Towards the eigenvalue rigidity of Zariski-dense subgroups

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Abstract. We discuss the notion of weak commensurability of Zariski-dense subgroups of semi-simple algebraic groups over fields of characteristic zero, which enables one to match in a convenient way the eigenvalues of semi-simple elements of these subgroups. The analysis of weakly commensurable arithmetic groups has led to a resolution of some long-standing problems about isospectral locally symmetric spaces. This work has also initiated a number of questions in the theory of algebraic groups dealing with the characterization of absolutely almost simple simply connected algebraic groups having the same isomorphism classes of maximal tori over the field of definition. The recent results in this direction provide evidence to support a new conjectural form of rigidity for arbitrary Zariski-dense subgroups in absolutely almost simple algebraic groups over fields of characteristic zero based on the eigenvalue information ("eigenvalue rigidity").

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1. Introduction

The purpose of my talk is two-fold. First, I would like to report on the results obtained in a series of papers written in collaboration with G. Prasad and other co-authors. In these papers, we introduced the notion of *weak commensurability* of Zariski-dense subgroups of semi-simple algebraic groups, determined the consequences of the weak commensurability of two S-arithmetic subgroups of absolutely almost simple algebraic groups over a field of characteristic zero, and applied these results to the analysis of length-commensurable isospectral locally symmetric spaces. Second, I would like to outline a variety of problems and results in the theory of algebraic groups and related areas that this work has led to. These problems have to do with the understanding of finite-dimensional division algebras having the same maximal subfields, and more generally, with the characterization of absolutely almost simple algebraic groups having the same isomorphism classes of maximal tori over the field of definition. The results in this new direction obtained in the last several years point to a new version of the rigidity phenomenon, some aspects of which apply not only in the classical case of lattices but in fact to arbitrary Zariski-dense subgroups. Its distinctive feature is that it is formulated in terms of the eigenvalues of semi-simple elements of a given Zariski-dense subgroup, which led us to call it *eigenvalue rigidity*. Its investigation is very much a work in progress, so along with available results, we will discuss several conjectures. Overall, the possibility of having some form of rigidity for arbitrary Zariski-dense subgroups

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(which may well be free) looks quite exciting, and I would like to begin with a discussion of what kinds of results one can or cannot expect in this generality.

In the theory of algebraic/Lie groups, the term "rigidity" in a very general sense is used to describe a situation where, given a semi-simple algebraic group G over a field F, the structure of a "large" subgroup Γ of $G(F)$ determines the group G as well as the "location" of Γ inside $G(F)$. More concretely, when F is a non-discrete locally compact field, then under appropriate assumptions, any abstract isomorphism $\Gamma_1 \to \Gamma_2$ between two lattices $\Gamma_1, \Gamma_2 \subset G(F)$ extends to a rational automorphism of G (strong rigidity), or even any abstract representation $\Gamma \to \mathrm{GL}_n(F)$ (virtually) extends to a rational representation $G \to \mathrm{GL}_n$ (superrigidity). This implies, for example, that the entire geometry of a compact hyperbolic manifold of dimension $\geqslant 3$ (including its volume, the Laplace spectrum, the lengths of closed geodesics, etc.) is determined by the structure of its fundamental group. Among the algebraic consequences of structural rigidity, the following is most relevant for our discussion.

Let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$, where $n \geqslant 3$, and suppose we are given an absolutely almost simple simply connected algebraic group G over a number field K with ring of integers $\mathcal{O}$. If Γ is (virtually) isomorphic to $G(\mathcal{O})$ as an abstract group, then $K = \mathbb{Q}$ (and hence $\mathcal{O} = \mathbb{Z}$), and $G \simeq \mathrm{SL}_n$ as algebraic groups over $\mathbb{Q}$. Thus, the structure of a higher rank arithmetic group uniquely determines the *field of definition* and the *ambient group* as an algebraic group over this field. The results we will present suggest that one should be able to recover this data (in a somewhat weaker form) not just from a higher rank arithmetic group, but in fact from any finitely generated Zariski-dense subgroup if in place of structural information one uses information about the eigenvalues of elements, expressed in terms of *weak commensurability*. More precisely, we will see that in this set-up the field of definition can still be recovered uniquely (cf. Theorem 3.2), while the ambient algebraic group over this field is conjecturally determined up to finitely many possibilities (cf. Conjecture 6.1). The finiteness is known to hold when the field of definition is a number field, and is supported in the general case by, for example, results on division algebras having the same maximal subfields (cf. 6.5). Moreover, in many situations, S-arithmetic groups are unique (up to commensurability) in their weak commensurability class (cf. Theorem 6.3(1)), and thus are eigenvalue rigid in a strong sense. Just like structural rigidity, eigenvalue rigidity has geometric applications to isospectral locally symmetric spaces (see 2.2 and 4.4). There are other aspects of eigenvalue rigidity dealing with questions of whether various properties of Zariski-dense subgroups (such as discreteness, co-compactness, arithmeticity) can be characterized in terms of the eigenvalue information (see 4.3), but here we will focus almost exclusively on the question of to what extent the latter determines the ambient algebraic group. As we already mentioned, it is precisely shifting the focus from the structure to eigenvalues that makes results of this kind possible for arbitrary Zariski-dense subgroups.

Before discussing the results, we need to explain how we match the eigenvalues of elements of two Zariski-dense subgroups, and on the other hand, why we care about these eigenvalues.

2. Weak commensurability

The following definition, introduced in [40], provides a way of matching the eigenvalues of matrices of different sizes.

Definition 2.1. Let F be an infinite field.

(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple matrices*, and let

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2}$$

be their eigenvalues (in a fixed algebraic closure $\overline{F}$). We say that γ_1 and γ_2 are *weakly commensurable* if there exist $a_1, \ldots, a_{n_1}$, $b_1, \ldots, b_{n_2} \in \mathbb{Z}$ such that

$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} \;=\; \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \;\neq\; 1.$$

(2) Let $G_1 \subset \mathrm{GL}_{n_1}$ and $G_2 \subset \mathrm{GL}_{n_2}$ be reductive algebraic groups defined over F. Two *Zariski-dense subgroups* $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ are called *weakly commensurable* if every semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.

It should be noted that the definition of weak commensurability can be stated in several different ways. First, in the above notations, semi-simple elements $\gamma_1 \in G_1(F)$ and $\gamma_2 \in G_2(F)$ are weakly commensurable if and only if there exist maximal F-tori T_i of G_i for $i = 1, 2$ such that $\gamma_i \in T_i(F)$ and for some characters $\chi_i \in X(T_i)$ (defined over $\overline{F}$) we have

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

This reformulation shows that the notion of weak commensurability (of γ_1 and γ_2) does not depend on the choice of matrix realizations of G_1 and G_2, and is also more convenient for technical arguments.

Second, semi-simple elements $\gamma_1 \in G_1(F)$ and $\gamma_2 \in G_2(F)$ are weakly commensurable if and only if there exist F-rational representations

$$\rho_1 \colon G_1 \longrightarrow \mathrm{GL}_{m_1} \text{ and } \rho_2 \colon G_2 \longrightarrow \mathrm{GL}_{m_2}$$

such that $\rho_1(\gamma_1)$ and $\rho_2(\gamma_2)$ have a *nontrivial common eigenvalue* (these representations can vary from one element to another).

Informally speaking, weak commensurability appears to be a rather natural way (and perhaps even the only natural way) of matching the eigenvalues of (semi-simple) elements of two algebraic groups that does not depend on the choice of their matrix realizations. On the other hand, it is easy to construct examples of very different (certainly non-conjugate) matrices that are weakly commensurable, so one needs to discuss the utility of this notion. As we will see later, while being inconsequential for individual matrices and "small" (e.g., cyclic) subgroups, weak commensurability has remarkably strong consequences for "large" subgroups (viz., Zariski-dense and particularly S-arithmetic subgroups). Now, however, we would like to point out that the main motivation for the notion of weak commensurability in our work came from the famous problem in differential geometry about isospectral Riemannian manifolds best known as M.Kac's [30] question *Can one hear the shape of a drum?*

2.2. Geometric motivation. Let M be a Riemannian manifold. In differential geometry one considers the following sets of data associated with M:

- $\mathcal{E}(M)$ - spectrum of the Beltrami-Laplace operator;

- $L(M)$ - (weak) length spectrum, i.e. the collection of lengths of all closed geodesics in M.

Then one asks whether two Riemannian manifolds M_1 and M_2 are necessarily isometric if

(1) $\mathcal{E}(M_1) = \mathcal{E}(M_2)$ (i.e., M_1 and M_2 are *isospectral*);

(2) $L(M_1) = L(M_2)$ (i.e., M_1 and M_2 are *iso-length spectral*)?

When asking a question of this kind, one of course needs to specialize the class of manifolds being considered, and in our work we focused on locally symmetric spaces of semi-simple groups having nonpositive curvature (recall that these are endowed with the standard Riemannian structure coming from the Killing form); this class includes such geometrically important spaces as hyperbolic manifolds and, in particular, Riemann surfaces. It is important to point out that for *compact* locally symmetric spaces, questions (1) and (2) are *related*, viz.

$$\mathcal{E}(M_1) \ = \ \mathcal{E}(M_2) \ \ \Rightarrow \ \ L(M_1) \ = \ L(M_2), \tag{S}$$

but *both* have a negative answer. Counter-examples for (arithmetically defined) Riemann surfaces were given by Vigneras [53], and then a more general group-theoretic construction was offered by Sunada [50]. Both constructions always produce pairs of *commensurable* locally symmetric spaces. We recall that Riemannian manifolds M_1 and M_2 are called *commensurable* if they admit a common finite-sheeted cover M, i.e. if there is a diagram:

$$\begin{tikzcd} & M \arrow[dl, "\theta_1"'] \arrow[dr, "\theta_2"] & \\ M_1 & & M_2 \end{tikzcd}$$

in which M is a Riemannian manifold and θ_1, θ_2 are finite-sheeted locally isometric covering maps. This suggests that one should probably settle for a weaker version of the question, viz. whether M_1 and M_2 are necessarily *commensurable* given the fact that they are isospectral or iso-length-spectral. While this modified question still has a negative answer in the general case [35], our work, based on the analysis of weakly commensurable groups, shows that the answer is in the affirmative for many (arithmetically defined) locally symmetric spaces - cf. Theorem 4.5 (previously such results were available only for arithmetically defined Riemann surfaces [47] and hyperbolic 3-manifolds [12]). In fact, our results give the commensurability of pairs of locally symmetric spaces that satisfy the following condition:

(3) $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

This condition, called *length commensurability*, is conceivably much weaker than conditions (1) and (2), but surprisingly in most situations it has many of the same consequences. Its real advantage over (1) and (2) is that it is invariant under passing to commensurable manifolds.

The main point here is that the length-commensurability of finite volume locally symmetric spaces implies the weak commensurability of their fundamental groups. To give a precise statement, we need to fix some notations. Let G be an absolutely simple adjoint real algebraic group, let $\mathcal{G} = G(\mathbb{R})$ be the group of $\mathbb{R}$-points, considered as a real Lie group, and let $\mathfrak{X} = \mathcal{K} \backslash \mathcal{G}$, where $\mathcal{K}$ is a maximal compact subgroup of $\mathcal{G}$, be the associated symmetric space endowed with the Riemannian metric coming from the Killing form on the Lie algebra of $\mathcal{G}$. Furthermore, given a torsion-free discrete subgroup Γ of $\mathcal{G}$, we let $\mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$ denote the corresponding locally symmetric space; we say that $\mathfrak{X}_\Gamma$ is *arithmetically defined* if the subgroup Γ is *arithmetic*[1]. Finally, given two simple real algebraic groups G_i $(i = 1, 2)$, we will denote the symmetric spaces of the groups $\mathcal{G}_i = G_i(\mathbb{R})$ by $\mathfrak{X}_i$, and the locally symmetric spaces obtained as quotients by torsion-free discrete subgroups Γ_i of $\mathcal{G}_i$ by $\mathfrak{X}_{\Gamma_i}$.

Theorem 2.3 ([43], Corollary 2.8). *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be two locally symmetric spaces having finite volume, of absolutely simple real algebraic groups G_1 and G_2. If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, then Γ_1 and Γ_2 are weakly commensurable.*

While this result is straightforward for Riemann surfaces (see [43, 2.1]), its proof in the general case relies on the formula for the length of a closed geodesic c_γ in $\mathfrak{X}_\Gamma$ corresponding to a nontrivial semi-simple element $\gamma \in \Gamma$ as a function of the logarithms of eigenvalues of γ in the adjoint representation - see [40, Proposition 8.5(ii)] (note that this formula also explains why we care about the eigenvalues of semi-simple elements of discrete subgroups). So, to prove the weak commensurability of Γ_1 and Γ_2, we need to sort out the logarithms appearing in this formula, which requires transcendental number theory. More precisely, for rank one locally symmetric spaces of dimension > 2, we use the famous result of Gel'fond and Schneider that settled Hilbert's seventh problem - cf. [4]. In all other cases, our argument assumes the truth of Schanuel's conjecture (cf. [3]). This means that while all of our results on weak commensurability are, of course, unconditional, their geometric consequences are *conditional* (at least for locally symmetric spaces of rank > 1).

Since the locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable if and only if the subgroups Γ_1 and Γ_2 are commensurable as groups up to an isomorphism between G_1 and G_2 (see 3.4 below for the details of this notion), we see that in order to prove the commensurability of length-commensurable (in particular, isospectral or iso-length spectral) locally symmetric space, we need to answer the following question:

(C) *When does the weak commensurability of Γ_1 and Γ_2 imply their commensurability?*

3. First signs of eigenvalue rigidity

Before providing a rather definitive answer to Question (C) for S-arithmetic subgroups (see §4), we would like to present a few results demonstrating that weak commensurability captures some important characteristics in the case of arbitrary Zariski-dense subgroups. So, let G_1 and G_2 be absolutely almost simple algebraic groups over a field F of *characteristic zero*, and let $\Gamma_i \subset G_i(F)$ be a *finitely generated* Zariski-dense subgroup for $i = 1, 2$.

Theorem 3.1 ([40], Theorem 1). *If Γ_1 and Γ_2 are weakly commensurable, then either G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_ℓ and the other of type C_ℓ for some $\ell \geqslant 3$.*

This result is already interesting because, in principle, Γ_1 and Γ_2 may very well be free groups, hence carry no structural information about the ambient algebraic groups. Note that what we really prove is that G_1 and G_2 have the same order of the Weyl group - it

[1]We recall that combining the celebrated results of Margulis [36] on the arithmeticity of higher rank irreducible lattices and of Corlette [15] and Gromov-Shoen [27], one obtains that a finite volume locally symmetric space $\mathfrak{X}_\Gamma$ of a simple real algebraic group is automatically arithmetically defined unless $\mathfrak{X}$ is either the real hyperbolic space $\mathbb{H}^n$ or the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$.

is known that this number uniquely determines the Killing-Cartan type of the group except for the ambiguity involving types B_ℓ and C_ℓ. As shown by Theorem 4.2 below, Zariski-dense, and even S-arthmetic, subgroups in groups of types B_ℓ and C_ℓ can indeed be weakly commensurable.

Now, given a Zariski-dense subgroup $\Gamma \subset G(F)$, where G is a semi-simple F-group, we let K_Γ denote the *trace field* of Γ, i.e. the subfield of F generated by the traces $\mathrm{tr}(\mathrm{Ad}\,\gamma)$ of all elements $\gamma \in \Gamma$ in the adjoint representation on the corresponding Lie algebra $\mathfrak{g} = L(G)$. By a result of Vinberg [51], the field $K = K_\Gamma$ is the minimal field of definition of $\mathrm{Ad}\ \Gamma$. This means that K is the minimal subfield of F such that all transformations in $\mathrm{Ad}\ \Gamma$ can be simultaneously represented by matrices over K in a suitable basis of $\mathfrak{g}$. If such a basis is chosen, then the Zariski closure of $\mathrm{Ad}\ \Gamma$ in $\mathrm{GL}(\mathfrak{g})$ is a semi-simple algebraic K-group $\mathcal{G}$. It is an F/K-form of the adjoint group $\overline{G}$, and we will call it the *algebraic hull* of $\mathrm{Ad}\ \Gamma$.

Theorem 3.2 ([40], Theorem 2). *Keep the notations and conventions introduced prior to Theorem 3.1. If Γ_1 and Γ_2 are weakly commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.*

Now let K be the common trace field of two weakly commensurable Zariski-dense subgroups Γ_1 and Γ_2 as above, and let $\mathcal{G}_i$ be the algebraic hull of $\mathrm{Ad}\ \Gamma_i$ for $i = 1, 2$. We denote by L_i the minimal Galois extension of K over which $\mathcal{G}_i$ becomes an inner form of a split group.

Proposition 3.3 (cf. [44], Lemma 5.2). *In the above notations, $L_1 = L_2$.*

(We would like to mention the following useful consequence of this proposition: *Let G_1 and G_2 be absolutely almost simple groups over a field F of characteristic zero, and let E_i be the minimal Galois extension of F over which G_i becomes an inner form. If there exist finitely generated Zariski-dense subgroups $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ that are weakly commensurable, then $E_1 = E_2$.* Indeed, $E_i = FL_i$ in the above notations.)

3.4. *S*-arithmetic subgroups. We will now specialize to the case of S-arithmetic subgroups. We recall that if G is an absolutely almost simple algebraic group over a field F of characteristic zero, then Zariski-dense S-arithmetic subgroups of $G(F)$ can be described in terms of triples $(\mathcal{G}, K, S)$, where K is a number field contained in F, $\mathcal{G}$ is a F/K-form of the adjoint group $\overline{G}$, and S is a finite set of places of K containing all archimedean ones; the subgroups corresponding to such triples are called $(\mathcal{G}, K, S)$*-arithmetic*. We refer to [40, §1] and [43, 3.3] for the details of this description, and only indicate here that for a $(\mathcal{G}, K, S)$-arithmetic Zariski-dense subgroup Γ, the field K coincides with the trace field K_Γ, and the group $\mathcal{G}$ with the algebraic hull of $\mathrm{Ad}\ \Gamma$.

Furthermore, given two absolutely almost simple F-groups G_1 and G_2, we say that the subgroups $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ are *commensurable up to an F-isomorphism* between the adjoint groups $\overline{G}_1$ and $\overline{G}_2$ if there exists an F-isomorphism $\sigma \colon \overline{G}_1 \to \overline{G}_2$ such that the subgroups $\sigma(\pi_1(\Gamma_1))$ and $\pi_2(\Gamma_2)$ are commensurable as subgroups of $\overline{G}_2(F)$ in the usual sense (i.e., their intersection is of finite index in each of them), where $\pi_i \colon G_i \to \overline{G}_i$ is the canonical isogeny for $i = 1, 2$. (This notion of commensurability is precisely what we need for geometric applications, cf. §2.)

The following result shows that the description of S-arithmetic subgroups of absolutely almost simple groups in terms of triples is very convenient in the analysis of their commensurability.

Theorem 3.5. *Let G_1 and G_2 be absolutely almost simple algebraic groups defined over a field F of characteristic zero, and for $i = 1, 2$, let Γ_i be a Zariski-dense $(\mathcal{G}_i, K_i, S_i)$-arithmetic subgroup of $G_i(F)$. Then*

(1) *Γ_1 and Γ_2 are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if and only if $K_1 = K_2$, $S_1 = S_2$, and $\mathcal{G}_1$ and $\mathcal{G}_2$ are K-isomorphic;*

(2) *if Γ_1 and Γ_2 are weakly commensurable, then $K_1 = K_2$ and $S_1 = S_2$.*

Thus, the study of the commensurability classes of weakly commensurable Zariski-dense S-arithmetic subgroups *is equivalent* to the study of K-forms $\mathcal{G}$ involved in their description. This leads to a complete resolution of question (C) for S-arithmetic subgroups that we will present in the next section.

4. Results for S-arithmetic groups and geometric consequences

The following two theorems summarize the main results dealing with the weak commensurability of S-arithmetic subgroups.

Theorem 4.1 (cf. Prasad-Rapinchuk [40, 43]). *Let G_1 and G_2 be absolutely almost simple algebraic groups over a field F of characteristic zero, and let $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ be Zarsiki-dense S-arithmetic subgroups.*

(1) *Assume that G_1 and G_2 are of the same Killing-Cartan type, which is different from A_n, D_{2n+1} $(n > 1)$, and E_6. If Γ_1 and Γ_2 are weakly commensurable, then they are commensurable.*

(2) *In all cases, S-arithmetic subgroups $\Gamma_2 \subset G_2(F)$ weakly commensurable to a given S-arithmetic subgroup $\Gamma_1 \subset G_1(F)$ form* finitely many *commensurability classes.*

(3) *If Γ_1 and Γ_2 as above are weakly commensurable, then Γ_1 contains unipotent elements if and only if Γ_2 does.*

(4) (arithmeticity theorem) *Let now F be a locally compact field of characteristic zero, and $\Gamma_1 \subset G_2(F)$ be an S-arithmetic lattice. If $\Gamma_2 \subset G_2(F)$ is a lattice weakly commensurable to Γ_1, then Γ_2 is also S-arithmetic.*

(In this theorem, "commensurability" means "commensurability up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$" as defined in 3.4.)

An interesting feature of this theorem is that for groups of type D_n, the answer to Question (C) is different depending on whether n is even or odd. Assertion (1) for type D_{2n} with $n > 2$ was originally proved in [41]. The case of type D_4 (including triality forms) was handled by Garibaldi [21] by a different method which applies to all groups of type D_{2n}. On the other hand, for each of the exceptional types A_n $(n > 1)$, D_{2n+1} $(n > 1)$, and E_6 one can construct weakly commensurable, but not commensurable, Zariski-dense S-arithmetic subgroups (see [40, §9])[2].

According to Theorem 3.1, to complete the investigation of weak commensurability for S-arithmetic subgroups, it remains to consider the case where one group is of type B_ℓ and the other of type C_ℓ for some $\ell \geqslant 3$.

[2]Note that these are precisely the types for which the multiplication by (-1) considered as an automorphism of the corresponding root system is *not* in the Weyl group.

Theorem 4.2 (Garibaldi-Rapinchuk [22]). *Let G_1 and G_2 be absolutely almost simple algebraic groups over a field F of characteristic zero having Killing-Cartan types B_ℓ and C_ℓ $(\ell \geqslant 3)$, respectively, and let Γ_i be a Zariski-dense $(\mathcal{G}_i, K, S)$-arithmetic subgroup of $G_i(F)$. Then Γ_1 and Γ_2 are weakly commensurable if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ are* twins*, i.e.*

- *$\mathcal{G}_1$ and $\mathcal{G}_2$ are both split over all nonarchimedean places of K;*
- *$\mathcal{G}_1$ and $\mathcal{G}_2$ are* simultaneously *either split or anisotropic over all archimedean valuations of K.*

In §5, we will review some of the techniques involved in the proof of Theorems 4.1 and 4.2. But from a very general perspective, the essence of the argument is to obtain information about the algebraic hull $\mathcal{G}$ of an S-arithmetic group Γ that is weakly commensurable to a given S-arithmetic group – recall that according to Theorem 3.5, $\mathcal{G}$ uniquely determines Γ up to commensurability. So, to establish assertion (1) of Theorem 4.1, we prove that the algebraic hull $\mathcal{G}$ is itself unique when the type if different from A_n, D_{2n+1} $(n > 1)$, and E_6. Furthermore, for assertion (2), we prove that there are only finitely many possibilities for the $\mathcal{G}$'s. In §§6-7 we will indicate that the latter property is expected to hold not only for S-arithmetic, but in fact for arbitrary finitely generated Zariski-dense subgroups (see Conjecture 6.2). This phenomenon, if confirmed, would be a rather strong form of eigenvalue rigidity. We will now, however, briefly discuss a few other questions that one can ask in the context of weak commensurability.

4.3. Some other aspects of eigenvalue rigidity. First, it is easy to construct examples showing that a Zariski-dense subgroup weakly commensurable to a *rank-one* arithmetic subgroup need not be arithmetic (see [40, Remark 5.5]); in other words, assertion (4) of Theorem 4.1 fails if we drop the assumption that Γ_2 is a lattice. So, it would be extremely interesting to determine if a Zariski-dense subgroup weakly commensurable to a *higher rank* S-arithmetic subgroup is itself S-arithmetic (see Problem 10.1 in [43] and the subsequent discussion). This can potentially provide a new characterization of higher rank S-arithmetic subgroups involving weak commensurability (i.e., ultimately the eigenvalue information).

Second, one can ask if weak commensurability can be used to characterize the discreteness of Zariski-dense subgroups. More precisely, let G_1 and G_2 be connected absolutely almost simple algebraic groups over a nondiscrete locally compact field F, and let Γ_i be a finitely generated Zariski-dense subgroup of $G_i(F)$ for $i = 1, 2$. Assume that Γ_1 and Γ_2 are weakly commensurable. *Does the discreteness of Γ_1 imply the discreteness of Γ_2?* (Problem 10.2 in [43]). An affirmative answer to this question was given in [40, Proposition 5.6] for the case where F is a *nonarchimedean* local field, but the case $F = \mathbb{R}$ or $\mathbb{C}$ remains open.

Third, one can also ask if weak commensurability preserves cocompactness of lattices. Namely, let again G_1 and G_2 be connected absolutely almost simple algebraic groups over $F = \mathbb{R}$ or $\mathbb{C}$, and let $\Gamma_i \subset G_i(F)$ be a lattice for $i = 1, 2$. Assume that Γ_1 and Γ_2 are weakly commensurable. *Does the compactness of $G_1(F)/\Gamma_1$ imply the compactness of $G_2(F)/\Gamma_2$?* (Problem 10.3 in [43]). (We note that if G is a semi-simple algebraic group over a nonarchimedean local field F of characteristic zero, then any lattice $\Gamma \subset G(F)$ is automatically cocompact, and the problem in this case becomes vacuous.) We recall that the cocompactness of a lattice in a semi-simple real Lie group is equivalent to the absence of nontrivial unipotent elements in it, see [45, Corollary 11.13]. So, the above question is equivalent to the question of whether for two weakly commensurable *lattices*, the existence of nontrivial unipotent elements in one of them implies their existence in the other. The combination of parts (3) and (4) of Theorem 4.1 provides an affirmative answer if one of the lattices is arithmetic. On the other hand, in this form the question itself is meaningful for arbitrary Zariski-dense subgroups (not necessarily discrete or of finite covolume), but no other cases have been considered so far.

4.4. Geometric applications. Combining Theorem 2.3, which reduced the length - commensurability of locally symmetric spaces to the weak commensurability of their fundamental groups, with the analysis of weak commensurability in Theorem 4.1, we obtain the following geometric result.

Theorem 4.5 ([40], Theorem 8.16). *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and set $\mathcal{G}_i = G_i(\mathbb{R})$, for $i = 1, 2$. Then the set of arithmetically defined locally symmetric spaces $\mathfrak{X}_{\Gamma_2}$ of $\mathcal{G}_2$, which are length-commensurable to a given arithmetically defined locally symmetric space $\mathfrak{X}_{\Gamma_1}$ of $\mathcal{G}_1$, is a union of finitely many commensurability classes. It in fact consists of a single commensurability class if G_1 and G_2 have the same type different from A_n, D_{2n+1}, with $n > 1$, or E_6.*

This statement applies in various concrete geometric situations. For example, here is what it yields for hyperbolic manifolds.

Corollary 4.6. *Let M_1 and M_2 be arithmetically defined real hyperbolic d-manifolds where d is either even or is $\equiv 3 (\mathrm{mod}\, 4)$ and $d > 3$. If M_1 and M_2 are length-commensurable (in particular, compact and isospectral), then they are commensurable.*

Previously, this was known only for $d = 2$ [47] and $d = 3$ [12]. Length-commensurability implies commensurability also for all quaternionic hyperbolic manifolds. On the other hand, in the case of real hyperbolic manifolds of dimension $\equiv 1 (\mathrm{mod}\, 4)$ or of complex hyperbolic manifolds, one can construct examples of arithmetically defined length-commensurable, but not commensurable spaces. Furthermore, using Theorem 3.1 (and Proposition 3.3 to handle the isomorphism $\mathsf{A}_3 = \mathsf{D}_3$), one proves that an arithmetically defined complex hyperbolic space cannot be length-commensurable to either a real or a quaternionic arithmetically defined hyperbolic space. Employing Theorem 4.2, one also proves that arithmetically defined real and quaternionic hyperbolic spaces cannot be length-commensurable. (In fact, assuming Schanuel's conjecture in all cases, one can get rid of the arithmeticity assumption in these two statement, see [42], particularly Remark 8.5, and the discussion after Theorem 4.8 below.)

Next, parts (3) and (4) of Theorem 4.1, in conjunction with Theorem 2.3, imply the following rather surprising result which has so far defied all attempts to find a purely geometric proof.

Theorem 4.7 ([40], Theorem 8.19). *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be locally symmetric spaces of finite volume. Assume that one of the spaces is arithmetically defined. If the spaces are length-commensurable, then the other space is also arithmetically defined, and the compactness of one of the spaces implies the compactness of the other.*

In fact, if one of the spaces is compact and the other is not, the length spectra $L(\mathfrak{X}_{\Gamma_1})$ and $L(\mathfrak{X}_{\Gamma_2})$ are quite different - see [42, Theorem 5.9]. The question of whether the arithmeticity assumption in this theorem can be dropped boils down to one of the problems we discussed in 4.3.

Last but not least, implication (S) from 2.2 enables us to apply the above results to isospectral compact locally symmetric spaces. We then obtain the following.

Theorem 4.8 ([40], §10). *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be compact locally symmetric spaces, and assume that they are isospectral.*

(1) *If $\mathfrak{X}_{\Gamma_1}$ is arithmetically defined, then $\mathfrak{X}_{\Gamma_2}$ is also arithmetically defined.*
(2) *$G_1 = G_2 =: G$, hence $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ have the same universal cover.*
(3) *If at least one of the subgroups Γ_1 and Γ_2 is arithmetic, then unless G is of type A_n $(n > 1)$, D_{2n+1} $(n > 1)$ and E_6, the spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ commensurable.*

We note that part (2) was proved in [40] (with the help of a result of Sai-Kee Yeung [55]) under the assumption that at least one of the groups Γ_1 or Γ_2 is arithmetic. Suppose now that both Γ_1 and Γ_2 are nonarithmetic. Then each space $\mathfrak{X}_i$ $(i = 1, 2)$ is either the real hyperbolic space $\mathbb{H}^{n_i}$ or the complex hyperbolic space $\mathbb{H}^{n_i}_{\mathbb{C}}$ for some $n_i \geqslant 2$, and the corresponding real adjoint algebraic group G_i is, respectively, either $\mathrm{PSO}(n_i, 1)$ or $\mathrm{PSU}(n_i, 1)$ in the standard notations. It follows from Theorem 3.1 that the isospectrality, hence length-commensurability, of $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ implies that either G_1 and G_2 must be of the same Cartan-Killing type, or one of them is of type B_ℓ and the other of type C_ℓ for some $\ell \geqslant 3$. In our situation, this can happen only if either $G_1 = G_2$ or (after a possible switch) $G_1 = \mathrm{PSO}(5, 1)$ and $G_2 = \mathrm{PSU}(3, 1)$ (of common type $\mathsf{D}_3 = \mathsf{A}_3$). In the latter case, $\mathfrak{X}_{\Gamma_1}$ is 5-dimensional, and $\mathfrak{X}_{\Gamma_2}$ is 6-dimensional. But according to Weyl's Law (see, for example, [24]) isospectral Riemannian manifolds are always of the same dimension. So, in this case $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ cannot be isospectral[3], leaving us with the only option $G_1 = G_2$, as required.

5. Generic elements. Isogeny Theorem

In this section, we would like to discuss two ingredients involved in the proofs of Theorems 4.1 and 4.2: the existence of generic elements in Zariski-dense subgroups and the Isogeny Theorem.

5.1. Generic elements. First, we need to recall the notion of a *generic K-torus*. Let G be a connected semi-simple algebraic group defined over an infinite field K. Fix a maximal K-torus T of G, and, as usual, let $\Phi = \Phi(G, T)$ denote the corresponding root system, and let $W(G, T)$ be its Weyl group. Furthermore, we let K_T denote the (minimal) splitting field of T in a fixed algebraic closure $\overline{K}$ of K. Then the natural action of the Galois group $\mathrm{Gal}(K_T/K)$ on the character group $X(T)$ of T induces an injective homomorphism

$$\theta_T \colon \mathrm{Gal}(K_T/K) \to \mathrm{Aut}(\Phi(G, T)).$$

We say that T is *generic* (over K) if

$$\theta_T(\mathrm{Gal}(K_T/K)) \supset W(G, T). \tag{5.1}$$

For example, any maximal K-torus of $G = \mathrm{SL}_{n,K}$ is of the form $T = \mathrm{R}^{(1)}_{E/K}(\mathbb{G}_{m,E})$ for some n-dimensional commutative étale K-algebra E. Then such a torus is generic over K if and only if E is a separable field extension of K and the Galois group of the normal closure L of E over K is isomorphic to the symmetric group S_n.

[3]In fact, it follows from the remark made after Proposition 3.3 that in this case $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ cannot even be length-commensurable as G_1 is an inner form of a split group over $\mathbb{R}$, and G_2 is an outer form.

Definition 5.2. Let G be a connected semi-simple algebraic group defined over a field K. A regular semi-simple element $g \in G(K)$ is called *generic* (over K) if the torus $T = Z_G(g)^\circ$ is generic (over K) as defined above (note that T is a K-torus, cf. [7, 9.1]).

Generic elements play a crucial role in our work, but they have also been used in a variety of other problems, including the study of the rigidity of actions (cf. [32, 37]) and the Auslander problem [1].

Theorem 5.3 (cf. [39], Theorem 3). *Let G be a connected absolutely almost simple algebraic group over a finitely generated field K of characteristic zero, and let $\Gamma \subset G(K)$ be a Zariski-dense subgroup. Then Γ contains a regular generic element (over K) of infinite order.*

Basically, our proof (which in fact applies to all connected semi-simple groups) shows that given a finitely generated Zariski-dense subgroup $\Gamma \subset G(K)$, one can produce a finite system of congruences (defined in terms of suitable valuations of K) such that the set of elements $\gamma \in \Gamma$ satisfying this system of congruences consists entirely of generic elements (and additionally this set is in fact a coset of a finite index subgroup in Γ, in particular, it is Zariski-dense in G). Recently, Gorodnik-Nevo [26], Jouve-Kowalski-Zywina [29], and Lubotzky-Rosenzweig [34] have developed different *quantitative* ways of showing that generic elements exist in abundance (in fact, these results demonstrate that "most" elements in Γ are generic). More precisely, the result of [26] gives the asymptotics of the number of generic elements of a given height in an arithmetic group, while the results of [34], generalizing earlier results of [29], are formulated in terms of random walks on groups and apply to arbitrary Zariski-dense subgroups in not necessarily connected semi-simple groups. These papers introduce several new ideas and techniques, but at the same time employ some elements of the argument from [39]. We also note that the proof of Theorems 4.1 and 4.2 uses not only Theorem 5.3 itself but also its different variants that provide generic elements with additional properties, e.g. having prescribed local behavior.

5.4. The Isogeny Theorem and its consequences. An important step in the proofs of Theorems 4.1 and 4.2 is the passage from the weak commensurability of two semi-simple elements to an isogeny, and in most cases even to an isomorphism, of the tori containing these elements. This is done with the help of the following technical statement which we called the *Isogeny Theorem*. After the theorem, we give a (less technical) corollary that, to a significant degree, reduces the analysis of weak commensurability to the investigation of absolutely almost simple algebraic groups having the same isomorphism/isogeny classes of maximal tori over the base field; this problem, together with some variations, will be discussed in the concluding §§6-7. We recall that a K-torus T is called (K-)*irreducible* if it does not contain any proper K-subtori; note that a maximal K-torus of an absolutely almost simple algebraic K-group which is generic over K, is automatically K-irreducible.

Theorem 5.5 ([40], Theorem 4.2). *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over an infinite field K, and let L_i be the minimal Galois extension of K over which G_i becomes an inner form of a split group. Suppose that for $i = 1, 2$, we are given a semi-simple element $\gamma_i \in G_i(K)$ contained in a maximal K-torus T_i of G_i. Assume that*

(i) *G_1 and G_2 are either of the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n;*

(ii) *γ_1 has infinite order;*

(iii) *T_1 is K-irreducible; and*

(iv) *γ_1 and γ_2 are weakly commensurable.*

Then:

(1) *there exists a K-isogeny $\pi\colon T_2 \to T_1$ which carries $\gamma_2^{m_2}$ to $\gamma_1^{m_1}$ for some integers $m_1, m_2 \geqslant 1$;*

(2) *if $L_1 = L_2 =: L$ and $\theta_{T_1}(\mathrm{Gal}(L_{T_1}/L)) \supset W(G_1, T_1)$, then*

$$\pi^*\colon X(T_1) \otimes_{\mathbb{Z}} \mathbb{Q} \to X(T_2) \otimes_{\mathbb{Z}} \mathbb{Q}$$

has the property that $\pi^*(\mathbb{Q} \cdot \Phi(G_1, T_1)) = \mathbb{Q} \cdot \Phi(G_2, T_2)$. Moreover, if G_1 and G_2 are of the same Killing-Cartan type different from $\mathsf{B}_2 = \mathsf{C}_2$, F_4 or G_2, then a suitable rational multiple of π^* maps $\Phi(G_1, T_1)$ onto $\Phi(G_2, T_2)$, and if G_1 is of type B_n and G_2 is of type C_n, with $n > 2$, then a suitable rational multiple λ of π^* takes the long roots in $\Phi(G_1, T_1)$ to the short roots in $\Phi(G_2, T_2)$, while 2λ takes the short roots in $\Phi(G_1, T_1)$ to the long roots in $\Phi(G_2, T_2)$.

It follows that in the situations where π^* can be, and has been, scaled so that $\pi^*(\Phi(G_1, T_1)) = \Phi(G_2, T_2)$, it induces K-isomorphisms $\tilde{\pi}\colon \widetilde{T}_2 \to \widetilde{T}_1$ and $\overline{\pi}\colon \overline{T}_2 \to \overline{T}_1$ between the corresponding tori in the simply connected and adjoint groups $\widetilde{G}_i$ and $\overline{G}_i$, respectively, that extend to $\overline{K}$-isomorphisms $\widetilde{G}_2 \to \widetilde{G}_1$ and $\overline{G}_2 \to \overline{G}_1$.

Furthermore, if G_1 and G_2 are absolutely almost simple algebraic groups over a field K of characteristic zero and $\Gamma_1 \subset G_1(K)$ and $\Gamma_2 \subset G_2(K)$ are weakly commensurable finitely generated Zariski-dense subgroups, then we already know that either G_1 and G_2 have the same type or one of them is of type B_ℓ and the other of type C_ℓ for some $\ell \geqslant 3$ (Theorem 3.1), and $L_1 = L_2$ (remark after Proposition 3.3). Thus, the important assumptions in Theorem 5.5 are satisfied automatically, and its application yields the following.

Corollary 5.6. *In the above situation, every generic maximal K-torus T_1 of G_1 whose intersection with Γ_1 contains an element of infinite order, is K-isogenous, and if both G_1 and G_2 are either simply connected or adjoint of the same type different from $\mathsf{B}_2 = \mathsf{C}_2$, F_4 and G_2, even K-isomorphic, to a generic maximal K-torus T_2 of G_2 whose intersection with Γ_2 contains an element of infinite order, and vice versa.*

If finitely generated Zariski-dense subgroups $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ are weakly commensurable, then by Theorem 3.2, they have a common trace field $K_{\Gamma_1} = K_{\Gamma_2} =: K$, which is finitely generated. Then Theorem 5.3 and its variants guarantee the existence in Γ_1 and Γ_2 of elements that are generic over K and its suitable finite extensions, and satisfy some additional conditions. Applying Theorem 5.5 and/or Corollary 5.6, we obtain that the algebraic hulls $\mathcal{G}_1$ and $\mathcal{G}_2$ of Γ_1 and Γ_2, respectively, share large families of maximal K-tori. In the case where Γ_1 and Γ_2 are S-arithmetic, this information about the common maximal tori turns out to be sufficient to prove Theorems 4.1 and 4.2. In the final two sections we will discuss the implementation of this approach for arbitrary Zariski-dense subgroups.

6. Arbitrary Zariski-dense subgroups

As we already explained, the results for arithmetic groups were obtained by analyzing the algebraic hulls of arithmetic groups which are weakly commensurable to a given one. While general Zariski-dense subgroups are not determined up to commensurability by their algebraic hull (even if they are lattices, cf. [52]), the latter remains an important invariant. At the same time, the results in the arithmetic case as well as some very recent results over general fields concerning simple algebraic groups with the same maximal tori and division algebras with the same maximal subfields, which we will discuss in the rest of this article, have led us to believe that the algebraic hull itself is *almost* determined by the presence of a Zariski-dense subgroup weakly commensurable to a given one in all situations. More precisely, we would like to propose the following *Finiteness Conjecture*.

Conjecture 6.1. *Let G_1 and G_2 be absolutely simple (adjoint) algebraic groups over a field F of characteristic zero, and let $\Gamma_1 \subset G_1(F)$ be a finitely generated Zariski-dense subgroup with trace field $K_{\Gamma_1} = K$. Then there exists a* finite *collection $\mathcal{G}_1^{(2)}, \ldots, \mathcal{G}_r^{(2)}$ of F/K-forms of G_2 such that if $\Gamma_2 \subset G_2(F)$ is a finitely generated Zariski-dense subgroup that is weakly commensurable to Γ_1, then it is conjugate to a subgroup of one of the $\mathcal{G}_i^{(2)}(K)$'s ($\subset G_2(F)$).*

We already know that two weakly commensurable finitely generated Zariski-dense subgroups have the same trace field (Theorem 3.2). The above conjecture takes this result much farther by claiming that a finitely generated Zariski-dense subgroup weakly commensurable to a given one can exist only in finitely many simple algebraic groups over this field. (For example, if $G_0 = \mathrm{SO}_n(q_0)$, where q_0 is a nondegenerate quadratic form of dimension $n \geqslant 3$, $n \neq 4$, over a finitely generated field K of characteristic zero, and $\Gamma_0 \subset G_0(K)$ is a finitely generated Zariski-dense subgroup with trace field K, then according to the conjecture, there should exists a *finite* collection $q_1, \ldots, q_r$ of nondegenerate n-dimensional quadratic forms over K such that if $G(K)$ for $G = \mathrm{SO}_n(q)$, with q a nondegenerate n-dimensional quadratic form over K, contains a finitely generated Zariski-dense subgroup that is weakly commensurable to Γ_0, then q *must* be similar to one of the q_i's, $i = 1, \ldots, r$.)

Based on our results for S-arithmetic groups (cf., for example, Theorem 4.1(1)) and the results concerning division algebras algebras of exponent two having the same maximal subfields (see Corollary 6.8 and Theorem 7.10), one expects that in some situations one should be able to show that actually $r = 1$, which informally means that the ambient algebraic group is *uniquely* determined by the eigenvalue information of semi-simple elements in a finitely generated Zariski-dense subgroup.

Conjecture 6.1 is known to be true if K is a number field even when Γ_1 is not S-arithmetic (cf. [44, Theorem 5.1]) and also over general fields when G_1 is of type A_1. We recall that given a connected absolutely almost simple real algebraic subgroup of SL_n such that $\mathcal{G} = G(\mathbb{R})$ is noncompact and is not locally isomorphic to $\mathrm{SL}_2(\mathbb{R})$ and a lattice Γ in $\mathcal{G}$, then there exists a number field $K \subset \mathbb{R}$ such that Γ can be conjugated into $\mathrm{SL}_n(K)$, cf. [45, 7.67 and 7.68]. Combining these results, we conclude that Conjecture 6.1 is true when Γ_1 is a lattice in the group of real points of an absolutely almost simple real algebraic group. More evidence supporting this conjecture comes from the investigation of another natural problem in the theory of algebraic group — namely, characterizing absolutely almost simple algebraic K-groups having the same isomorphism/isogeny classes of maximal K-tori. The connection between the two is based on the Isogeny Theorem 5.5 and Corollary 5.6. While these two problems are not equivalent, their investigation usually involves many common elements. To comment on these common aspects, we will temporarily shift the focus to the second problem. We will later see how the finiteness statements in the context of both problems fit into some more general conjectures about algebraic groups with reductive reduction - see Conjectures 7.5 and 7.8.

6.2. Simple algebraic groups over number fields with the same maximal tori. The tools used to prove Theorems 4.1 and 4.2 can be used to characterize absolutely almost simple algebraic groups over number fields having the same isomorphism/isogeny classes of maximal tori. We give the statements of these results below in order to demonstrate their complete similarity to the corresponding results concerning weak commensurability.

Theorem 6.3 (cf. [40], Theorem 7.5).

(1) *Let G_1 and G_2 be connected absolutely almost simple algebraic groups defined over a number field K, and let L_i be the smallest Galois extension of K over which G_i becomes an inner form of a split group. If G_1 and G_2 have the same K-isogeny classes of maximal K-tori, then either G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n, and moreover, $L_1 = L_2$.*

(2) *Fix an absolutely almost simple K-group G. Then the set of isomorphism classes of all absolutely almost simple K-groups G' having the same K-isogeny classes of maximal K-tori is finite.*

(3) *Fix an absolutely almost simple simply connected K-group G whose Killing-Cartan type is different from A_n, D_{2n+1} $(n > 1)$ or E_6. Then any K-form G' of G (in other words, any absolutely almost simple simply connected K-group G' of the* same *type as G) that has the same K-isogeny classes of maximal K-tori as G, is isomorphic to G.*

The construction described in [40, §9] shows that the types excluded in (3) are honest exceptions, i.e., for each of those types one can construct non-isomorphic absolutely almost simple simply connected K-groups G_1 and G_2 of this type over a number field K that have the same isomorphism classes of maximal K-tori.

The case where G_1 and G_2 are of types B_ℓ and C_ℓ, respectively, is treated in the following theorem.

Theorem 6.4 ([22], Theorem 1.4). *Let G_1 and G_2 be absolutely almost simple algebraic groups over a number field K of types B_ℓ and C_ℓ, respectively, for some $\ell \geqslant 3$.*

(1) *The groups G_1 and G_2 have the same* isogeny *classes of maximal K-tori if and only if they are twins.*

(2) *The groups G_1 and G_2 have the same* isomorphism *classes of maximal K-tori if and only if they are twins, G_1 is adjoint, and G_2 is simply connected.*

We note that some aspects of the general problem of characterizing absolutely almost simple algebraic groups over local and global fields having the same isomorphism classes of maximal tori were considered in [20] and [31]. Another direction of research, which has already generated a number of results (cf. [5], [6] [21], [33], [41]) is the investigation of local-global principles for embedding tori into absolutely almost simple algebraic groups as maximal tori (in particular, for embedding of commutative étale algebras with involutive automorphisms into simple algebras with involution); some number-theoretic applications of these results can be found, for example, in [17].

In order to get a better idea of what kind of results can be obtained over general fields, it is helpful to consider first the related problem of characterizing finite-dimensional division algebras having the same maximal subfields, which is somewhat reminiscent of Amitsur's famous theorem about central simple algebras having the same *generic* splitting fields (cf. [2], [25]).

6.5. Division algebras with the same maximal subfields. Let D_1 and D_2 be central division algebras of the same degree n over a field K. We say that D_1 and D_2 have the *same maximal subfields* if a degree n field extension L/K admits a K-embedding $L \hookrightarrow D_1$ if and only if it admits a K-embedding $L \hookrightarrow D_2$. We also let $\mathrm{Br}(K)$ denote the Brauer group of K, and for a (finite-dimensional) central simple K-algebra A, we let $[A] \in \mathrm{Br}(K)$ denote the corresponding Brauer class.

Definition 6.6. Let D be a central division K-algebra of degree n. The *genus* of D is defined to be

$$\mathbf{gen}(D) = \{\, [D'] \mid D' \text{is a central division} K\text{-algebra with the same maximal subfields as } D\}.$$

Two basic questions about the genus are:

(I) *When does* $\mathbf{gen}(D)$ *reduce to a single element?* (Then D is uniquely determined by its maximal subfields.)

(II) *What can one say about the size of* $\mathbf{gen}(D)$ *in the general case? In particular, when is* $\mathbf{gen}(D)$ *finite?*

We note that since the opposite algebra D^{op} has the same maximal subfields as D, the genus $\mathbf{gen}(D)$ can reduce to one element only if $D \simeq D^{\mathrm{op}}$, i.e. if $[D]$ has exponent 2 in $\mathrm{Br}(K)$. If K is a global field, then any central division algebra over K of exponent 2 is a quaternion algebra and furthermore it follows from the theorem of Albert-Hasse-Brauer-Noether (ABHN) that for any quaternion division K-algebra D we have $|\mathbf{gen}(D)| = 1$. Another consequence of (AHBN) is that $\mathbf{gen}(D)$ is finite for any central division algebra D over a global field K (see [9, 3.6]).

On the other hand, a construction proposed by M. Rost, M. Schacher, A. Wadsworth, and others (cf. [23, Example 2.1]), enables one to produce quaternion algebras over infinitely generated fields with nontrivial, and even infinite (see [38]), genus. So, both questions become nontrivial over fields more general than global fields, and the following two theorems, obtained jointly with V. Chernousov and I. Rapinchuk [8], [9], [46], contain some recent results in that direction.

The first theorem expands the variety of examples where the genus is trivial. We will say that a field F satisfies property $(*)$ if for any central division F-algebra D of exponent 2, the genus $\mathbf{gen}(D)$ reduces to a single element.

Theorem 6.7 (Stability theorem, [9, 46]). *If a field k of characteristic $\neq 2$ satisfies $(*)$, then so does the field of rational functions $k(x)$.*

(The stability property in characteristic 2 has not been investigated yet.)

Corollary 6.8. *Let k be either a finite field of characteristic $\neq 2$ or a number field, and let $K = k(x_1, \ldots, x_t)$ be a finitely generated purely transcendental extension of k. Then for any central division K-algebra of exponent 2, we have $|\mathbf{gen}(D)| = 1$.*

The second theorem establishes the finiteness of the genus over finitely generated fields.

Theorem 6.9. *Let D be a central division algebra of degree n over a finitely generated field K of characteristic not dividing n. Then the genus $\mathbf{gen}(D)$ is finite.*

Both theorems are based on an analysis of the ramification properties of division algebras in the genus. More precisely, given a discrete valuation v of K, we let $\mathcal{O}_{K,v}$ and $\overline{K}_v$ denote the corresponding valuation ring and residue field, respectively. Fix an integer $n > 1$ (which will later be either the degree or the exponent of D), and suppose that V is a set of discrete valuations of K that satisfy the following three conditions:

(A) *For any $a \in K^{\times}$, the set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite;*

(B) *There exists a* finite *subset $V' \subset V$ such that the field of fractions of*

$$\mathcal{O} = \bigcap_{v \in V \setminus V'} \mathcal{O}_{K,v},$$

coincides with K;

(C) *For any $v \in V$, the characteristic of the residue field $\overline{K}_v$ is prime to n.*

(We note that if K is finitely generated, which will be the case in most of our applications, then (B) automatically follows from (A).) Due to (C), we can define for each $v \in V$ the corresponding *residue map*

$$\rho_v \colon {}_n\mathrm{Br}(K) \longrightarrow \mathrm{Hom}(\mathcal{G}^{(v)}\ ,\ \mathbb{Z}/n\mathbb{Z}), \tag{R}$$

where ${}_n\mathrm{Br}(K)$ is the n-torsion in the Brauer group and $\mathcal{G}^{(v)}$ is the absolute Galois group of $\overline{K}_v$ (cf., for example, [48, §10] or [49, Ch.II, Appendix]). A class $[A] \in {}_n\mathrm{Br}(K)$ (or a finite-dimensional central simple K-algebra A representing this class) is said to be *unramified* at v if $\rho_v([A]) = 1$, and *ramified* otherwise. We let $\mathrm{Ram}_V(A)$ denote the set of all $v \in V$ where A is ramified; one shows that this set is always finite. We also define the unramified part of ${}_n\mathrm{Br}(K)$ with respect to V to be

$${}_n\mathrm{Br}(K)_V = \bigcap_{v \in V} \mathrm{Ker}\, \rho_v.$$

Then one shows [9, Theorem 2.2] that if ${}_n\mathrm{Br}(K)_V$ is finite, then for a central division algebra K-algebra D of degree n one has

$$|\mathbf{gen}(D)| \leqslant |{}_n\mathrm{Br}(K)_V| \cdot \varphi(n)^r, \text{ with } r = |\mathrm{Ram}_V(D)|. \tag{U}$$

Thus, to prove Theorem 6.9, one needs to show that for a finitely generated field K whose characteristic is prime to a given integer $n > 1$, there exists a set V of discrete valuations of K satisfying the above conditions (A)-(C) and such that the unramified Brauer group ${}_n\mathrm{Br}(K)_V$ is finite. This was first done by an explicit construction based on an analysis of the standard exact sequence for the Brauer group of a curve; this approach enables one to give some explicit estimations on the size of the unramified Brauer group, hence of the genus, cf. [9, §4], [10]. Subsequently, a more general argument was pointed out to us J.-L. Colliot-Thélène (cf. [8]). More precisely, suppose our finitely generated field K is realized as the field of rational functions on a regular integral scheme X of finite type over $\mathrm{Spec}\ A$, where A is either a finite field or the ring of S-integers in a number field for some finite set S of its places, with n invertible in A, and let V be the set of discrete valuations of K associated with the divisors on X. Then the finiteness of ${}_n\mathrm{Br}(K)_V$ follows from Deligne's finiteness theorem for the étale cohomology of constructible sheaves [16] and Gabber's purity theorem [19]. The proof of Theorem 6.7 relies on the fact that if V is the set of all geometric places of the field of rational functions $k(x)$ (i.e., those that are trivial on k), where k is a field of characteristic $\neq 2$, then ${}_2\mathrm{Br}(k(x))_V$ reduces to ${}_2\mathrm{Br}(k)$ (cf. [25, Corollary 6.4.6]).

7. The genus of a simple algebraic group. Groups with reductive reduction.

In this concluding section, we will describe the ongoing project (cf. [11]) of obtaining the analogs of results from 6.5 for arbitrary absolutely almost simple algebraic groups, and connect this activity back to the Finiteness Conjecture 6.1. First, we need to extend Definition 6.6.

Definition 7.1. Let G be an absolutely almost simple simply connected algebraic group over a field K. The (K-)*genus* $\mathbf{gen}_K(G)$ (or simply $\mathbf{gen}(G)$ if this does not lead to any confusion) of G is the collection of K-isomorphism classes of K-forms G' of G that have the same K-isomorphism classes of maximal K-tori as G.

One can alternatively define the genus using "K-isogeny classes" of maximal tori in place of "K-isomorphism classes." While the exact relationship between these notions of genus has not been investigated, the Isogeny Theorem 5.5 and subsequent remarks strongly suggest that they should lead to the same qualitative results in most cases. On the other hand, A.S. Merkurjev proposed a different (in a way, more functorial) definition of the genus of an absolutely almost simple algebraic K-group G as the set of K-isomorphism classes of K-forms G' of G that have the same isomorphism/isogeny classes of maximal tori not only over K, but also over *any* field extension F/K. The results of Izboldin, Vishik and Karpenko indicate a connection between this genus for the spinor group $G = \mathrm{Spin}_n(q)$ of a quadratic form q and the motive of the projective quadric $q = 0$ in the category of Chow motives, so it makes sense to call this genus *motivic* (see [9, Remark 5.6] for more details). In this article, however, we will only use the notion of genus given in Definition 7.1.

Building on Theorem 6.9, it is natural to make the following conjecture.

Conjecture 7.2. *Let G be an absolutely almost simple simply connected algebraic group over a finitely generated field K of good characteristic*[4]*. Then $\mathbf{gen}_K(G)$ is finite.*

This conjecture is true over global fields (Theorem 6.3) and also for inner forms of type A_ℓ in the general case (see Theorem 7.6 below). While Conjecture 7.2 does not automatically imply our main Conjecture 6.1, we will now outline an approach that can potentially lead

[4]For each type, the following characteristics are defined to be *bad*: type A_ℓ - all primes dividing $(\ell+1)$, and also $p=2$ for outer forms; types B_ℓ, C_ℓ, D_ℓ - $p=2$, and also $p=3$ for ${}^{3,6}\mathsf{D}_4$; for type E_6 - $p=2,3,5$; for types E_7, E_8 - $p=2,3,5,7$; for types F_4, G_2 - $p=2,3$. All other characteristics for a given type are *good*.

to the proof of both conjectures, and also have some other implications. The considerations in 6.5 were based on an analysis of the ramification properties of central simple algebras at discrete valuations of the center. An adequate replacement of the notion of an unramified algebra for arbitrary absolutely almost simple groups is the notion of a group with *reductive reduction*. Let G be a connected absolutely almost simple simply connected algebraic group over a field K. Given a discrete valuation v of K, we let K_v denote the corresponding completion with valuation ring $\mathcal{O}_v$, valuation ideal $\mathfrak{p}_v$, and residue field $\overline{K}_v = \mathcal{O}_v/\mathfrak{p}_v$. One says that G has reductive reduction at v if there exists a reductive group scheme $\mathcal{G}$ over $\mathcal{O}_v$ with generic fiber $G \otimes_K K_v$. Then the reduction $\mathcal{G} \otimes_{\mathcal{O}_v} \overline{K}_v$ modulo $\mathfrak{p}_v$ will be denoted $\underline{G}^{(v)}$. A crucial point in the proof of the estimate (U) in 6.5, which reduces the finiteness of the genus $\mathbf{gen}(D)$ to the finiteness of the unramified Brauer group, was the fact that if $D' \in \mathbf{gen}(D)$, and $\chi = \rho_v([D])$, $\chi' = \rho_v([D'])$, where ρ_v is the residue map at v (cf. (R) in 6.5), then $\mathrm{Ker}\ \chi = \mathrm{Ker}\ \chi'$. In particular, if D is unramified at v then so is D' (thus, the property of being unramified is determined by maximal subfields). We have been able to prove the following analog of the latter fact for arbitrary absolutely almost simple groups.

Theorem 7.3 ([11]). *Assume that the residue field $\overline{K}_v$ is finitely generated and that G has reductive reduction at v. Then any $G' \in \mathbf{gen}_K(G)$ also has reductive reduction at v. Furthermore, the reduction $\underline{G}'^{(v)}$ lies in the genus $\mathbf{gen}_{\overline{K}_v}(\underline{G}^{(v)})$.*

Assume now that the field K is equipped with a set V of discrete valuations that satisfies the following two conditions

(A′) for any $a \in K^{\times}$, the set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite;

(B′) for any $v \in V$, the residue field $\overline{K}^{(v)}$ is finitely generated.

Corollary 7.4. *If K and V satisfy conditions (A′) and (B′), then for any absolutely almost simple simply connected algebraic K-group G, there exists a finite subset $V_0 \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has reductive reduction at all $v \in V \setminus V_0$.*

The other ingredient of the proof of the finiteness of $\mathbf{gen}(D)$ in 6.5 was the finiteness of the unramified Brauer group ${}_n\mathrm{Br}(K)_V$ for a suitable set V of discrete valuations of K. One can expect the following general statement to be valid for the same sets V of valuations as in 6.5. Let again X be a regular integral scheme of finite type over $\mathrm{Spec}\ A$, where A is either a finite field or the ring of S-integers in a number field for some finite set S of its places, let K be the field of rational functions on X, and let V be the set of discrete valuations of K associated with the prime divisors on X (obviously, V satisfies conditions (A′) and (B′)).

Conjecture 7.5. *Let K and V be as above, and let G be an absolutely almost simple simply connected algebraic K-group such that* $\mathrm{char}\ K$ *is good for G. Then for any finite subset $V_0 \subset V$, the set of K-isomorphism classes of (inner) K-forms G' of G that have reductive reduction at all $v \in V \setminus V_0$, is finite.*

Over a number field K, the assertion of Conjecture 7.5 is an easy consequence of the finiteness results for Galois cohomology, cf. [49, Ch. III, 4.6]. (Interestingly, there are absolutely almost simple nonsplit algebraic groups over $\mathbb{Q}$ that have reductive reduction at all primes, see [28], [14], but there are no $\mathbb{Q}$-defined abelian varieties with smooth reduction at all primes [18].) At the time of this writing, our knowledge about the conjecture is limited to the following two theorems.

Theorem 7.6 (cf. [9], Theorem 5.3). *Conjectures 7.2 and 7.5 (for inner forms) are true for $G = \mathrm{SL}_{1,A}$ where A is a central simple K-algebra.*

Assume that $\mathrm{char}\ K \neq 2$ and let $\mu_2 = \{\pm 1\}$. Then for any discrete valuation v of K such that $\mathrm{char}\ \overline{K}_v \neq 2$ and any $i \geqslant 1$, one can define the residue map in Galois cohomology

$$\rho_v^i \colon H^i(K, \mu_2) \to H^{i-1}(\overline{K}_v, \mu_2)$$

extending (R) in 6.5 to all dimensions (see, for example, [13, 3.3] or [25, 6.8] for the details). Then for any set V of discrete valuations of K such that $\mathrm{char}\ \overline{K}_v \neq 2$ for all $v \in V$, one defines the unramified part $H^i(K, \mu_2)_V$ to be $\bigcap_{v \in V} \mathrm{Ker}\ \rho_v^i$ (of course, $H^2(K, \mu_2)_V = {}_2\mathrm{Br}(K)_V$).

Theorem 7.7 ([11]). *Let K be a finitely generated field of characteristic $\neq 2$, and let V be a set of discrete valuations of K as in Conjecture 7.5 such that* $\mathrm{char}\ \overline{K}_v \neq 2$ *for all $v \in V$. Assume that for any finite subset $V_0 \subset V$, the unramified cohomology groups $H^i(K, \mu_2)_{V \setminus V_0}$ are finite for all $i \geqslant 1$. Then for any $n \geqslant 5$, the set of K-isomorphism classes of the spinor groups* $\mathrm{Spin}_n(q)$*, where q is a nondegenerate n-dimensional quadratic form, that have reductive reduction at all $v \in V$, is finite.*

Now, our Finiteness Conjecture 6.1 would be a consequence of Conjecture 7.5 and the following.

Conjecture 7.8. *Let K and V be as in Conjecture 7.5, and assume that* $\mathrm{char}\ K = 0$. *Furthermore, let G_1 and G_2 be absolutely almost simple algebraic groups defined over a field $F \supset K$, and let $\Gamma_1 \subset G_1(F)$ be a Zariski-dense subgroup with trace field $K_\Gamma = K$. Then there exists a finite subset $V_0 \subset V$ (depending on Γ_1) such that if $\Gamma_2 \subset G_2(F)$ is weakly commensurable to Γ_1, then the (simply connected cover of the) algebraic hull $\mathcal{G}_2$ of Γ_2 has reductive reduction at all $v \in V \setminus V_0$.*

At this point, Conjecture 7.8 has been established for groups of type A_1 using the strong approximation theorem of Weisfeiler [54]. It seems that the same method should also be applicable in the general case.

The potential implications of Conjecture 7.5 reach beyond eigenvalue rigidity, e.g., it would also imply the finiteness of the Tate-Shafarevich set in some situations. More precisely, let K and V be as in Conjecture 7.5, and let G be an absolutely almost simple simply connected K-group. Consider the Tate-Shafarevich set

$$Ш(\overline{G}) := \mathrm{Ker}\left(H^1(K, \overline{G}) \longrightarrow \prod_{v \in V} H^1(K_v, \overline{G}) \right)$$

for the corresponding adjoint group $\overline{G}$. We can pick a finite subset $V_0 \subset V$ so that G has reductive reduction at all $v \in V \setminus V_0$. Now, let $\xi \in Ш(\overline{G})$, and let $G' = {}_\xi G$ be the corresponding twisted group. Then $G' \simeq G$ over K_v for all $v \in V$, and in particular, G' has reductive reduction at all $v \in V \setminus V_0$. Assuming Conjecture 7.5, we would have that the groups ${}_\xi G$ for $\xi \in Ш(\overline{G})$ form finitely many K-isomorphism classes; in other words, the image of $Ш(\overline{G})$ under the canonical map $H^1(K, \overline{G}) \xrightarrow{\lambda} H^1(K, \mathrm{Aut}\, G)$ is finite. But since $\overline{G} \simeq \mathrm{Int}\, G$ is of finite index in $\mathrm{Aut}\, G$, the map λ has finite fibers, so we obtain the finiteness of $Ш(\overline{G})$. In particular, Theorem 7.6 yields the following.

Corollary 7.9. *Let K and V be as in Conjecture 7.5, and let A be a central simple K-algebra of degree n not divisible by* $\mathrm{char}\ K$*. Then for $\overline{G} = \mathrm{PSL}_{1,A}$, the Tate-Shafarevich set $\text{Ш}(\overline{G})$ is finite.*

Finally, we would to point out that the techniques involved in Theorem 7.3 are instrumental not only for proving the finiteness of $\mathbf{gen}_K(G)$, but also for its quantitative analysis. For example, they give yet another instance where a K-form is *uniquely* determined by its maximal K-tori.

Theorem 7.10. *Let $K = k(x)$, where k is a global field of characteristic $\neq 2$. For any K-group G of type G_2, the genus $\mathbf{gen}_K(G)$ reduces to a single element.*

One expects a similar statement to hold over such a field K also for groups of types B_ℓ, C_ℓ ($\ell \geqslant 2$) and F_4 (maybe under some additional assumptions).

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