Arithmetic and Zariski-dense subgroups:

weak commensurability, eigenvalue rigidity, and applications to locally symmetric spaces

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1. Eigenvalue rigidity and hearing the shape of a drum
   - Classical vs. Eigenvalue Rigidity
2. Hearing the Shape
   - 1-dimensional case
   - Flat tori of dimension > 1
   - Weyl's Law and its Consequences
3. Locally symmetric spaces
   - Laplace-Beltrami operator
   - Isospectral non-isometric manifolds
   - Our results

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• We call this phenomenon *eigenvalue rigidity*.

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- Why do we care about eigenvalues?

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$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \neq 1.$$

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Example. Let

$$A = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1/24 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/12 \end{pmatrix} \in \mathrm{SL}_3(\mathbb{C}).$$

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Then A and B are *weakly commensurable* because

$$\lambda_1 = 12 = 4 \cdot 3 = \mu_1 \cdot \mu_2 \quad (\text{or} \quad \lambda_1 = \mu_3^{-1}).$$

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- **Reason:** these subgroups contain special elements, called *generic.*

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CAN ONE HEAR THE SHAPE OF A DRUM?

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To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

> "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

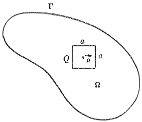

FIG. 1

1. And now to the theme and the title.

It has been known for well over a century that if a membrane Ω, held fixed along its boundary Γ (see Fig. 1), is set in motion its displacement (in the direction perpendicular to its original plane)

$$F(x, y; t) \equiv F(\vec{\rho}; t)$$

obeys the wave equation

$$\frac{\partial^2 F}{\partial t^2} = c^2 \nabla^2 F,$$

where c is a certain constant depending on the physical properties of the membrane and on the tension under which the membrane is held.

I shall choose units to make $c^2 = \frac{1}{2}$.

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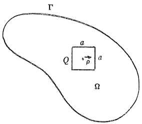

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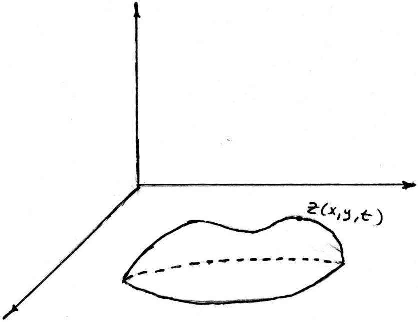

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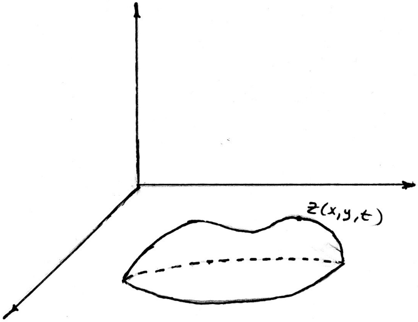

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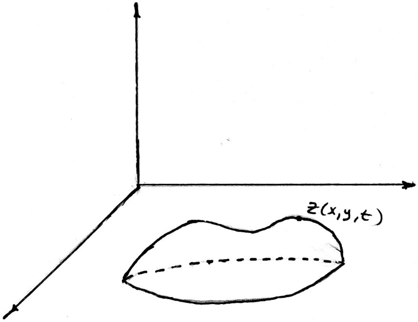

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(For simplicity, we take $c = 1$ in the sequel.)

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(**then** ω is one of *overtones* (or *harmonics*) of membrane.)

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Thus, *harmonics* in sound produced by membrane have to do with

eigenvalues of Laplacian.

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(Both conditions result in same *eigenvalues*.)

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Each λ_n $(n > 0)$ has *multiplicity* 2 with eigenfunctions

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- **So**, one **can** hear shape of circumference (or string) (even if one misses a couple of low overtones).

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(In fact, there are *infinite families* of such tori having same dimension **but** pairwise different volumes.)

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- *Laplace spectrum* = set of eigenvalues of Laplacian **with** multiplicities (assuming that these are finite)
- Two geometric objects are *isospectral* **if** they have same Laplace spectra.

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- corresponding quadratic forms q_1 and q_2 have same *discriminant*.

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- M. KNESER: *Finiteness* holds for any lattice (not necessarily integral-valued).

Bad news

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

BY J. MILNOR

PRINCETON UNIVERSITY

Communicated February 6, 1964

To every compact Riemannian manifold M there corresponds the sequence $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ of eigenvalues for the Laplace operator on M. It is not known just how much information about M can be extracted from this sequence.[1] This note will show that the sequence does not characterize M completely, by exhibiting two 16-dimensional toruses which are distinct as Riemannian manifolds but have the same sequence of eigenvalues.

By a *flat torus* is meant a Riemannian quotient manifold of the form R^n/L, where L is a lattice (= discrete additive subgroup) of rank n. Let L^* denote the dual lattice, consisting of all $y \in R^n$ such that $x \cdot y$ is an integer for all $x \in L$. Then each $y \in L^*$ determines an eigenfunction $f(x) = \exp(2\pi\, ix \cdot y)$ for the Laplace operator on R^n/L. The corresponding eigenvalue λ is equal to $(2\pi)^2 y \cdot y$. Hence, the number of eigenvalues less than or equal to $(2\pi r)^2$ is equal to the number of points of L^* lying within a ball of radius r about the origin.

According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

In an attempt to distinguish R^{16}/L_1 from R^{16}/L_2 one might consider the eigenvalues of the Hodge-Laplace operator $\Delta = d\delta + \delta d$, applied to the space of differential p-forms. However, both manifolds are flat and parallelizable, so the identity

$$\Delta(f\, dx_{i_1} \wedge \ldots \wedge dx_{i_p}) = (\Delta f) dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

shows that one obtains simply the old eigenvalues, each repeated $\binom{16}{p}$ times.

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According to Witt[2] there exist two self-dual lattices $L_1, L_2 \subset R^{16}$ which are distinct, in the sense that no rotation of R^{16} carries L_1 to L_2, such that each ball about the origin contains exactly as many points of L_1 as of L_2. It follows that the Riemannian manifolds R^{16}/L_1 and R^{16}/L_2 are not isometric, but do have the same sequence of eigenvalues.

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Bad news

EIGENVALUES OF THE LAPLACE OPERATOR ON CERTAIN MANIFOLDS

BY J. MILNOR

PRINCETON UNIVERSITY

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1. Eigenvalue rigidity and hearing the shape of a drum
   - Classical vs. Eigenvalue Rigidity
2. Hearing the Shape
   - 1-dimensional case
   - Flat tori of dimension > 1
   - Weyl's Law and its Consequences
3. Locally symmetric spaces
   - Laplace-Beltrami operator
   - Isospectral non-isometric manifolds
   - Our results

Let M be a Riemannian d-manifold.

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Laplace - Beltrami operator in local coordinates $x_1, \ldots, x_d$:

$$\Delta(u) = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(g^{ij} \sqrt{\det g} \frac{\partial u}{\partial x_j} \right), \qquad \text{(L)}$$

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Example.

For upper-half plane $\mathbb{H} = \{x + iy \mid y > 0\}$, $ds^2 = y^{-2}(dx^2 + dy^2)$, we have

$$\Delta_{\mathbb{H}} = y^{-2} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Properties of Δ

- Δ is 2nd order linear differential operator that commutes with isometries;

- Δ is self-adjoint and *negative* definite

- eigenvalues have finite multiplicities and form a discrete set of nonnegative numbers:

$$0 = \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \cdots$$

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However, eigenvalues are extremely hard to compute!

Selberg's 1/4 Conjecture (1965)

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\Gamma(m)$ be congruence subgroup mod m. Then all nonzero eigenvalues of Laplacian on $\mathbb{H}/\Gamma(m)$ are

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We will describe techniques to *bypass* explicit computations!

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BOTH constructions result in commensurable manifolds.

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Previously, results were available only for hyperbolic 2- and 3-manifolds. (A. Reid et al.)

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Now, let G_1 and G_2 be *absolutely almost simple* $\mathbb{R}$-groups,

$\Gamma_i \subset \mathcal{G}_i = G_i(\mathbb{R})$ be a discrete torsion-free subgroup,

$\mathfrak{X}_{\Gamma_i}$ - corresponding locally symmetric space, $i = 1, 2$.

Theorem 1 (Prasad-R.)

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(3) *If at least one of the groups* Γ_1 *or* Γ_2 *is* *arithmetic*, *then unless* G *is of type* A_n $(n > 1)$, D_{2n+1} $(n > 1)$ *or* E_6, *spaces* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are commensurable.*

Geometric applications

Corollary

Let M_1 and M_2 be arithmetically defined hyperbolic manifolds of dimension $d \not\equiv 1 \pmod 4$. If M_1 and M_2 are isospectral then they are commensurable.

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Our techniques apply to locally symmetric spaces that share a different set of geometric data, viz. *length spectrum*.

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Length spectrum $L(M)$ = set of length of all closed geodesics.

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It has *less* geometric content, **but** may be *easier* to figure out, and it is invariant under passing to a commensurable manifold.

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Most of our geometric results rely only on assumption that locally symmetric spaces are *length-commensurable*.

Length-commensurability is translated into *weak commensurability* of fundamental groups.