u^+(m)$ and $u^-(m)$ generate a Zariski-dense subgroup of infinite index in $\mathrm{SL}_2(\mathbb{Z})$, which is not arithmetic (*thin*)

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Here a subset $\Phi \subset \Gamma$ is called *free* if inclusion $\Phi \hookrightarrow \Gamma$ extends to *injective* homomorphism of free group on Φ to Γ.

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We would like to extend definition of some attributes (such as K and $\mathcal{G}$) from *arithmetic groups* to *arbitrary Zariski-dense subgroups*.

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Let V be a vector space over a field F, and let $\Gamma \subset \mathrm{GL}(V)$ be a subgroup.

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We let K_Γ denote the trace field of Γ, i.e., subfield of F generated by

$$\mathrm{Tr}\,\mathrm{Ad}_G(\gamma), \quad \gamma \in \Gamma.$$

Theorem (E.B. Vinberg)

K_Γ is the minimal field of definition for $\mathrm{Ad}_G(\Gamma) \subset \mathrm{GL}(\mathfrak{g})$.

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Clearly, $\mathcal{G}$ is F/K-form of $\overline{G}$.

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So, K_Γ and $\mathcal{G}(\Gamma)$ are *direct analogs* of K and $\mathcal{G}$ for general Zariski-dense subgroups.

These are important invariants of *commensurability class* of Γ even though they do *not* determine this class uniquely in the general case.