Towards the eigenvalue rigidity of Zariski-dense subgroups

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Seoul August 2014

MAIN THEMES:

- results on isospectral and length-commensurable locally symmetric spaces (joint with G. Prasad);
- problems in the theory of algebraic groups this work has led to;
- new (conjectural) form of rigidity: *eigenvalue rigidity*.

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• We call this phenomenon the *eigenvalue rigidity*.

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• Why do we care about the eigenvalues?

1. Weak commensurability
   - Definition
   - Geometric motivation
2. First signs of eigenvalue rigidity
3. Arithmetic groups
   - Results
   - Geometric applications
4. Algebraic groups with the same maximal tori
   - Division algebras with the same maximal subfields
   - Groups with reductive reduction

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Definition.

(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple* matrices, let

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2} \quad (\in \overline{F})$$

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be their eigenvalues. Then γ_1 and γ_2 are *weakly commensurable* if $\exists\ a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2} \in \mathbb{Z}$ such that

$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \neq 1.$$

Let $G_1 \subset \mathrm{GL}_{n_1}$ and $G_2 \subset \mathrm{GL}_{n_2}$ be reductive F-groups, $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ be Zariski-dense subgroups.

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(2) Γ_1 and Γ_2 are *weakly commensurable* if

every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order

is weakly commensurable to

some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order,

and vice versa.

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Why the notion of weak commensurability?

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- M_1 and M_2 are commensurable if they have a common *finite-sheeted* cover:

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ M_1 & & M_2 \end{array}$$

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Can one hear the shape of a drum? (M. Kac)

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- Prior to our work, this was done only for arithmetically defined Riemann surfaces (A. Reid) and hyperbolic 3-manifolds (A. Reid et al.).

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$\mathfrak{X}_\Gamma$ is *arithmetically defined* **if** Γ is arithmetic.

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Theorem

Let $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *be locally symmetric spaces having* finite volume, *of absolutely simple real algebraic groups* G_1 *and* G_2.

If $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are* *length-commensurable,* **then** Γ_1 *and* Γ_2 *are* *weakly commensurable.*

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The proof relies on results and conjectures from *transcendental number theory.*

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(leading to the concept of *eigenvalue rigidity* ...)

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Theorem 1

If Γ_1 *and* Γ_2 *are weakly commensurable,* **then** *either* G_1 *and* G_2 *have same Killing-Cartan type, or one of them is of type* B_ℓ *and the other of type* C_ℓ $(\ell \geqslant 3)$.

For a Zariski-dense subgroup $\Gamma \subset G(F)$, let

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- $\mathcal{G}$ is a K-defined algebraic group (in fact, an F/K-form of $\overline{G}$)
- $\mathcal{G}$ is an *important characteristic* of Γ; it *determines* Γ if it is arithmetic

Theorem 2

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(Additionally, one expects that $r = 1$ in certain situations ...)

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FINITENESS CONJECTURE $\Rightarrow$ There are only finitely many c.s.a. A' such that for $G' = \mathrm{PSL}_{1,A'}$,

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- Similar consequences for orthogonal groups of quadratic forms etc.

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General case is work in progress ...

Theorem 3 (G. Prasad, A.R.)

Let

- G_1 *and* G_2 *be absolutely almost simple F-groups,* $\mathrm{char}\, F = 0$;
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(1) *Assume* G_1 *and* G_2 *are of* **same type**, *different from* A_n*,* D_{2n+1} $(n > 1)$*, and* E_6*.*

If Γ_1 *and* Γ_2 *are* weakly commensurable, *then they are* commensurable.

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(2) *In* **all** *cases,* S*-arithmetic* $\Gamma_2 \subset G_2(F)$ *weakly commensurable to a given* S*-arithmetic* $\Gamma_1 \subset G_1(F)$*, form* finitely many *commensurability classes.*

(cont.)

(3) **If** Γ_1 *and* Γ_2 *are weakly commensurable,* **then**

$$\Gamma_1 \text{ contains nontrivial unipotents} \Leftrightarrow \Gamma_2 \text{ does.}$$

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(4) (arithmeticity theorem) *Let now* F *be a locally compact field, and let* $\Gamma_1 \subset G_1(F)$ *be an* S*-arithmetic lattice.*

If $\Gamma_2 \subset G_2(F)$ *is a lattice weakly commensurable to* Γ_1, **then** Γ_2 *is also* S*-arithmetic.*

Theorem 4 (R. Garibaldi, A.R.)

Let

- G_1 *and* G_2 *be absolutely almost simple F-groups of types* B_ℓ *and* C_ℓ $(\ell \geqslant 3)$;
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- $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are both split over all nonarchimedean places of K;*
- $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are* simultaneously *either split or anisotropic over all archimedean places.*

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Let (as above)

- $\mathfrak{X}_{\Gamma_1}$ *be an* arithmetically defined *locally symmetric space,*
- $\mathfrak{X}_{\Gamma_2}$ *be a locally symmetric space of* finite volume.

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(1) $\mathfrak{X}_{\Gamma_2}$ *is arithmetically defined;*

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- *The set of* $\mathfrak{X}_{\Gamma_2}$*'s length-commensurable to* $\mathfrak{X}_{\Gamma_1}$ *is a union of finitely many commensurability classes.*

It consists of single commensurability class G_1 *and* G_2 *are of same type different from* A_n, D_{2n+1} $(n > 1)$, *or* E_6.

Corollary

Let M_1 and M_2 be arithmetically defined hyperbolic d-manifolds, where $d \neq 3$ is even or $\equiv 3 \pmod 4$.

If *M_1 and M_2 are length-commensurable,* **then** *they are commensurable.*

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(In fact, their length spectra are *very* different ...)

- A *complex* hyperbolic manifold cannot be length-commensurable to a *real* or *quaternionic* hyperbolic manifold, etc.

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• *Assume that at least one of* Γ_1 *and* Γ_2 *is arithmetic.*

If G *is of type different from* A_n, D_{2n+1} $(n > 1)$, *and* E_6,
then $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are commensurable.*

- Investigation of weak commensurability is related to understanding algebraic groups *having same isomorphism/ isogeny classes of maximal tori.*

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Definition

Let G_1 and G_2 be absolutely almost simple K-groups.

G_1 and G_2 have same isomorphism/isogeny classes of maximal tori **if**

for every maximal K-torus T_1 of G_1 there exists a maximal K-torus T_2 of G_2

with K-defined isomorphism/isogeny $T_1 \to T_2$,

and vice versa.

- Results on weakly commensurable arithmetic groups $\Rightarrow$ description of groups with same tori over number fields.

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Definiton

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(I) *When does* $\mathbf{gen}_K(G)$ *reduce to a* single *element?*

(II) *When is* $\mathbf{gen}_K(G)$ *finite?*

1. Weak commensurability
   - Definition
   - Geometric motivation

2. First signs of eigenvalue rigidity

3. Arithmetic groups
   - Results
   - Geometric applications

4. Algebraic groups with the same maximal tori
   - Division algebras with the same maximal subfields
   - Groups with reductive reduction

(Finite-dimensional) central division K-algebras D_1 and D_2 have *same maximal subfields* **if**

- they have same degree n,
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Over a *number field* K:

- $|\mathbf{gen}(D)| = 1$ for a quaternion algebra D;
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Theorem 7 (V. Chernousov, I. Rapinchuk, A.R.)

Let

- $K = k(x_1, \ldots, x_r)$, *field of rational functions, where* k *is either a number field, or a finite field of* char $\neq 2$,
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Then $|\mathbf{gen}(D)| = 1$.

Note that $|\mathbf{gen}(D)| = 1$ is *only possible* when D is of exponent 2.

Theorem 8 ($C + R^2$)

Let D be a central division algebra of degree n over a finitely generated *field K with* $\mathrm{char}\, K \nmid n$.

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Note that over *infinitely generated* fields, there are quaternion algebras with *nontrivial*, and even *infinite*, genus

(Rost, Schacher, Wadsworth, Meyer ...)

Both theorems rely on analysis of *ramification*.

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In particular: *if D is unramified w.r.t. a discrete valuation v of K, then every $D' \in \mathbf{gen}(D)$ is unramified at v.*

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In particular: *if D is unramified w.r.t. a discrete valuation v of K, then every $D' \in \mathbf{gen}(D)$ is unramified at v.*

So, finiteness of $\mathbf{gen}(D)$ reduces to finiteness of n-torsion of a certain *unramified Brauer group*.

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In lieu of notion of *unramified division algebra*, one uses notion of a group with *reductive reduction*.

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G has reductive reduction at v **if** there exists a *reductive group scheme* $\mathcal{G}$ over $\mathcal{O}_v$ with

$$\mathcal{G} \otimes_{\mathcal{O}_v} K_v = G \otimes_K K_v.$$

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$\underline{G}^{(v)} = \mathcal{G} \otimes_{\mathcal{O}_v} \overline{K}_v$

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Furthermore, $\underline{G}'^{(v)} \in \mathbf{gen}_{\overline{K}_v}(\underline{G}^{(v)})$.

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CONJECTURE 2

Assume that $\operatorname{char} K$ *is good for* G*.*

Then *set of* K*-isomorphism classes of (inner)* K*-forms* G' *of* G *that have reductive reduction at all* $v \in V$*, is finite.*

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There are also *conditional* results for spinor groups.