

# ARITHMETIC AND ZARISKI-DENSE SUBGROUPS:

weak commensurability, eigenvalue rigidity, and  
applications to locally symmetric spaces

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## 1 Algebraic groups and their arithmetic and Zariski-dense subgroups

- Basic definitions
- Field of definition
- Algebraic groups: important classes and structure theory
- Basic results about arithmetic groups
- Arithmetic lattices in simple Lie groups
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such that

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**Thus**, algebraic groups = Zariski-closed subgroups of  $\mathrm{GL}_n(\mathbb{C})$ .

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such that  $\varphi(X) = (f_{k\ell}(X))$  for all  $X = (x_{ij}) \in G$ .

# Example of algebraic groups

①  $G = \mathrm{SL}_n(\mathbb{C})$  is given by  $\det(X) - 1 = 0$  where  $X = (x_{ij})$ .

②  $G = \mathrm{Sp}_{2m}(\mathbb{C})$  is given by  ${}^t X E X = E$ , where

$$E = \begin{pmatrix} O & I_m \\ -I_m & O \end{pmatrix}$$

③  $G = \mathrm{O}_n(q)$ ,  $q$  a nondegenerate quadratic form.

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①  $\varphi: \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathrm{GL}_1(\mathbb{C}), X \mapsto \det(X).$

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$$\text{Ad}: \text{GL}_n(\mathbb{C}) \rightarrow \text{Aut}(M_n(\mathbb{C})) = \text{GL}_{n^2}(\mathbb{C}), \quad g \mapsto i_g$$

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*Analytically:*  $G$  can be considered as a complex Lie group, and then we take its Lie algebra in this (analytic) context.

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(2) A morphism of  $F$ -groups  $\varphi: G \rightarrow H$  is  $F$ -defined if it can be given by functions from

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# Examples

- 1  $SL_n$  and  $Sp_{2m}$  are defined over  $\mathbb{Q}$ .
- 2  $O_n(q)$  and  $SO_n(q)$  are defined over any  $F \subset \mathbb{C}$  that contains coefficients of  $q$ .
- 3 Determinant morphism  $GL_n \rightarrow GL_1$ ,  $g \mapsto \det(g)$ , is defined over  $\mathbb{Q}$ .
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Determinant restricts to character

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mapsto x^2 + y^2$$



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Another character  $\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mapsto x + iy$   
is **not** defined over  $\mathbb{Q}$  (or  $\mathbb{R}$ ) but over  $\mathbb{Q}(i)$ .

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(The group of integral points depends on basis in  $\mathbb{Q}^n$ .)

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**Note:** conclusion remains valid for any *surjective*  $\varphi$ , but proof requires *reduction theory*.



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- For our purposes, we will need to **generalize** notion of arithmetic group for *real* algebraic/Lie groups *without* canonical  $\mathbb{Q}$ -structure (i.e., a realization as  $\mathbb{Q}$ -group).
- In order to formulate basic results about (usual) arithmetic groups, we need to review important classes of algebraic groups.

- 1 Algebraic groups and their arithmetic and Zariski-dense subgroups
  - Basic definitions
  - Field of definition
  - Algebraic groups: important classes and structure theory
  - Basic results about arithmetic groups
  - Arithmetic lattices in simple Lie groups
  - Zariski-dense subgroups



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A connected diagonalizable group is an **algebraic torus**.

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If this is possible (equivalently, there is an  $F$ -isomorphism  $T \simeq \mathbb{D}_m$ ), then  $T$  is **split** over  $F$ .

- $T$  is  $F$ -split  $\Leftrightarrow$  all characters of  $T$  are  $F$ -defined.

In general, for  $T$  of rank  $m$  all characters form an abelian group  $X(T)$  isomorphic to  $\mathbb{Z}^m$ .

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- $\dim T_s = \text{rk}_F T$  ( $F$ -rank of  $T$ ).

# Example

$G = \left\{ \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \right\}$  is defined over  $F = \mathbb{R}$  (even  $\mathbb{Q}$ ),  
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**Using characters:** We have following characters

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \xrightarrow{\chi_1} x + iy \quad \text{and} \quad \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \xrightarrow{\chi_2} x - iy$$

(which actually form a basis of  $X(T) \simeq \mathbb{Z}^2$ ).

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- **So**,  $\mathrm{rk} G = 2$  (absolute rank) and  $\mathrm{rk}_{\mathbb{R}} G = 1$  ( $\mathbb{R}$ -rank).



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- Complex semi-simple algebraic groups are (almost) classified by their *root systems*.

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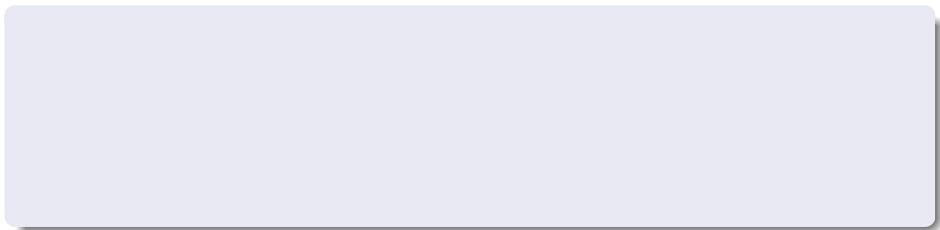
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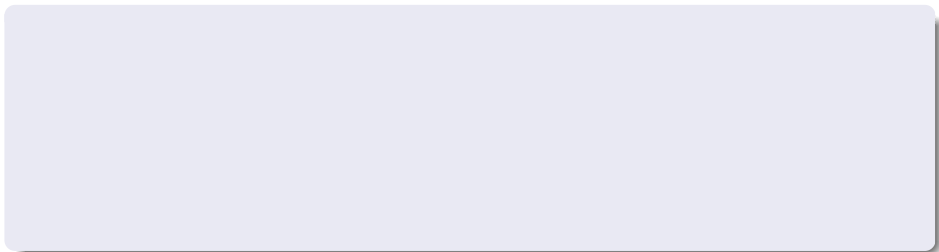


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*a subgroup  $\Gamma \subset \mathcal{G}$  is arithmetic if there is  $\mathbb{R}/\mathbb{Q}$ -form  $G'$  of  $G$  and  $\mathbb{R}$ -isomorphism  $\varphi: G' \rightarrow G$  such that  $\Gamma$  is commensurable with  $\varphi(G'(\mathbb{Z}))$ .*

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Then each of the *rational* quadratic forms

$$q_1 = x^2 + y^2 - 3z^2 \quad \text{and} \quad q_2 = x^2 + y^2 - 7z^2,$$

being equivalent to  $q$  over  $\mathbb{R}$ , defines a family of arithmetic subgroups of  $\mathcal{G}$ .

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**So,**  $\Gamma = \mathrm{SO}_3(q_3)(\mathbb{Z}[\sqrt{2}])$  embeds as a *discrete* subgroup in

$$\mathcal{H} = \mathcal{G}_3 \times \mathcal{G}'_3$$

where  $\mathcal{G}_3 = \mathrm{SO}_3(q_3)(\mathbb{R})$ ,  $\mathcal{G}'_3 = \mathrm{SO}_3(q'_3)(\mathbb{R})$ ,  $q'_3 = x^2 + y^2 + \sqrt{2}z^2$ .

By *restriction of scalars*, one constructs a semi-simple  $\mathbb{Q}$ -group  $H$  such that

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**So**, a “reasonable definition” of an arithmetic group/lattice must include groups that arise from rings of algebraic integers other than  $\mathbb{Z}$ .

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If  $G$  is not adjoint and  $\pi: G \rightarrow \overline{G}$  is  $F$ -isogeny onto adjoint group, then  $\Gamma \subset G(F)$  is  $(K, \mathcal{G})$ -arithmetic if  $\pi(\Gamma) \subset \overline{G}(F)$  is such.

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### Proposition (G.Prasad - A.R.)

*Let  $G_1$  and  $G_2$  be simple algebraic  $F$ -groups, and let  $\Gamma_i \subset G_i(F)$  be Zariski-dense  $(K_i, \mathfrak{G}_i)$ -arithmetic subgroup of  $G_i(F)$ .*

**Then**  $\Gamma_1$  and  $\Gamma_2$  are commensurable up  $F$ -isomorphism between  $\bar{G}_1$  and  $\bar{G}_2 \Leftrightarrow K_1 = K_2 =: K$  and  $\mathfrak{G}_1$  and  $\mathfrak{G}_2$  are  $K$ -isomorphic.



So,

$$(\mathbb{Q}, \mathrm{SO}_3(q_1)) \quad , \quad (\mathbb{Q}, \mathrm{SO}_3(q_2)) \quad \text{and} \quad (\mathbb{Q}(\sqrt{2}), \mathrm{SO}_3(q_3))$$

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This is part of **eigenvalue rigidity**.



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and multiplication table

$$i^2 = a, j^2 = b, k = ij = -ji \text{ etc.}$$





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**Then**  $N(z'z'') = N(z')N(z'')$  (also for  $z', z'' \in D \otimes_K F$ ).

It follows (assuming  $K \subset \mathbb{C}$ ) that

$$G := \{ z = z_0 + z_1i + z_2j + z_3k \in D \otimes_F \mathbb{C} \mid z_0^2 - az_1^2 - bz_2^2 + abz_3^2 = 1 \}$$

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(For computations, one still uses realization of  $G$  as hypersurface.)



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- May assume that  $a, b \in \mathcal{O}$  (ring of integers in  $K$ ), then

$$\mathcal{A} = \mathcal{O} + \mathcal{O}i + \mathcal{O}j + \mathcal{O}k$$

is a subring of  $D$  (called an  $\mathcal{O}$ -**order** in  $D$ ).



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  - Basic definitions
  - Field of definition
  - Algebraic groups: important classes and structure theory
  - Basic results about arithmetic groups
  - Arithmetic lattices in simple Lie groups
  - Zariski-dense subgroups

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**Then:**

- $u^+(1)$  and  $u^-(1)$  generate  $\mathrm{SL}_2(\mathbb{Z})$  which is arithmetic;
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- for  $m \geq 3$ ,  $u^+(m)$  and  $u^-(m)$  generate a Zariski-dense subgroup of **infinite** index in  $\mathrm{SL}_2(\mathbb{Z})$ , which is **not** arithmetic (*thin*).

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Here a subset  $\Phi \subset \Gamma$  is called *free* if inclusion  $\Phi \hookrightarrow \Gamma$  extends to *injective* homomorphism of free group on  $\Phi$  to  $\Gamma$ .

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**Definition.**

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$$\mathrm{Tr} \mathrm{Ad}_G(\gamma), \quad \gamma \in \Gamma.$$

## Theorem (E.B. Vinberg)

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Clearly,  $\mathcal{G}$  is  $F/K$ -form of  $\overline{G}$ .



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So,  $K_\Gamma$  and  $\mathcal{G}(\Gamma)$  are *direct analogs* of  $K$  and  $\mathcal{G}$  for general Zariski-dense subgroups.

These are **important invariants** of *commensurability class* of  $\Gamma$  even though they do *not* determine this class uniquely in the general case.