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Number-theoretic Techniques in the Theory of Lie Groups and Differential Geometry

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Abstract. We will present a survey of our recent results on length commensurable and isospectral locally symmetric spaces. The geometric questions discussed below led us to a new notion of "weak commensurabilty" of two Zariski-dense subgroups. We have shown that for arithmetic groups, weak commensurabilty has surprisingly strong consequences. Our proofs make use of p-adic techniques and results from algebraic and transcendental number theory. For the current publication, some revisions have been made to the original article presented at the ICCM 2007 to reflect the results obtained in the last couple of years.

Introduction

The aim of this article is to give a brief survey of the results obtained in the series of papers [**19**]–[**25**]. These papers deal with a variety of problems, but have a common feature: they all rely in a very essential way on number-theoretic techniques (including p-adic techniques), and use results from algebraic and transcendental number theory. The fact that number-theoretic techniques turned out to be crucial for tackling certain problems originating in the theory of (real) Lie groups and differential geometry was very exciting. We hope that these techniques will become an integral part of the repertoire of mathematicians working in these areas.

To keep the size of this article within a reasonable limit, we will focus primarily on the paper [**23**] and its sequel [**24, 25**], and only briefly mention the results of [**19**]–[**22**] as well as some other related results in the last section. The work in [**23**], which was originally motivated by questions

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in differential geometry dealing with *length-commensurable* and *isospectral* locally symmetric spaces (cf. §1), led us to define a new relationship between Zariski-dense subgroups of semi-simple algebraic groups which we call *weak commensurability* (cf. §2). The results of [**23, 24**] give an almost complete characterization of weakly commensurable arithmetic groups, but there remain quite a few natural questions (some of which are mentioned below) for general Zariski-dense subgroups. We hope that the notion of weak commensurability will be useful in investigation of (discrete) subgroups of Lie groups, geometry and ergodic theory.

1. Length-commensurable and isospectral manifolds

Let M be a Riemannian manifold. In differential geometry, one associates to M the following sets of data: the *length spectrum* $\mathcal{L}(M)$ (the set of lengths of all closed geodesics *with* multiplicities), the *weak length spectrum* $L(M)$ (the set of lengths of all closed geodesics *without* multiplicities), the *spectrum of the Laplace operator* $\mathcal{E}(M)$ (the set of eigenvalues of the Laplacian Δ_M with multiplicities). The fundamental question is to what extent do $L(M)$, $\mathcal{L}(M)$ and $\mathcal{E}(M)$ determine M? In analyzing this question, the following terminology will be used: two Riemannian manifolds M_1 and M_2 are said to be *isospectral* if $\mathcal{E}(M_1) = \mathcal{E}(M_2)$, and *iso-length-spectral* if $\mathcal{L}(M_1) = \mathcal{L}(M_2)$.

First, it should be pointed out that the conditions like isospectrality, iso-length-spectrality are related to each other. For example, for compact hyperbolic 2-manifolds M_1 and M_2, we have $\mathcal{L}(M_1) = \mathcal{L}(M_2)$ if and only if $\mathcal{E}(M_1) = \mathcal{E}(M_2)$ (cf. [**12**]), and two hyperbolic 3-manifolds are isospectral if and only if they have the same *complex-length* spectrum (for its definition see the footnote later in this section), cf. [**6**] or [**8**]. Furthermore, for compact locally symmetric spaces M_1 and M_2 of nonpositive curvature, if $\mathcal{E}(M_1) = \mathcal{E}(M_2)$, then $L(M_1) = L(M_2)$ (see [**23**], Theorem 10.1). (Notice that all these results rely on some kind of trace formula.)

Second, neither of $\mathcal{L}(M)$, $L(M)$ or $\mathcal{E}(M)$ determines M up to isometry. In fact, in 1980, Vignéras [**31**] constructed examples of isospectral, but nonisometric, hyperbolic 2 and 3-manifolds. This construction relied on arithmetic properties of orders in a quaternion algebra D. More precisely, her crucial observation was that it is possible to choose D so that it contains orders $\mathcal{O}_1$ and $\mathcal{O}_2$ with the property that the corresponding groups $\mathcal{O}_1^{(1)}$ and $\mathcal{O}_2^{(1)}$, of elements with reduced norm one, are not conjugate, but their closures in the completions are conjugate, for all nonarchimedean places of the center. Five years later, Sunada [**29**] gave a very general, and purely group-theoretic, method for constructing isospectral, but nonisometric, manifolds. His construction goes as follows: *Let M be a Riemannian manifold with the fundamental group $\Gamma := \pi_1(M)$. Assume that Γ has a finite quotient G with the following property: there are nonconjugate subgroups H_1, H_2 of G such that $|C \cap H_1| = |C \cap H_2|$ for all conjugacy classes C of G. Let M_i be the* *finite-sheeted cover of M corresponding to the pull-back of H_i in Γ. Then (under appropriate assumptions), M_1 and M_2 are nonisometric isospectral (or iso-length-spectral) manifolds.*

Since its inception, Sunada's method and its variants have been used to construct examples of nonisometric manifolds with same invariants. In particular, Alan Reid [**28**] constructed examples of nonisometric iso-length-spectral hyperbolic 3-manifolds, and recently, in a joint paper [**10**], Leninger, McReynolds, Neumann and Reid gave examples of hyperbolic manifolds with the same weak length spectrum, but different volumes. These, and other, examples demonstrate that it is not possible to characterize Riemannian manifolds (even hyperbolic ones) up to isometry by their spectrum or length spectrum. On the other hand, it is worth noting that the manifolds furnished by Vignéras, and the ones obtained using Sunada's method are always *commensurable,* i.e., have a common finite-sheeted cover. This suggests that the following is perhaps a more reasonable question.

QUESTION 1. *Let M_1 and M_2 be two (hyperbolic) manifolds (of finite volume or even compact). Suppose $L(M_1) = L(M_2)$. Are M_1 and M_2 necessarily commensurable?*

(Of course, the same question can be asked for other classes of manifolds, e.g. for general locally symmetric spaces of finite volume.)

The answer even to this modified question turns out to be "no" in general: Lubotzky, Samuels and Vishne [**11**] have given examples of isospectral (hence, with same weak length spectrum) compact locally symmetric spaces that are *not* commensurable. At the same time, some positive results have emerged. Namely, Reid [**28**] and Chinburg, Hamilton, Long and Reid [**7**] have given a positive answer to Question 1 for arithmetically defined hyperbolic 2- and 3-manifolds, respectively. Our results in [**23**] provide an almost complete answer to Question 1 for arithmetically defined locally symmetric spaces of arbitrary absolutely simple Lie groups. In fact, in [**23**] we analyze when two locally symmetric spaces are commensurable given that they satisfy a much weaker condition than iso-length-spectrality, which we termed *length-commensurability.* We observe that not only does the use of this condition produce stronger results, but the condition itself is more suitable for analyzing Question 1 as it allows one to replace the manifolds under consideration with commensurable manifolds.

DEFINITION. Two Riemannian manifolds M_1 and M_2 are said to be *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Now, we are in a position to formulate precisely the question which is central to [**23**].

QUESTION 2. *Suppose M_1 and M_2 are length-commensurable. Are they commensurable?*

In [**23**], we have been able to answer this question for arithmetically defined locally symmetric spaces of absolutely simple Lie groups. The precise formulations will be given in §3, after introducing appropriate definitions. The following theorem, however, is fully representative of these results.

Theorem. *Let M_1 and M_2 be two* arithmetically defined hyperbolic *d-manifolds.*

(1) *Assume that d is either even or $\equiv 3 \pmod 4$, $d \neq 7$. If M_1 and M_2 are not commensurable, then, after a possible interchange of M_1 and M_2, there exists $\lambda_1 \in L(M_1)$ such that for* any *$\lambda_2 \in L(M_2)$, the ratio λ_1/λ_2 is* transcendental*. In particular, M_1 and M_2 are not length-commensurable.*

(2) *For any dimension $d \equiv 1 \pmod 4$, there exist length-commensurable, but not commensurable, arithmetically defined hyperbolic d-manifolds.*

We have proved similar results for arithmetically defined locally symmetric spaces of absolutely simple real Lie groups of all types; see [**23**]. For example, for hyperbolic spaces modeled on Hamiltonian quaternions we have an assertion similar to (1) (i.e., Question 2 has an affirmative answer); but for complex hyperbolic spaces we have an assertion similar to (2) (i.e., Question 2 has a negative answer).

The key ingredient of our approach is the new notion of *weak commensurability* of Zariski-dense subgroups of an algebraic group, and the relationship between the length-commensurability of locally symmetric spaces and the weak commensurability of their fundamental groups. To motivate the definition of weak commensurability, we consider the following simple example.

Let $\mathfrak{H} = \{x + iy \mid y > 0\}$ be the upper half-plane with the standard hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$. Then $t \mapsto ie^t$ is a geodesic in $\mathfrak{H}$, whose piece $\tilde{c}$ connecting i to ai, where $a > 1$, has length $\ell(\tilde{c}) = \log a$. Now, let $\Gamma \subset SL_2(\mathbb{R})$ be a discrete torsion-free subgroup, and $\pi \colon \mathfrak{H} \to \mathfrak{H}/\Gamma$ be the canonical projection. If $c := \pi(\tilde{c})$ is a closed geodesic in $\mathfrak{H}/\Gamma$ (traced once), then it is not difficult to see that for $\lambda = \sqrt{a}$, the element

$$\gamma = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

lies in Γ. Then the length $\ell(c)$ of c equals $\log a = 2\log\lambda$. This shows that the lengths of closed geodesics in the hyperbolic 2-manifold $\mathfrak{H}/\Gamma$ are (multiples of) the logarithms of the eigenvalues of semi-simple elements of the fundamental group Γ (cf. §3 below, and Proposition 8.2 in [**23**] for a general statement that applies to arbitrary locally symmetric spaces)[1]. Furthermore, let c_i for $i = 1, 2$, be closed geodesics in $\mathfrak{H}/\Gamma$ that in the above notation correspond to semi-simple elements $\gamma_i \in \Gamma$ having the eigenvalue

[1] In the above construction if we replace $\mathrm{SL}_2(\mathbb{R})$ with $\mathrm{SL}_2(\mathbb{C})$, then the collection of (principal values) of the logarithms of the eigenvalues of semi-simple elements is known as the *complex length spectrum* of the corresponding hyperbolic 3-manifold.

$\lambda_i > 1$. Then

$$\ell(c_1)/\ell(c_2) = m/n \quad \Leftrightarrow \quad \lambda_1^n = \lambda_2^m.$$

Notice that the condition on the right-hand side can be reformulated as follows: *If T_i is a torus of* SL_2 *such that $\gamma_i \in T_i(\mathbb{R})$, then there exist $\chi_i \in X(T_i)$ with*

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

The above discussion suggests the following.

Definition. *Let G_1 and G_2 be two semi-simple algebraic groups defined over a field F. Semi-simple elements $\gamma_i \in G_i(F)$, where $i = 1, 2$, are weakly commensurable if there exist maximal F-tori T_i of G_i such that $\gamma_i \in T_i(F)$, and for some characters χ_i of T_i (defined over an algebraic closure $\overline{F}$ of F), we have*

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

As we have seen, weak commensurability adequately reflects length-commensu-rability of hyperbolic 2-manifolds. In fact, it remains relevant for length-commensu-rability of arbitrary locally symmetric spaces. This is easy to see for rank one spaces but is less obvious for higher rank spaces - cf. §3 below, and [**23**], §8.

2. Weakly commensurable arithmetic subgroups

We observe that for G_1 and G_2 different from SL_2, weak commensurability of $\gamma_i \in G_i(F)$, where $i = 1, 2$, may not relate these elements to each other in a significant way (in particular, the F-tori T_i of G_i used in the preceding definition may be very different). So, to get meaningful consequences of weak commensurability, one needs to extend this notion from individual elements to "large" (in particular, Zariski-dense) subgroups.

Definition. *(Zariski-dense) subgroups Γ_i of $G_i(F)$, for $i = 1, 2$, are weakly commensurable if every semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.*

It was discovered in [**23**] that weak commensurability has some important consequences even for completely general finitely generated Zariski-dense subgroups. For simplicity, we will assume henceforth that all our fields are of characteristic zero.

Theorem A. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. Assume that for $i = 1, 2$, there exist finitely generated Zariski-dense subgroups Γ_i of $G_i(F)$ which are weakly commensurable. Then either G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n.*

(We notice that split groups G_1 and G_2 of types B_n and C_n respectively indeed contain weakly commensurable arithmetic subgroups, cf. Example 6.7 in [**23**].)

To formulate our next result, we need one additional notation: given a subgroup Γ of $G(F)$, where G is an absolutely almost simple algebraic F-group, we let K_Γ denote the subfield of F generated by the traces $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$ for all $\gamma \in \Gamma$, where Ad denotes the adjoint representation of G. We recall that according to a result of Vinberg [**32**], for a Zariski-dense subgroup Γ of G, the field K_Γ is precisely the field of definition of $\mathrm{Ad}\,\Gamma$, i.e., it is the minimal subfield K of F such that all elements of $\mathrm{Ad}\,\Gamma$ can be represented simultaneously by matrices with entries in K, in a certain basis of the Lie algebra $\mathfrak{g}$ of G.

THEOREM B. *Let G_1 and G_2 be two connected absolutely almost simple groups defined over a field F of characteristic zero. For $i = 1, 2$, let Γ_i be a finitely generated Zariski-dense subgroup of $G_i(F)$. If Γ_1 and Γ_2 are weakly commensurable, then $K_{\Gamma_1} = K_{\Gamma_2}$.*

Much stronger results are available for the case of arithmetic subgroups. To formulate these, we need to describe the terminology we use regarding arithmetic subgroups. Let G be a connected absolutely almost simple algebraic group defined over a field F of characteristic zero, $\overline{G}$ be its adjoint group, and $\pi\colon G \to \overline{G}$ be the natural isogeny. Suppose we are given:

- a number field K with an embedding $K \hookrightarrow F$;
- an algebraic K-group $\mathcal{G}$ such that for the F-group ${}_F\mathcal{G}$ obtained from it by extension of scalars $K \hookrightarrow F$, there is an F-isomorphism ${}_F\mathcal{G} \overset{\iota}{\simeq} \overline{G}$ (i.e., $\mathcal{G}$ is an F/K-form of $\overline{G}$);
- a (finite) subset S of places of K that contains all the archimedean places but does not contain any nonarchimedean places where $\mathcal{G}$ is anisotropic.

In this situation, we have the natural embedding $\mathcal{G}(K) \hookrightarrow \overline{G}(F)$ induced by ι, and then a subgroup Γ of $G(F)$ is called $(\mathcal{G}, K, S)$-*arithmetic* if $\pi(\Gamma)$ is commensurable with $\sigma(\iota(\mathcal{G}(\mathcal{O}_K(S))))$ for some F-automorphism σ of $\overline{G}$, where $\mathcal{O}_K(S)$ is the ring of S-integers in K. Notice that in this definition we *do* fix an embedding of K into F (in other words, isomorphic, but distinct, subfields of F are treated as different fields), but we *do not* fix an F-isomorphism ι, so by varying it we generate a class of subgroups *invariant under F-automorphisms of $\overline{G}$*. For this reason, by "commensurability" we will mean "commensurability *up to an F-isomorphism* of the corresponding adjoint groups." More precisely, given connected absolutely almost simple F-groups G_i for $i = 1, 2$, and the isogenies $\pi_i\colon G_i \to \overline{G}_i$ onto the corresponding adjoint groups, two subgroups Γ_i of $G_i(F)$ are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if there exists an F-isomorphism $\sigma\colon \overline{G}_1 \to \overline{G}_2$ such that $\sigma(\pi_1(\Gamma_1))$ and $\pi_2(\Gamma_2)$ are commensurable in the usual sense, i.e., their intersection has finite index in both of them.

The groups G_1 and G_2 in Theorems C–G are assumed to be connected and absolutely almost simple.

THEOREM C. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. If Zariski-dense $(\mathcal{G}_i, K_i, S_i)$-arithmetic subgroups Γ_i of $G_i(F)$ are weakly commensurable for $i = 1, 2$, then $K_1 = K_2$ and $S_1 = S_2$.*

One shows that Γ_1 and Γ_2 as in Theorem C are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if and only if $K_1 = K_2$, $S_1 = S_2$ and $\mathcal{G}_1 \simeq \mathcal{G}_2$ over $K := K_1 = K_2$ (cf. Proposition 2.5 in [**23**]). So, according to Theorem C, the weak commensurability of Γ_1 and Γ_2 implies that the first two of these three conditions do hold true. In general, however, $\mathcal{G}_1$ and $\mathcal{G}_2$ do not have to be K-isomorphic. Our next theorem describes the situations where it can be asserted that $\mathcal{G}_1$ and $\mathcal{G}_2$ are K-isomorphic.

THEOREM D. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero, of the same type different from A_n, D_{2n+1}, with $n > 1$, D_4 and E_6. If for $i = 1, 2$, $G_i(F)$ contain Zariski-dense weakly commensurable $(\mathcal{G}_i, K, S)$-arithmetic subgroups Γ_i, then $\mathcal{G}_1 \simeq \mathcal{G}_2$ over K, and hence Γ_1 and Γ_2 are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$.*

In earlier versions of [**23**] as well as in the original version of this article presented at the ICCM 2007, the case of groups of type D_{2n} in Theorem D was left open. This case was later settled in [**24**] using techniques of [**23**] in conjunction with new results on embeddings of fields with involutive automorphisms into simple algebras with involutions.

In the general case, we have the following finiteness result.

THEOREM E. *Let G_1 and G_2 be two connected absolutely almost simple groups defined over a field F of characteristic zero. Let Γ_1 be a Zariski-dense $(\mathcal{G}_1, K, S)$-arithmetic subgroup of $G_1(F)$. Then the set of K-isomorphism classes of K-forms $\mathcal{G}_2$ of $\overline{G}_2$ such that $G_2(F)$ contains a Zariski-dense $(\mathcal{G}_2, K, S)$-arithmetic subgroup weakly commensurable to Γ_1, is finite. In other words, the set of all Zariski-dense (K, S)-arithmetic subgroups of $G_2(F)$ which are weakly commensurable to a given Zariski-dense (K, S)-arithmetic subgroup of $G_1(F)$, is a union of finitely many commensurability classes.*

Note that for the types A_n, D_{2n+1} $(n > 1)$ and E_6 excluded in Theorem D, the number of commensurability classes in Theorem E may not be bounded by an absolute constant depending, say, on G, K and S, viz. as one varies Γ_1 (or, equivalently, $\mathcal{G}_1$), this number changes and typically grows to infinity. To explain what happens for groups of these types, let us consider the following example.

Fix any $n > 1$ and pick four nonarchimedean places v_1, v_2, v_3, $v_4 \in V^K$. Next, consider central division K-algebras D_1 and D_2 of degree $d = n+1 > 2$

with local invariants ($\in \mathbb{Q}/\mathbb{Z}$):

$$n_v^{(1)} = \begin{cases} 0 & , \quad v \neq v_i, i \leqslant 4 \\ 1/d & , \quad v = v_1 \text{ or } v_2 \\ -1/d & , \quad v = v_3 \text{ or } v_4 \end{cases}$$

and

$$n_v^{(2)} = \begin{cases} 0 & , \quad v \neq v_i, i \leqslant 4 \\ 1/d & , \quad v = v_1 \text{ or } v_3 \\ -1/d & , \quad v = v_2 \text{ or } v_4. \end{cases}$$

Then the algebras D_1 and D_2 are neither isomorphic nor anti-isomorphic, implying that the algebraic groups $G_1 = \mathrm{SL}_{1,D_1}$ and $G_2 = \mathrm{SL}_{1,D_2}$ (which are anisotropic inner forms of type A_n) are not K-isomorphic. On the other hand, D_1 and D_2 have exactly the same maximal subfields, which means that G_1 and G_2 have the same maximal K-tori. It follows that for any S, the corresponding S-arithmetic subgroups are weakly commensurable, but not commensurable. Furthermore, by increasing the number of places in this construction, one can construct an arbitrarily large number of central division K-algebras of degree d with the above properties. Then the associated S-arithmetic groups will all be weakly commensurable, but will constitute an arbitrarily large number of commensurability classes.

In [**23**], Example 6.6, we described how a similar construction can be given for *some* outer form of type A_n (i.e., for special unitary groups). The construction relies on a local-global principle for embeddings of fields with an involutive automorphism into simple algebras with involutions which in full generality was recently established in [**24**], §4 (previously, it was known only in the case where $d = n+1$ is odd, cf. Proposition A.2 in [**18**]). It is easiest to implement this construction for d odd (which is the case considered in *loc. cit.*) as then the corresponding special unitary group is automatically quasi-split at every nonarchimedean place where it remains an outer form, but the case of d even can also be worked out.

However, construction of nonisomorphic K-groups with the same K-tori was not known for types D_{2n+1} $(n > 1)$ and E_6. We have given a construction, using Galois cohomology, which works uniformly for types A_n, D_{2n+1} $(n > 1)$ and E_6 (cf. [**23**], §9). Towards this end, we established a new local-global principle for the existence of an embedding of a given K-torus as a maximal torus in a given absolutely simple simply connected K-group. This construction, of course, allows one to produce examples of noncommensurable weakly commensurable S-arithmetic subgroups in groups of types A_n, D_{2n+1} $(n > 1)$ and E_6, and in fact, show that the number of commensurability classes is unbounded. This construction may also be useful elsewhere, for example, in the Langlands program. The triality forms ${}^{3,6}D_4$ remain to be studied.

Even though the definition of weak commensurability involves only semi-simple elements, it detects the presence of unipotent elements; in fact it detects K-rank.

THEOREM F. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a field F of characteristic zero. For $i = 1, 2$, let Γ_i be a Zariski-dense $(\mathcal{G}_i, K, S)$-arithmetic subgroup of $G_i(F)$. Assume that Γ_1 and Γ_2 are weakly commensurable. Then $\mathrm{rk}_K\, \mathcal{G}_1 = \mathrm{rk}_K\, \mathcal{G}_2$ (in particular, if $\mathcal{G}_1$ is K-isotropic, then so is $\mathcal{G}_2$). If G_1 and G_2 are of the same type, then the Tits indices of $\mathcal{G}_1/K$ and $\mathcal{G}_2/K$, and for every place v of K, the Tits indices of $\mathcal{G}_1/K_v$ and $\mathcal{G}_2/K_v$, are isomorphic.*

The above results provide an almost complete picture of weak commensurability among S-arithmetic subgroups. In view of the connection of weak commensurability with length-commensurability of locally symmetric spaces (cf. §3), one would like to extend these results to not necessarily arithmetic Zariski-dense subgroups. We conclude this section with an arithmeticity theorem in which only one subgroup is assumed to be arithmetic, and a discussion of some open questions.

THEOREM G. *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups defined over a nondiscrete locally compact field F of characteristic zero, and for $i = 1, 2$, let Γ_i be a Zariski-dense lattice in $G_i(F)$. Assume that Γ_1 is a (K, S)-arithmetic subgroup of $G_1(F)$. If Γ_1 and Γ_2 are weakly commensurable, then Γ_2 is a (K, S)-arithmetic subgroup of $G_2(F)$.*

REMARKS: (i) The assumption that *both* Γ_1 and Γ_2 are lattices cannot be omitted. For example, let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a torsion-free subgroup of finite index, and Γ^n be the subgroup generated by the n-th powers of elements in Γ. Then Γ^n is weakly commensurable with Γ for all n. On the other hand, $[\Gamma : \Gamma^n] = \infty$ for all sufficiently large n, and then Γ^n is not arithmetic. The same remark applies to all hyperbolic groups. However, we do not know what happens in the higher rank situation.

(ii) The case of lattices in products of real and p-adic groups has not been fully investigated.

(iii) Yet another interesting question is whether or not the discreteness of one of the two weakly commensurable finitely generated Zariski-dense subgroups $\Gamma_i \subset G_i(F)$, where $i = 1, 2$, implies the discreteness of the other. The affirmative answer was given in [**23**], Proposition 5.6, for the case where G_1 and G_2 are absolutely almost simple and F is nonarchimedean. For general semi-simple real groups, the question remains open.

Further analysis of weak commensurability of general Zariski-dense subgroups of $G(F)$ for an arbitrary field F would require information about classification of forms of G over general fields, which is not yet available. For example, even the following basic question seems to be open.

Question 3. *Let D_1 and D_2 be two quaternion division algebras over a field K. Assume that D_1 and D_2 have the* same maximal subfields. *Are they isomorphic?*

M. Rost has informed us that over large fields (like those used in the proof of the Merkurjev-Suslin theorem) the answer can be "no" (apparently, the same observation was independently made by A. Wadsworth and some others). Recently, Garibaldi and Saltman [**9**] have shown that if the unramified Brauer group $\mathrm{Br}_u(K)$ is trivial, then the answer to Question 3 is in the affirmative. This result (and its variants) yield an affirmative answer for $K = k(x_1, \ldots, x_r)$, a purely transcendental extension of a number field k. However, for general finitely generated fields (and the fields that arise in the investigation of weakly commensurable finitely generated subgroups are finitely generated), the answer is unknown. Furthermore, if the answer turns out to be negative in general, we would like to know if the number of isomorphism classes of quaternion algebras over a given finitely generated field, and containing the same maximal subfields, is finite (this may be useful for extending the finiteness result of Theorem E to some nonarithmetic subgroups such as the fundamental groups of general compact Riemann surfaces). Of course, one can ask similar questions for other types of algebraic groups.

3. Length-commensurable locally symmetric spaces

Let G be a connected semi-simple algebraic $\mathbb{R}$-group, $\mathcal{G} = G(\mathbb{R})$. We let $\mathcal{K}$ denote a maximal compact subgroup of $\mathcal{G}$, and let $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$ be the corresponding symmetric space of $\mathcal{G}$. For a discrete torsion-free subgroup Γ of $\mathcal{G}$, we let $\mathfrak{X}_\Gamma$ denote the locally symmetric space $\mathfrak{X}/\Gamma$ with the fundamental group Γ. We say that $\mathfrak{X}_\Gamma$ is *arithmetically defined* if Γ is arithmetic (with S the set of archimedean places of K) in the sense specified in §2.

Our goal now is to relate length-commensurability of locally symmetric spaces to weak commensurability of their fundamental groups. We need to recall some basic facts about closed geodesics on $\mathfrak{X}_\Gamma$ (cf. [**22**], or [**23**], §8). The closed geodesics on $\mathfrak{X}_\Gamma$ correspond to semi-simple elements of Γ. For a semi-simple element $\gamma \in \Gamma$, let c_γ be the closed geodesic corresponding to γ. Its length is given by the following formula (see [**23**], Proposition 8.5):

$$\ell_\Gamma(c_\gamma)^2 = (1/n_\gamma^2)\left(\sum (\log|\alpha(\gamma)|)^2\right), \tag{3.1}$$

where n_γ is an integer, and the sum is over all roots α of G with respect to a maximal $\mathbb{R}$-torus T such that $\gamma \in T(\mathbb{R})$. (We notice that for the upper half-plane $\mathfrak{H} = \mathrm{SO}_2\backslash\mathrm{SL}_2(\mathbb{R})$ this metric differs from the standard hyperbolic metric, considered in §1, by a factor of $\sqrt{2}$, which, of course, does not affect length commensurability.)

For our purposes, we need to recast (3.1) using the notion of a *positive real character*. Given a real torus T, a real character χ of T is called *positive* if $\chi(t) > 0$ for all $t \in T(\mathbb{R})$. We notice that for any character χ of T we have

$$|\chi(t)|^2 = \chi(t) \cdot \overline{\chi(t)} = (\chi + \overline{\chi})(t) = \chi_0(t),$$

where χ_0 is a positive real character. Hence,

$$\ell_\Gamma(c_\gamma)^2 = (1/n_\gamma^2) \sum_{i=1}^{p} s_i (\log \chi^{(i)}(\gamma))^2, \tag{3.2}$$

where $s_i \in \mathbb{Q}$, and $\chi^{(i)}$ are positive real characters.

The right-hand side of (3.2) is easiest to analyze when $\mathrm{rk}_{\mathbb{R}}\, G = 1$, which we will now assume. Let χ be a generator of the group of positive real characters of a maximal $\mathbb{R}$-torus T containing γ. Then

$$\ell_\Gamma(c_\gamma) = (s/n_\gamma) \cdot |\log \chi(\gamma)|, \tag{3.3}$$

where s is the square root of an integer which is independent of γ and T (because any two maximal $\mathbb{R}$-tori of real rank one are conjugate to each other by an element of $G(\mathbb{R})$).

Now, for $i = 1, 2$, let G_i be a connected semi-simple real group, $\mathcal{G}_i = G_i(\mathbb{R})$, and Γ_i be a Zariski-dense discrete torsion-free subgroup of $\mathcal{G}_i$. Set $\mathfrak{X}_{\Gamma_i} = \mathfrak{X}_i/\Gamma_i$, where $\mathfrak{X}_i$ is the symmetric space of $\mathcal{G}_i$. Then $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable as Riemannian manifolds (i.e., have a common finite-sheeted cover) if and only if Γ_1 and Γ_2 are commensurable up to $\mathbb{R}$-isomorphism between $\overline{G}_1$ and $\overline{G}_2$.

The relationship between the length-commensurability of $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ is particularly easy to observe if $G_1 = G_2$ is a group of real rank one. Indeed, if semi-simple elements $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ are *not* weakly commensurable, then it follows from (3.3) that

$$\ell_{\Gamma_1}(c_{\gamma_1})/\ell_{\Gamma_2}(c_{\gamma_2}) = (n_{\gamma_2}/n_{\gamma_1}) \cdot \left(\pm \frac{\log \chi_1(\gamma_1)}{\log \chi_2(\gamma_2)}\right) \notin \mathbb{Q}. \tag{3.4}$$

Therefore, if Γ_1 and Γ_2 are *not weakly commensurable*, then $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are *not length-commensurable.* Thus, the connection noted in §1 for hyperbolic 2-manifolds remains valid for arbitrary locally symmetric spaces of rank one. In fact, one can make a stronger statement assuming that Γ_1 and Γ_2 are arithmetic (or, more generally, can be conjugated into $\mathrm{SL}_n(\overline{\mathbb{Q}})$), even when G_1 and G_2 are non necessarily isomorphic groups of real rank one (in which case the factors s in (3.3) for G_1 and G_2 may be different). Then $\chi_i(\gamma_i) \in \overline{\mathbb{Q}}^\times$ for $i = 1, 2$. But according to a theorem proved independently by Gelfond and Schneider in 1934, if α and β are algebraic numbers such that $\log \alpha/\log \beta$ is irrational, then it is transcendental over $\mathbb{Q}$ (cf. [**4**]). So, it follows from (3.3) that if Γ_1 and Γ_2 are as above, and $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ are not weakly commensurable, then

$$\ell_{\Gamma_1}(c_{\gamma_1})/\ell_{\Gamma_2}(c_{\gamma_2})$$

is transcendental over $\mathbb{Q}$.

To relate length-commensurability of locally symmetric spaces of higher rank with the notion of weak commensurability of their fundamental groups, we need to invoke the Schanuel's Conjecture from transcendental number theory (cf. [**4**]).

SCHANUEL'S CONJECTURE. *If $z_1, \ldots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the transcendence degree over $\mathbb{Q}$ of the field generated by*

$$z_1, \ldots, z_n;\ e^{z_1}, \ldots, e^{z_n}$$

is $\geqslant n$.

What we need is the following corollary of Schanuel's conjecture. Let $\alpha_1, \ldots,\ \alpha_n$ be nonzero *algebraic* numbers, and set $z_i = \log \alpha_i$. Applying Schanuel's conjecture, we obtain that $\log \alpha_1, \ldots, \log \alpha_n$ are *algebraically independent* as soon as they are *linearly independent* (over $\mathbb{Q}$), i.e., whenever $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent.

Before we proceed, we would like to point out that our results for locally symmetric spaces of rank > 1 depend on the truth of Schanuel's conjecture (hence are *conditional*). Analyzing the right hand side of equation (3.2) with the help of the above consequence of Schanuel's conjecture, we show that if both Γ_1 and Γ_2 can be conjugated into $\mathrm{SL}_n(\overline{\mathbb{Q}})$, for non-weakly commensurable $\gamma_i \in \Gamma_i$, $\ell_{\Gamma_1}(c_{\gamma_1})$ and $\ell_{\Gamma_2}(c_{\gamma_2})$ are algebraically independent over $\mathbb{Q}$. Thus, we obtain the following.

PROPOSITION. *For $i = 1, 2$, let Γ_i be a discrete torsion-free subgroup of $\mathcal{G}_i = G_i(\mathbb{R})$, where G_1 and G_2 are semi-simple $\mathbb{R}$-subgroups of SL_n. Under each of the following sets of assumptions*

(i) *G_1 and G_2 are of real rank 1, and either Γ_1 and Γ_2 can be conjugated into $\mathrm{SL}_n(\overline{\mathbb{Q}})$, or $G_1 = G_2$;*

(ii) *Γ_1 and Γ_2 can be conjugated into $\mathrm{SL}_n(\overline{\mathbb{Q}})$, and Schanuel's conjecture holds,*

the length-commensurability of $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ implies the weak commensurability of Γ_1 and Γ_2.

We recall that if Γ is a lattice in $\mathcal{G} = G(\mathbb{R})$, where G is a connected absolutely simple real algebraic group, not isogenous to SL_2, then there exists a real number field K such that G is defined over K and $\Gamma \subset G(K)$, see [**26**], Proposition 6.6. Thus, if G is not isogenous to SL_2 and $\mathfrak{X}_\Gamma$ has finite volume, then Γ can always be conjugated into $\mathrm{SL}_n(\overline{\mathbb{Q}})$, and the corresponding assumptions in the above proposition become redundant.

Henceforth, we will assume that G_1 and G_2 are connected and absolutely simple. We will refer to the following situation as the *exceptional case:*

($\mathcal{E}$) One of the locally symmetric spaces, say, $\mathfrak{X}_{\Gamma_1}$, is 2-dimensional and the corresponding discrete subgroup Γ_1 cannot be conjugated into $\mathrm{PGL}_2(\overline{\mathbb{Q}})$, and the other space, $\mathfrak{X}_{\Gamma_2}$, has dimension > 2.

The following is an immediate consequence of the above proposition and subsequent remarks.

COROLLARY. *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and let $\mathfrak{X}_{\Gamma_i}$ be a locally symmetric space of finite volume, of $\mathcal{G}_i = G_i(\mathbb{R})$, for $i = 1, 2$. Assume that we are not in the exceptional case $(\mathcal{E})$. If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, then Γ_1 and Γ_2 are weakly commensurable.*

Now, Theorems A and B immediately imply the following.

THEOREM 1. *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and let $\mathfrak{X}_{\Gamma_i}$ be a locally symmetric space of finite volume, of $\mathcal{G}_i$, for $i = 1, 2$. Assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable, and we are not in the exceptional case $(\mathcal{E})$. Then* (i) *either G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n,* (ii) *$K_{\Gamma_1} = K_{\Gamma_2}$, where K_{Γ_i} denotes the field generated by the traces* $\mathrm{Tr}\,\mathrm{Ad}\,\gamma$ *for $\gamma \in \Gamma_i$.*

We now turn to arithmetically defined locally symmetric spaces. Combining Theorems D and E with the above proposition, we obtain the following.

THEOREM 2. *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and let $\mathcal{G}_i = G_i(\mathbb{R})$, for $i = 1, 2$. Then the set of arithmetically defined locally symmetric spaces $\mathfrak{X}_{\Gamma_2}$ of $\mathcal{G}_2$, which are length-commensurable to a given arithmetically defined locally symmetric space $\mathfrak{X}_{\Gamma_1}$ of $\mathcal{G}_1$, is a union of finitely many commensurability classes. It in fact consists of a single class if G_1 and G_2 have the same type different from A_n, D_{2n+1}, with $n > 1$, D_4 and E_6.*

Next, Theorems F and G imply

THEOREM 3. *Let G_1 and G_2 be connected absolutely simple real algebraic groups, and let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be length-commensurable locally symmetric spaces of $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively, of finite volume. Assume that at least one of the spaces is arithmetically defined and that we are not in the exceptional case $(\mathcal{E})$. Then the other space is also arithmetically defined, and the compactness of one of the spaces implies the compactness of the other.*

We now recall that isospectral compact locally symmetric spaces have same weak length spectrum ([**23**], Theorem 10.1). Using this fact in conjunction with Theorems 2 and 3, we obtain the following results, which apparently do not follow directly from the spectral theory.

THEOREM 4. *For $i = 1, 2$, let G_i be a connected absolutely simple adjoint real algebraic group, and Γ_i be a discrete torsion-free subgroup of $\mathcal{G}_i$ such that the corresponding locally symmetric space $\mathfrak{X}_{\Gamma_i}$ is compact. If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral, and Γ_1 is arithmetic, then so is Γ_2.*

Theorem 5. *Notations and assumptions as in Theorem 4, assume that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are isospectral, and at least one of the subgroups Γ_1 and Γ_2 is arithmetic. Then $G_1 = G_2 =: G$. Moreover, unless G is of type A_n, D_{2n+1} $(n > 1)$, D_4 or E_6, the spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable.*

Finally, we want to address the question as to how different $L(\mathfrak{X}_{\Gamma_1})$ and $L(\mathfrak{X}_{\Gamma_2})$ are, given that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are not length-commensurable. Assuming Schanuel's conjecture, we prove in [**25**] the following statement.

Theorem 6. *Let G_1 and G_2 be absolutely simple real algebraic groups of the same type different from A_n $(n > 1)$, D_n $(n \geqslant 4)$, and E_6. Given discrete torsion-free subgroups Γ_i of $\mathcal{G}_i$ for $i = 1, 2$, we let $\mathcal{L}_i$ denote the subfield of $\mathbb{R}$ generated by $L(\mathfrak{X}_{\Gamma_i})$. If Γ_1 and Γ_2 are arithmetic, and $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are not length-commensurable, then $\mathcal{L} := \mathcal{L}_1\mathcal{L}_2$ has infinite transcendence degree over either $\mathcal{L}_1$ or $\mathcal{L}_2$.*

It follows from Theorem 6 for the locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ considered therein, the fact that these are not length-commensurable implies that after a possible interchange of the spaces, we will have $L(\mathfrak{X}_{\Gamma_1}) \not\subset \mathbb{Q} \cdot \Omega \cdot L(\mathfrak{X}_{\Gamma_2})$ for *any* finite set Ω of real numbers. In particular, $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ cannot be made length-commensurable by scaling the metric on one of them. It would be interesting to determine if Theorem 6 and its consequences mentioned above remain valid for all arithmetically defined locally symmetric spaces.

(Using this opportunity, we would like to provide a list of some corrections to [**23**]: (i) p. 115, l. 12 - replace $\pi(\Gamma_2)$ with $\pi_2(\Gamma_2)$. (ii) In assertion (2) of Theorem 4.2, replace the condition "*if* $L_1 = L_2$," by "*if* $L_1 = L_2 =: L$, *and* $\theta_{T_1}(\mathrm{Gal}(\mathrm{L_{T_1}/L})) \supset \mathrm{W(G_1, T_1)}$,". (iii) In footnote 9 on p. 142, replace "(18)" with "$(*)$". (iv) In the proof of Proposition 5.6, after the proof of Lemma 5.7, replace "G", occurring without a subscript, with "G_2" everywhere. (v) In the fourth line of the proof of Theorem 4 (in §6), replace "G" with "G_1", and in the next line, replace "obtained from $\mathcal{G}$" by "obtained from $\overline{\mathcal{G}}$". (vi) In the sentence prior to the statement of Theorem 8.15 (p. 166), the reference to Theorem 7 is not needed.)

4. Proofs: p-adic techniques

Given two arithmetic subgroups, or, more generally, two Zariski-dense subgroups, the proofs of Theorems A–G ultimately rely on the possibility of constructing semi-simple elements in one subgroup whose spectra are quite different from the spectra of *all* semi-simple in the other subgroup unless certain strong conditions, relating these subgroups, hold. The relevant existence results fit into a broader project of constructing elements with special properties in a given Zariski-dense subgroup dealt with in our papers [**19**], [**21, 22**]. The starting point of this project was the following question asked independently by G.A. Margulis and R. Spatzier: *Let Γ be a Zariski-dense* *subgroup of $G(K)$, where G is a connected absolutely almost simple algebraic group defined over a finitely generated field K of characteristic zero. Does there exist a regular semisimple $\gamma \in \Gamma$ such that $\langle\gamma\rangle$ is Zariski-dense in $T := Z_G(\gamma)^\circ$?* It should be pointed out that the existence of such an element is by no means obvious. For example, if $\varepsilon \in \mathbb{C}^\times$ is any element of infinite order, then the subgroup $\langle\varepsilon\rangle \times \langle\varepsilon\rangle \subset \mathbb{C}^\times \times \mathbb{C}^\times$ is Zariski-dense, but it contains no Zariski-dense cyclic subgroup. Elaborating on this observation, one can construct a $\mathbb{Q}$-torus T such that $T(\mathbb{Z})$ is Zariski-dense in T, but no element of $T(\mathbb{Z})$ generates a Zariski-dense subgroup of T. Something similar may also happen in the semi-simple situation. Namely, let G be an absolutely almost simple $\mathbb{Q}$-group with $\mathrm{rk}_{\mathbb{R}}\, G = 1$. Then if a $\mathbb{Q}$-subtorus T of G has a nontrivial decomposition into an almost direct product $T = T_1 \cdot T_2$ over $\mathbb{Q}$ (and such a decomposition exists if T has a nontrivial $\mathbb{Q}$-subtorus), no element of $T(\mathbb{Z})$ generates a Zariski-dense subgroup of T. The latter example shows that the fact that a given torus contains a proper subtorus is an obstruction to the existence of an element with the desired property. So, in [**19**] we singled out tori which were called "irreducible", and used them to provide an affirmative answer to the question of Margulis and Spatzier.

DEFINITION. *A K-torus T is K-irreducible if it does not contain any proper K-subtori.*

The point is that if T is K-irreducible, then *any* $t \in T(K)$ of infinite order generates a Zariski-dense subgroup. So, to answer the above question of Margulis and Spatzier in the affirmative, it would suffice to prove that Γ contains a regular semi-simple element γ of infinite order such that the torus $T := Z_G(\gamma)^\circ$ is K-irreducible.

We will now outline a general procedure for constructing irreducible tori. Let T be a K-torus, $\mathcal{G}_T = \mathrm{Gal}(K_T/K)$, where K_T is the splitting field of T. Then T is K-irreducible if and only if $\mathcal{G}_T$ acts irreducibly on $X(T)\otimes_{\mathbb{Z}}\mathbb{Q}$. Now, if T is a maximal K-torus of G, then $\mathcal{G}_T$ acts faithfully on the root system $\Phi(G,T)$, which allows us to identify $\mathcal{G}_T$ with a subgroup of $\mathrm{Aut}(\Phi(G,T))$. If under this identification $\mathcal{G}_T$ contains the Weyl group $W(G,T)$, then T is K-irreducible (provided that G is absolutely almost simple). Therefore, it would be enough to find a $\gamma \in \Gamma$ such that for $T = Z_G(\gamma)^\circ$ as above, $\mathcal{G}_T \supset W(G,T)$.

Our proof of the existence in [**19**] for S-arithmetic subgroups of $G(K)$, K a number field, used the so-called "generic tori". It was shown by V. E. Voskresenskii [**33**] that G has a maximal torus $\mathcal{T}$ defined over a purely transcendental extension $\mathcal{K} = K(x_1, \ldots, x_n)$ such that $\mathcal{G}_{\mathcal{T}} \supset W(G,\mathcal{T})$. Then, using Hilbert's Irreducibility Theorem, one can specialize parameters to get (plenty of) maximal K-tori T of G such that $\mathcal{G}_T \supset W(G,T)$. In fact, we can construct such tori with prescribed behavior at finitely many places of K, using which it is easy to ensure that for the resulting torus T, the group $T(\mathcal{O}_K(S))$ is infinite, and then any element $\gamma \in T(\mathcal{O}_K(S))$ of infinite order has the desired property.

Some time later, Margulis and G.A. Soifer asked us a different version of the original question which arose in their joint work with H. Abels on the Auslander problem: *Let G be an absolutely almost simple real algebraic group, Γ be a finitely generated Zariski-dense subgroup of $G(\mathbb{R})$. Is there a regular semi-simple element γ in Γ which generates a Zariski-dense subgroup of $T = Z_G(\gamma)^\circ$ and which is also $\mathbb{R}$-regular?* We recall that $\gamma \in G(\mathbb{R})$ is *$\mathbb{R}$-regular* if the number of eigenvalues, counted with multiplicity, of modulus 1 of $\mathrm{Ad}\,\gamma$ is minimal possible (cf. [**17**]). It should be noted that even the existence of an $\mathbb{R}$-regular element without any additional requirement in an arbitrary Zariski-dense subgroup Γ is a nontrivial matter: this was established by Benoist and Labourie [**5**] using the multiplicative ergodic theorem, and then by Prasad [**16**] by a direct argument; we will not, however, discuss this aspect here. The real problem is that the above argument for the existence of a regular semisimle element in Γ which generates a Zariski-dense subgroup of its centralizer does not extend to the case where Γ is not arithmetic. More precisely, since Γ is finitely generated, we can choose a finitely generated subfield K of $\mathbb{R}$ such that G is defined over K and $\Gamma \subset G(K)$. Then we can construct a maximal K-torus T of G which is irreducible over K. However, it is not clear at all why $T(K)$ should contain an element of Γ of infinite order if the latter is not of "arithmetic type". Nevertheless, the answer to the question of Margulis and Soifer turned out to be in the affirmative.

Theorem 7 ([**21**]). *Let G be a connected semi-simple real algebraic group. Then any Zariski-dense subsemigroup Γ of $G(\mathbb{R})$ contains a regular $\mathbb{R}$-regular element γ such that the cyclic subgroup generated by it is a Zariski-dense subgroup of the maximal torus $T = Z_G(\gamma)^\circ$.*

The proof of the theorem, which we will now sketch, used a rather interesting technique involving p-adic embeddings. We begin by recalling the following proposition.

Proposition ([**21**]). *Let $\mathcal{K}$ be a finitely generated field of characteristic zero, $\mathcal{R} \subset \mathcal{K}$ be a finitely generated ring. Then there exists an infinite set of primes Π such that for each $p \in \Pi$, there exists an embedding $\varepsilon_p \colon \mathcal{K} \hookrightarrow \mathbb{Q}_p$ with the property: $\varepsilon_p(\mathcal{R}) \subset \mathbb{Z}_p$.*

We will only show that Γ contains an element γ which is "irreducible" over the field K chosen above, i.e., a regular semi-simple element whose centralizer T is a K-irreducible maximal torus of G. For this, we fix a matrix realization $G \hookrightarrow \mathrm{SL}_n$ and pick a finitely generated subring $\mathcal{R}$ of K so that $\Gamma \subset G(\mathcal{R}) := G(K) \cap \mathrm{SL}_n(\mathcal{R})$. We then choose a finitely generated field extension $\mathcal{K}$ of K over which G splits, and fix a $\mathcal{K}$-split maximal torus T_0 of G. We now let $C_1, \ldots, C_r$ denote the nontrivial conjugacy classes in the Weyl group $W(G, T_0)$. Using the above proposition, we pick r primes $p_1, \ldots, p_r$ such that for each p_i there is an embedding $\mathcal{K} \hookrightarrow \mathbb{Q}_{p_i}$ for which $\mathcal{R} \hookrightarrow \mathbb{Z}_{p_i}$. We then employ results on Galois cohomology of semi-simple groups over local fields to construct, for each $i = 1, \dots, r$, an open set $\Omega_{p_i}(C_i) \subset G(\mathbb{Q}_{p_i})$ such that any $\omega \in \Omega_{p_i}(C_i)$ is regular semi-simple of infinite order, and for $T_\omega = Z_G(\omega)^\circ$, the Galois group $\mathcal{G}_{T_\omega}$ contains an element from the image of C_i under the natural identification $W(G, T_0) \simeq W(G, T_\omega)$. To conclude the argument, we show that

$$\bigcap_{i=1}^{r} (\Gamma \cap \Omega_{p_i}(C_i)) \neq \emptyset,$$

and any element γ of this intersection has the property that for $T = Z_G(\gamma)^\circ$, the inclusion $\mathcal{G}_T \supset W(G, T)$ holds, as required.

Theorem 7 was already used in [**1**]. Furthermore, its suitable generalizations were instrumental in settling a number of questions about Zariski-dense subgroups of Lie groups posed by Y. Benoist, T.J. Hitchman and R. Spatzier (cf. [**22**]). As we already mentioned, the elements constructed in Theorem 7 play a crucial role in the proof of Theorems A–G.

We conclude this article with a brief survey of other applications of p-adic embeddings. To our knowledge, Platonov [**14**] was the first to use p-adic embeddings in the context of algebraic groups. He proved the following.

Theorem 8 ([**14**]). *If $\pi \colon \widetilde{G} \to G$ is a nontrivial isogeny of connected semi-simple groups over a finitely generated field K of characteristic zero, then $\pi(\widetilde{G}(K)) \neq G(K)$.*

It is enough to show that if $\pi \colon \widetilde{T} \to T$ is a nontrivial isogeny of K-tori, then $\pi(\widetilde{T}(K)) \neq T(K)$. For this, we pick a finitely generated extension $\mathcal{K}$ of K so that $\widetilde{T}$ and T split over $\mathcal{K}$, and every element of $\operatorname{Ker} \pi$ is $\mathcal{K}$-rational. Then, using the above proposition, one finds an embedding $\mathcal{K} \hookrightarrow \mathbb{Q}_p$ for some p. To conclude the argument, one shows that $\pi(\widetilde{T}(\mathcal{K})) = T(\mathcal{K})$ would imply $\pi(\widetilde{T}(\mathbb{Q}_p)) = T(\mathbb{Q}_p)$, which is obviously false.

Another application is representation-theoretic rigidity of groups with bounded generation (cf. [**27**], and [**15**], Appendix A.2). We recall that an abstract group Γ has *bounded generation* if there are elements $\gamma_1, \dots, \gamma_d \in \Gamma$ such that

$$\Gamma = \langle \gamma_1 \rangle \cdots \langle \gamma_d \rangle,$$

where $\langle \gamma_i \rangle$ is the cyclic subgroup generated by γ_i.

Theorem 9 ([**27**]). *Let Γ be a group with bounded generation satisfying the following condition*

$(*)$ $\Gamma'/[\Gamma', \Gamma']$ *is finite for every subgroup Γ' of Γ of finite index.*

Then for any $n \geqslant 1$, there are only finitely many inequivalent completely reducible representations $\rho \colon \Gamma \longrightarrow \mathrm{GL}_n(\mathbb{C})$.

The proof is based on the following strengthening of the above proposition: given $\mathcal{K}$ and $\mathcal{R}$ as above, there exists an infinite set of primes Π such that for each $p \in \Pi$ there are embeddings $\varepsilon_p^{(i)} \colon \mathcal{K} \to \mathbb{Q}_p$, where $i = 1, 2, \ldots$, such that $\varepsilon_p^{(i)}(\mathcal{R}) \subset \mathbb{Z}_p$ for all i, and $\varepsilon_p^{(i)}(\mathcal{R}) \cap \varepsilon_p^{(j)}(\mathcal{R})$ consists of algebraic numbers for all $i \neq j$. The usual argument using representation varieties show that it is enough to show that for any $\rho \colon \Gamma \longrightarrow \mathrm{GL}_n(\mathbb{C})$, the traces Tr $\rho(\gamma)$ are algebraic numbers, for all $\gamma \in \Gamma$. For this we pick a finitely generated subring $\mathcal{R}$ of $\mathbb{C}$ for which $\rho(\Gamma) \subset \mathrm{GL}_n(\mathcal{R})$, and then fix a prime p for which there are embeddings $\varepsilon_p^{(i)} \colon \mathcal{R} \to \mathbb{Z}_p$ as above. Let $\rho^{(i)} \colon \Gamma \longrightarrow \mathrm{GL}_n(\mathbb{Z}_p)$ be the representation obtained by composing ρ with the embedding $\mathrm{GL}_n(\mathcal{R}) \to \mathrm{GL}_n(\mathbb{Z}_p)$ induced by $\varepsilon_p^{(i)}$. One then observes that bounded generation of Γ implies that for any subgroup Γ' of Γ of finite index, the pro-p completion Γ'_p of Γ' is a p-adic analytic group. Moreover, $(*)$ implies that the corresponding Lie algebra is a semi-direct product of a semi-simple algebra and a nilpotent one such that the former acts on the latter without fixed points. Using the fact that a semi-simple algebra has only finitely many inequivalent representations in any dimension, one derives that there are $i \neq j$ such that Tr $\rho^{(i)}(\gamma) =$ Tr $\rho^{(j)}(\gamma)$ for all γ in a suitable subgroup Γ' of Γ of finite index. Then it follows from our construction that the traces Tr $\rho(\gamma)$ are algebraic for $\gamma \in \Gamma'$, and consequently all traces Tr $\rho(\gamma)$ for $\gamma \in \Gamma$, are algebraic, as required. (We notice that Theorem 9 was extended in [**2**] to representations of boundedly generated groups over fields of positive characteristic.)

Finally, we would like to mention the following theorem which provides a far-reaching generalization of the results of [**3**] and [**13**].

THEOREM 10 ([**20**]). *Let G be a connected reductive group over an infinite field K. Then no noncentral subnormal subgroup of $G(K)$ can be contained in a finitely generated subgroup of $G(K)$.*

(In fact, a similar result is available in the situation where $G(K)$ is replaced by the group of points over a semi-local subring of K.) To avoid technicalities, let us assume that G is absolutely almost simple, and let N be a noncentral normal (rather than subnormal) subgroup of $G(K)$. Assume that N is contained in a finitely generated subgroup of $G(K)$. Then, after fixing a matrix realization $G \subset \mathrm{SL}_n$, one can pick a finitely generated subring $\mathcal{R}$ of K so that $N \subset G(\mathcal{R}) := G(K) \cap \mathrm{SL}_n(\mathcal{R})$. Let $\mathcal{K}$ be a finitely generated field that contains $\mathcal{R}$, and such that G is defined and split over $\mathcal{K}$. Now, choose an embedding $\varepsilon_p \colon \mathcal{K} \hookrightarrow \mathbb{Q}_p$ so that $\varepsilon_p(\mathcal{R}) \subset \mathbb{Z}_p$, and consider the closures $\Delta = \overline{N}$ and $\mathcal{G} = \overline{G(K)}$ in $G(\mathbb{Q}_p)$. Then $\Delta \subset G(\mathbb{Z}_p)$, hence it is compact, and at the same time it is normal in $\mathcal{G}$. On the other hand, $\mathcal{G}$ is essentially $G(\mathbb{Q}_p)$. However, $G(\mathbb{Q}_p)$ does not have any noncentral compact normal subgroups (in fact, the subgroup $G(\mathbb{Q}_p)^+$ of $G(\mathbb{Q}_p)$, generated by all unipotents, is a normal subgroup of finite index which does not contain *any* noncentral normal subgroups, cf. [**30**]). A contradiction.

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