On the notion of genus for division algebras and algebraic groups

(joint work with V. Chernousov and I. Rapinchuk)

Andrei S. Rapinchuk
University of Virginia

PU/IAS November 2, 2017

We are interested in the following question:

We are interested in the following question:

(∗) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related* **if** *they have* <u>*same*</u> *maximal subfields?*

We are interested in the following question:

(∗) *Let* D_1 *and* D_2 *be finite-dimensional central division algebras over a field* K. *How are* D_1 *and* D_2 *related* **if** *they have* <u>*same*</u> *maximal subfields?*

- D_1 and D_2 have same maximal subfields **if**

1. $\deg D_1 = \deg D_2 =: n$;
2. for P/K of degree n, $P \hookrightarrow D_1 \iff P \hookrightarrow D_2$.

We are interested in the following question:

(∗) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related* **if** *they have* <u>*same*</u> *maximal subfields?*

• D_1 and D_2 have same maximal subfields **if**

1. $\deg D_1 = \deg D_2 =: n$;
2. for P/K of degree n, $P \hookrightarrow D_1 \Leftrightarrow P \hookrightarrow D_2$.

We are interested in the following question:

(∗) *Let D_1 and D_2 be finite-dimensional central division algebras over a field K. How are D_1 and D_2 related* **if** *they have* <u>*same*</u> *maximal subfields?*

- D_1 and D_2 have same maximal subfields **if**

1. $\deg D_1 = \deg D_2 =: n$;
2. for P/K of degree n, $P \hookrightarrow D_1 \Leftrightarrow P \hookrightarrow D_2$.

Geometry

G. Prasad-A.R. (Publ. math. IHES **109**(2009)): *In many (although not all) situations, two arithmetically defined locally symmetric spaces having same lengths of closed geodesics are commensurable.*

Geometry

G. Prasad-A.R. (Publ. math. IHES **109**(2009)): *In many (although not all) situations, two arithmetically defined locally symmetric spaces having same lengths of closed geodesics are commensurable.*

Arithmetic Riemann surfaces were considered by A. Reid.

Geometry

G. Prasad-A.R. (Publ. math. IHES **109**(2009)): *In many (although not all) situations, two arithmetically defined locally symmetric spaces having same lengths of closed geodesics are commensurable.*

Arithmetic Riemann surfaces were considered by A. Reid.

Underlying algebraic fact:

Let D_1 *and* D_2 *be two quaternion division algebras over a number field* K*.* **If** D_1 *and* D_2 *have same maximal subfields* **then** $D_1 \simeq D_2$.

However, most Riemann surfaces are not arithmetic

However, most Riemann surfaces are not arithmetic $\Rightarrow$
One needs to understand to what degree this fact extends to more general fields

However, most Riemann surfaces are not arithmetic $\Rightarrow$
One needs to understand to what degree this fact extends to more general fields

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

However, most Riemann surfaces are not arithmetic $\Rightarrow$
One needs to understand to what degree this fact extends to more general fields

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

"Most" Riemann surfaces are of the form:

$$M = \mathbb{H}/\Gamma$$

However, most Riemann surfaces are not arithmetic $\Rightarrow$
One needs to understand to what degree this fact extends to more general fields

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

"Most" Riemann surfaces are of the form:

$$M = \mathbb{H}/\Gamma$$

where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a *discrete torsion free subgroup*.

However, most Riemann surfaces are not arithmetic $\Rightarrow$
One needs to understand to what degree this fact extends to more general fields

- Let $\mathbb{H} = \{ x + iy \mid y > 0 \}$.

"Most" Riemann surfaces are of the form:

$$M = \mathbb{H}/\Gamma$$

where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a *discrete torsion free subgroup*.

- Some properties of M can be understood in terms of the

associated quaternion algebra.

Let

- $\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$;

Let

- $\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$;
- $\tilde{\Gamma} = \pi^{-1}(\Gamma) \subset \mathrm{M}_2(\mathbb{R})$.

Let

- $\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$;
- $\tilde{\Gamma} = \pi^{-1}(\Gamma) \subset \mathrm{M}_2(\mathbb{R})$.

Set $A_\Gamma = \mathbb{Q}[\tilde{\Gamma}^{(2)}]$, $\tilde{\Gamma}^{(2)} \subset \tilde{\Gamma}$ generated by squares.

Let

- $\pi : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$;
- $\tilde{\Gamma} = \pi^{-1}(\Gamma) \subset \mathrm{M}_2(\mathbb{R})$.

Set $A_\Gamma = \mathbb{Q}[\tilde{\Gamma}^{(2)}]$, $\tilde{\Gamma}^{(2)} \subset \tilde{\Gamma}$ generated by squares.

One shows: A_Γ is a quaternion algebra with center

$$K_\Gamma = \mathbb{Q}(\operatorname{tr}\gamma \mid \gamma \in \Gamma^{(2)})$$

(trace field).

Let

- $\pi : SL_2(\mathbb{R}) \to PSL_2(\mathbb{R})$;
- $\tilde{\Gamma} = \pi^{-1}(\Gamma) \subset M_2(\mathbb{R})$.

Set $A_\Gamma = \mathbb{Q}[\tilde{\Gamma}^{(2)}]$, $\tilde{\Gamma}^{(2)} \subset \tilde{\Gamma}$ generated by squares.

One shows: A_Γ is a quaternion algebra with center

$$K_\Gamma = \mathbb{Q}(\operatorname{tr}\gamma \mid \gamma \in \Gamma^{(2)})$$

(trace field).

(**Note** that for general Fuchsian groups, K_Γ is not necessarily a number field.)

- **If** Γ is *arithmetic,* **then** A_Γ is <u>the</u> quaternion algebra involved in its description;

- **If** Γ is *arithmetic*, **then** A_Γ is the quaternion algebra involved in its description;

- In general, A_Γ **does not** determine Γ, **but** is an invariant of the commensurability class of Γ.

- **If** Γ is *arithmetic*, **then** A_Γ is the quaternion algebra involved in its description;

- In general, A_Γ **does not** determine Γ, **but** is an invariant of the commensurability class of Γ.

To a (nontrivial) semi-simple $\gamma \in \tilde{\Gamma}^{(2)}$ there corresponds:

- **If** Γ is *arithmetic*, **then** A_Γ is the quaternion algebra involved in its description;

- In general, A_Γ **does not** determine Γ, **but** is an invariant of the commensurability class of Γ.

To a (nontrivial) semi-simple $\gamma \in \tilde{\Gamma}^{(2)}$ there corresponds:

- *geometrically*: a closed geodesic $c_\gamma \subset M$,
  if $\gamma \sim \pm \begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$ $(t_\gamma > 1)$ then *length* $\ell(c_\gamma) = 2 \log t_\gamma$;

- **If** Γ is *arithmetic*, **then** A_Γ is the quaternion algebra involved in its description;

- In general, A_Γ **does not** determine Γ, **but** is an invariant of the commensurability class of Γ.

To a (nontrivial) semi-simple $\gamma \in \tilde{\Gamma}^{(2)}$ there corresponds:

- *geometrically*: a closed geodesic $c_\gamma \subset M$, if $\gamma \sim \pm \begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$ $(t_\gamma > 1)$ then *length* $\ell(c_\gamma) = 2 \log t_\gamma$;

- *algebraically*: a maximal etale subalgebra $K_\Gamma[\gamma] \subset A_\Gamma$.

For a Riemannian manifold M:

$L(M)$ = set of lengths of closed geodesics in M
((weak) length spectrum of M)

For a Riemannian manifold M:

$L(M)$ = set of lengths of closed geodesics in M

((weak) length spectrum of M)

Definition.

Riemannian manifolds M_1 and M_2 are

For a Riemannian manifold M:

$L(M)$ = set of lengths of closed geodesics in M
((weak) length spectrum of M)

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;

For a Riemannian manifold M:

$L(M)$ = set of lengths of closed geodesics in M
((weak) length spectrum of M)

Definition.

Riemannian manifolds M_1 and M_2 are

- *iso-length spectral* if $L(M_1) = L(M_2)$;
- *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$.

Let $M_i = \mathbb{H}/\Gamma_i\ (i = 1, 2)$ be Riemann surfaces.

1. $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
2. Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that
$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z})$$
elements γ_1^m and γ_2^n are conjugate $\Rightarrow$

Let $M_i = \mathbb{H}/\Gamma_i \; (i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

1. $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
2. Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that

$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z})$$

elements γ_1^m and γ_2^n are conjugate $\Rightarrow$

Let $M_i = \mathbb{H}/\Gamma_i \ (i = 1,2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

1. $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
2. Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1,2$ such that

$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z})$$

elements γ_1^m and γ_2^n are conjugate $\Rightarrow$

Let $M_i = \mathbb{H}/\Gamma_i \ (i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

1. $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
2. Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that

$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z})$$

   elements γ_1^m and γ_2^n are conjugate $\Rightarrow$

Let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

1. $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
2. Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that

$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z})$$

   elements γ_1^m and γ_2^n are conjugate $\Rightarrow$

   $K[\gamma_1] \subset A_{\Gamma_1}$ and $K[\gamma_2] \subset A_{\Gamma_2}$ are isomorphic.

Let $M_i = \mathbb{H}/\Gamma_i \ (i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

1. $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
2. Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that
$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z})$$
elements γ_1^m and γ_2^n are conjugate $\Rightarrow$
$K[\gamma_1] \subset A_{\Gamma_1}$ and $K[\gamma_2] \subset A_{\Gamma_2}$ are isomorphic.

So, A_{Γ_1} and A_{Γ_2} share "lots" of maximal etale subalgebras.

Let $M_i = \mathbb{H}/\Gamma_i$ $(i = 1, 2)$ be Riemann surfaces.

If M_1 and M_2 are length-commensurable **then:**

1. $K_{\Gamma_1} = K_{\Gamma_2} =: K$;
2. Given closed geodesics $c_{\gamma_i} \subset M_i$ for $i = 1, 2$ such that

$$\ell(c_{\gamma_2})/\ell(c_{\gamma_1}) = m/n \quad (m, n \in \mathbb{Z})$$

   elements γ_1^m and γ_2^n are conjugate $\Rightarrow$

   $K[\gamma_1] \subset A_{\Gamma_1}$ and $K[\gamma_2] \subset A_{\Gamma_2}$ are isomorphic.

So, A_{Γ_1} and A_{Γ_2} share "lots" of maximal etale subalgebras.

(Not all - but we will ignore it in this talk ...)

- For M_1 and M_2 to be commensurable, A_{Γ_1} and A_{Γ_2} must be isomorphic.

- For M_1 and M_2 to be commensurable, A_{Γ_1} and A_{Γ_2} must be isomorphic.

So, proving that length-commensurable M_1 and M_2 are commensurable implicitly involves answering a version of $(*)$.

- For M_1 and M_2 to be commensurable, A_{Γ_1} and A_{Γ_2} must be isomorphic.

So, proving that length-commensurable M_1 and M_2 are commensurable implicitly involves answering a version of $(*)$.

- **Recall:** If $M = \mathbb{H}/\Gamma$ is a compact Riemann surface then compact Riemann surfaces *isospectral* to M split into finitely many isometry classes.

- For M_1 and M_2 to be commensurable, A_{Γ_1} and A_{Γ_2} must be isomorphic.

So, proving that length-commensurable M_1 and M_2 are commensurable implicitly involves answering a version of $(*)$.

- **Recall:** If $M = \mathbb{H}/\Gamma$ is a compact Riemann surface then compact Riemann surfaces *isospectral* to M split into finitely many isometry classes.

What about *length-commensurable* Riemann surfaces?

- For M_1 and M_2 to be commensurable, A_{Γ_1} and A_{Γ_2} must be isomorphic.

So, proving that length-commensurable M_1 and M_2 are commensurable implicitly involves answering a version of $(*)$.

- **Recall:** If $M = \mathbb{H}/\Gamma$ is a compact Riemann surface then compact Riemann surfaces *isospectral* to M split into finitely many isometry classes.

What about *length-commensurable* Riemann surfaces?

Theorem

Let $M_i = \mathbb{H}/\Gamma_i \ (i \in I)$ be a family of length-commensurable Riemann surfaces where $\Gamma_i \subset \mathrm{PSL}_2(\mathbb{R})$ is Zariski-dense. Then quaternion algebras $A_{\Gamma_i} \ (i \in I)$ split into finitely many isomorphism classes (over common center).

Algebra

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.

Algebra

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.

If D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

Algebra

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.
If D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

Algebra

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K. **If** D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

- Proof of Amitsur's Theorem uses *generic splitting fields* (function fields of Severi-Brauer varieties),

Algebra

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K. **If** D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

- Proof of Amitsur's Theorem uses *generic splitting fields* (function fields of Severi-Brauer varieties), which are

  infinite extensions of K.

Algebra

Amitsur's Theorem

Let D_1 and D_2 be central division algebras over K.
If D_1 and D_2 have same splitting fields, i.e. for F/K we have

$$D_1 \otimes_K F \simeq M_{n_1}(F) \quad \Leftrightarrow \quad D_2 \otimes_K F \simeq M_{n_2}(F),$$

then $\langle [D_1] \rangle = \langle [D_2] \rangle$ in $\mathrm{Br}(K)$.

- Proof of Amitsur's Theorem uses *generic splitting fields* (function fields of Severi-Brauer varieties), which are

  infinite extensions of K.

- *What happens if one allows only splitting fields of finite degree, or just maximal subfields?*

- Amitsur's Theorem is no longer true in this setting.

- Amitsur's Theorem is no longer true in this setting. (Counterexamples can be found using cubic algebras over number fields.)

- Amitsur's Theorem is no longer true in this setting.
  (Counterexamples can be found using cubic algebras over number fields.)

This leads to question $(*)$ and its variations.

- Amitsur's Theorem is no longer true in this setting. (Counterexamples can be found using cubic algebras over number fields.)

This leads to question $(*)$ and its variations.

Question (Prasad-A.R.)

Are quaternion algebras over $K = \mathbb{Q}(x)$ *determined by their maximal subfields?*

- Amitsur's Theorem is no longer true in this setting.
  (Counterexamples can be found using cubic algebras over number fields.)

This leads to question $(*)$ and its variations.

Question (Prasad-A.R.)

Are quaternion algebras over $K = \mathbb{Q}(x)$ determined by their maximal subfields?

- Yes – D. Saltman

- Amitsur's Theorem is no longer true in this setting. (Counterexamples can be found using cubic algebras over number fields.)

This leads to question $(*)$ and its variations.

Question (Prasad-A.R.)

Are quaternion algebras over $K = \mathbb{Q}(x)$ determined by their maximal subfields?

- Yes – D. Saltman

- Same over $K = k(x)$, k a number field

(S. Garibaldi - D. Saltman)

Definition.

Let D be a finite-dimensional central division algebra over K.

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Over number fields:

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Over number fields:

- genus of any quaternion algebra reduces to one element;
- genus of any division algebra is finite.

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Over number fields:

- genus of any quaternion algebra reduces to one element;
- genus of any division algebra is finite.

Definition.

Let D be a finite-dimensional central division algebra over K. The *genus* of D is

$$\mathbf{gen}(D) = \{ [D'] \in \mathrm{Br}(K) \mid D' \text{ has same maximal subfields as } D \}$$

Question 1. *When does* $\mathbf{gen}(D)$ *reduce to a single element?*

(This means that D is uniquely determined by maximal subfields.)

Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

Over number fields:

- genus of any quaternion algebra reduces to one element;
- genus of any division algebra is finite.

(Follows from Albert-Hasse-Brauer-Noether Theorem.)

Theorem 1 (Stability Theorem)

Let $\operatorname{char} k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

Theorem 1 (Stability Theorem)

Let $\operatorname{char} k \neq 2$*. If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k*, then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$*.*

- Same statement is true for division algebras of *exponent 2*.

Theorem 1 (Stability Theorem)

Let $\mathrm{char}\, k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

- Same statement is true for division algebras of *exponent 2*.
- $|\mathbf{gen}(D)| > 1$ if D is <u>not</u> of exponent 2.

Theorem 1 (Stability Theorem)

Let char $k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

- Same statement is true for division algebras of *exponent 2*.
- $|\mathbf{gen}(D)| > 1$ if D is <u>not</u> of exponent 2.

- $\mathbf{gen}(D)$ can be infinite.

Theorem 1 (Stability Theorem)

Let $\mathrm{char}\, k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

- Same statement is true for division algebras of *exponent 2*.
- $|\mathbf{gen}(D)| > 1$ if D is not of exponent 2.

- $\mathbf{gen}(D)$ can be infinite.

Generalizing construction used by Schacher, Garibaldi, Saltman, ...

Theorem 1 (Stability Theorem)

Let $\mathrm{char}\, k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

- Same statement is true for division algebras of *exponent 2*.
- $|\mathbf{gen}(D)| > 1$ if D is not of exponent 2.

- $\mathbf{gen}(D)$ can be infinite.

Generalizing construction used by Schacher, Garibaldi, Saltman, ...
Meyer constructed quaternion algebras over "large" fields with infinite genus,

Theorem 1 (Stability Theorem)

Let $\operatorname{char} k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

- Same statement is true for division algebras of *exponent 2*.
- $|\mathbf{gen}(D)| > 1$ if D is not of exponent 2.

- $\mathbf{gen}(D)$ can be infinite.

Generalizing construction used by Schacher, Garibaldi, Saltman, ...

Meyer constructed quaternion algebras over "large" fields with infinite genus,

Tikhonov extended construction to algebras of prime degree.

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K ($\mathrm{char}\, K \neq 2$) having a common maximal subfield.

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K (char $K \neq 2$) having a common maximal subfield.

(E.g., take $D_1 = \left(\frac{-1, 3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1, 7}{\mathbb{Q}}\right)$ over $K = \mathbb{Q}$)

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K ($\mathrm{char}\, K \neq 2$) having a common maximal subfield.

(E.g., take $D_1 = \left(\frac{-1,3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1,7}{\mathbb{Q}}\right)$ over $K = \mathbb{Q}$)

- If D_1 and D_2 already have same maximal subfields, we are done.

  Otherwise, pick $K(\sqrt{d_1}) \hookrightarrow D_1$ such that $K(\sqrt{d_1}) \not\hookrightarrow D_2$.

Construction

- Start with nonisomorphic quaternion algebras D_1 and D_2 over K (char $K \neq 2$) having a common maximal subfield.

(E.g., take $D_1 = \left(\frac{-1, 3}{\mathbb{Q}}\right)$ and $D_2 = \left(\frac{-1, 7}{\mathbb{Q}}\right)$ over $K = \mathbb{Q}$)

- If D_1 and D_2 already have same maximal subfields, we are done.

  Otherwise, pick $K(\sqrt{d_1}) \hookrightarrow D_1$ such that $K(\sqrt{d_1}) \not\hookrightarrow D_2$.

(E.g., $\mathbb{Q}(\sqrt{11}) \hookrightarrow D_1$ but $\mathbb{Q}(\sqrt{11}) \not\hookrightarrow D_2$.)

- Find K_1/K such that

1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

- Find K_1/K such that

1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

- Find K_1/K such that

1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

- Find K_1/K such that

  1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
  2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

For K_1 one can take the function field of a quadric.

- Find K_1/K such that

1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

For K_1 one can take the function field of a quadric.

In our example, K_1 is function field of

$$-x_1^2 + 7x_2^2 + 7x_3^2 = 11x_4^2$$

- Find K_1/K such that
  1. $D_1 \otimes_K K_1 \not\simeq D_2 \otimes_K K_1$;
  2. $K_1(\sqrt{d_1}) \hookrightarrow D_2 \otimes_K K_1$.

For K_1 one can take the function field of a quadric.

In our example, K_1 is function field of

$$-x_1^2 + 7x_2^2 + 7x_3^2 = 11x_4^2$$

Then (2) is obvious, and (1) follows from the fact that

$$x_0^2 + x_1^2 - 21x_2^2 - 21x_3^2$$

remains anisotropic over K_1.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

This generates a tower $K \subset K_1 \subset K_2 \subset \cdots$

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

This generates a tower $K \subset K_1 \subset K_2 \subset \cdots$

Set $\mathcal{K} = \bigcup_{i=1}^{\infty} K_i$.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

This generates a tower $K \subset K_1 \subset K_2 \subset \cdots$

Set $\mathcal{K} = \bigcup_{i=1}^{\infty} K_i$.

- **Then** $D_1 \otimes_K \mathcal{K} \not\simeq D_2 \otimes_K \mathcal{K}$ and have same maximal subfields.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

This generates a tower $K \subset K_1 \subset K_2 \subset \cdots$

Set $\mathcal{K} = \bigcup_{i=1}^{\infty} K_i$.

- **Then** $D_1 \otimes_K \mathcal{K} \not\simeq D_2 \otimes_K \mathcal{K}$ and have same maximal subfields.

For *infinite* genus, one starts with $D_p = \left(\frac{-1, p}{\mathbb{Q}}\right)$, $p \equiv 3 (\mathrm{mod}\, 4)$.

- If there exists $K_1(\sqrt{d_2}) \hookrightarrow D_1 \otimes_K K_1$ and $K_1(\sqrt{d_2}) \not\hookrightarrow D_2 \otimes_K K_1$ we construct K_2/K_1 similarly.

This generates a tower $K \subset K_1 \subset K_2 \subset \cdots$

Set $\mathcal{K} = \bigcup_{i=1}^{\infty} K_i$.

- **Then** $D_1 \otimes_K \mathcal{K} \not\simeq D_2 \otimes_K \mathcal{K}$ and have same maximal subfields.

For *infinite* genus, one starts with $D_p = \left(\frac{-1, p}{\mathbb{Q}}\right)$, $p \equiv 3 \pmod 4$.

Note that $\mathcal{K}$ is infinitely generated.

Theorem 2.

Let K be a finitely generated field. Then for any central division K-algebra D of degree prime to $\mathrm{char}\, K$, the genus $\mathbf{gen}(D)$ is finite.

Theorem 2.

Let K be a finitely generated field. Then for any central division K-algebra D of degree prime to $\mathrm{char}\, K$*, the genus* $\mathbf{gen}(D)$ *is* *finite*.

- Proofs of both theorems use *analysis of ramification* and info about *unramified Brauer group*.

Theorem 2.

Let K be a finitely generated field. Then for any central division K-algebra D of degree prime to $\mathrm{char}\, K$, the genus $\mathbf{gen}(D)$ is finite.

- Proofs of both theorems use *analysis of ramification* and info about *unramified Brauer group*.

BASIC FACT: *Let v be a discrete valuation of K, and n be prime to characteristic of residue field $K^{(v)}$.*

Theorem 2.

Let K be a finitely generated field. Then for any central division K-algebra D of degree prime to $\mathrm{char}\, K$, *the genus* $\mathbf{gen}(D)$ *is finite.*

- Proofs of both theorems use *analysis of ramification* and info about *unramified Brauer group*.

BASIC FACT: *Let v be a discrete valuation of K, and n be prime to characteristic of residue field $K^{(v)}$.*

If D_1 and D_2 are central division K-algebras of degree n having same maximal subfields, then either both algebras are ramified *at v or both are* unramified.

- To prove finiteness of $\mathbf{gen}(D)$, one shows that a finitely generated field K can be equipped with a set V of discrete valuations such that

$$ {}_{n}\mathrm{Br}(K)_V \quad \text{is finite.} $$

- To prove finiteness of $\mathbf{gen}(D)$, one shows that a finitely generated field K can be equipped with a set V of discrete valuations such that

$$ {}_n\mathrm{Br}(K)_V \quad \text{is finite.} $$

Question. *Does there exist a quaternion division algebra D over $K = k(C)$, where C is a smooth geometrically integral curve over a number field k, such that*

$$ |\mathbf{gen}(D)| > 1? $$

- To prove finiteness of $\mathbf{gen}(D)$, one shows that a finitely generated field K can be equipped with a set V of discrete valuations such that

$$ {}_n\mathrm{Br}(K)_V \ \text{ is finite.} $$

Question. *Does there exist a quaternion division algebra D over $K = k(C)$, where C is a smooth geometrically integral curve over a number field k, such that*

$$ |\mathbf{gen}(D)| > 1? $$

- The answer is **not** known for *any* finitely generated K.

- To prove finiteness of $\mathbf{gen}(D)$, one shows that a finitely generated field K can be equipped with a set V of discrete valuations such that

$$ {}_n\mathrm{Br}(K)_V \ \text{ is finite.} $$

Question. *Does there exist a quaternion division algebra D over $K = k(C)$, where C is a smooth geometrically integral curve over a number field k, such that*

$$ |\mathbf{gen}(D)| > 1? $$

- The answer is **not** known for *any* finitely generated K.

- One can construct nonisomorphic D_1 and D_2 that have same ramification everywhere.

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- Let G_1 and G_2 be semi-simple groups over a field K.

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- Let G_1 and G_2 be semi-simple groups over a field K.

  G_1 & G_2 have *same isomorphism classes of maximal K-tori* **if** every maximal K-torus T_1 of G_1 is K-isomorphic to a maximal K-torus T_2 of G_2, and vice versa.

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- Let G_1 and G_2 be semi-simple groups over a field K.

  G_1 & G_2 have *same isomorphism classes of maximal K-tori* **if** every maximal K-torus T_1 of G_1 is K-isomorphic to a maximal K-torus T_2 of G_2, and vice versa.

- Let G be an absolutely almost simple K-group.

- To define the genus of an algebraic group, we replace maximal subfields with *maximal tori* in the definition of genus of division algebra.

- Let G_1 and G_2 be semi-simple groups over a field K.

  G_1 & G_2 have *same isomorphism classes of maximal K-tori* **if** every maximal K-torus T_1 of G_1 is K-isomorphic to a maximal K-torus T_2 of G_2, and vice versa.

- Let G be an absolutely almost simple K-group.

  $\mathbf{gen}_K(G) =$ set of isomorphism classes of K-forms G' of G having same K-isomorphism classes of maximal K-tori.

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 1$'$. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2$'$. *When is* $\mathbf{gen}_K(G)$ *finite?*

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 3 (Prasad-A.R.)

Let G be an absolutely almost simple simply connected algebraic group over a number field K.

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 3 (Prasad-A.R.)

Let G be an absolutely almost simple simply connected algebraic group over a number field K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 3 (Prasad-A.R.)

Let G be an absolutely almost simple simply connected algebraic group over a number field K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

(2) *If G is not of type A_n, D_{2n+1} or E_6, then* $|\mathbf{gen}_K(G)| = 1$.

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 3 (Prasad-A.R.)

Let G be an absolutely almost simple simply connected algebraic group over a number field K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

(2) *If G is not of type* A_n, D_{2n+1} *or* E_6, *then* $|\mathbf{gen}_K(G)| = 1$.

Conjecture. (1) *For* $K = k(x)$, *k a number field, and G an absolutely almost simple simply connected K-group with* $|Z(G)| \leqslant 2$, *we have* $|\mathbf{gen}_K(G)| = 1$;

Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

Question 2′. *When is* $\mathbf{gen}_K(G)$ *finite?*

Theorem 3 (Prasad-A.R.)

Let G be an absolutely almost simple simply connected algebraic group over a number field K.

(1) $\mathbf{gen}_K(G)$ *is finite;*

(2) *If G is not of type* A_n, D_{2n+1} *or* E_6, *then* $|\mathbf{gen}_K(G)| = 1$.

Conjecture. (1) *For* $K = k(x)$, *k a number field, and G an absolutely almost simple simply connected K-group with* $|Z(G)| \leqslant 2$, *we have* $|\mathbf{gen}_K(G)| = 1$;

(2) *If G is an absolutely almost simple group over a finitely generated field K of "good" characteristic then* $\mathbf{gen}_K(G)$ *is finite.*

- Results for division algebras do **not** automatically imply results for $G = \mathrm{SL}_{m,D}$.

- Results for division algebras do **not** automatically imply results for $G = \mathrm{SL}_{m,D}$.

Theorem 4.

(1) *Let D be a central division algebra of exponent 2 over $K = k(x_1, \ldots, x_r)$ where k is a number field or a finite field of characteristic $\neq 2$. Then for $G = \mathrm{SL}_{m,D}$ $(m \geqslant 1)$ we have $|\mathbf{gen}_K(G)| = 1$.*

- Results for division algebras do **not** automatically imply results for $G = \mathrm{SL}_{m,D}$.

Theorem 4.

(1) *Let* D *be a central division algebra of exponent 2 over* $K = k(x_1, \ldots, x_r)$ *where* k *is a number field or a finite field of characteristic* $\neq 2$. *Then for* $G = \mathrm{SL}_{m,D}$ $(m \geqslant 1)$ *we have* $|\mathbf{gen}_K(G)| = 1$.

(2) *Let* $G = \mathrm{SL}_{m,D}$, *where* D *is a central division algebra over a finitely generated field* K *with* $\mathrm{char}\, K$ *prime to degree of* D. *Then* $\mathbf{gen}_K(G)$ *is finite.*

Theorem 5.

Let $K = k(C)$ where C is a geometrically integral smooth curve over a number field k, and let G be either

- $\mathrm{Spin}_n(q)$, *q a quadratic form over K and n is odd, or*
- $\mathrm{SU}_n(h)$, *h a hermitian form over a quadratic extension L/K.*

Theorem 5.

Let $K = k(C)$ where C is a geometrically integral smooth curve over a number field k, and let G be either

- $\mathrm{Spin}_n(q)$, *q a quadratic form over K and n is odd, or*
- $\mathrm{SU}_n(h)$, *h a hermitian form over a quadratic extension L/K.*

Then $\mathbf{gen}_K(G)$ *is finite.*

Theorem 5.

Let $K = k(C)$ where C is a geometrically integral smooth curve over a number field k, and let G be either

- $\mathrm{Spin}_n(q)$, *q a quadratic form over K and n is odd, or*
- $\mathrm{SU}_n(h)$, *h a hermitian form over a quadratic extension L/K.*

Then $\mathbf{gen}_K(G)$ *is finite.*

Theorem 6.

Let G be a simple algebraic group of type G_2.

Theorem 5.

Let $K = k(C)$ *where* C *is a geometrically integral smooth curve over a number field* k*, and let* G *be either*

- $\mathrm{Spin}_n(q)$*,* q *a quadratic form over* K *and* n *is* *odd**, or*
- $\mathrm{SU}_n(h)$*,* h *a hermitian form over a quadratic extension* L/K.

Then $\mathbf{gen}_K(G)$ *is finite.*

Theorem 6.

Let G *be a simple algebraic group of type* G_2.

(1) *If* $K = k(x)$*, where* k *is a number field, then* $|\mathbf{gen}_K(G)| = 1$;

Theorem 5.

Let $K = k(C)$ where C is a geometrically integral smooth curve over a number field k, and let G be either

- $\mathrm{Spin}_n(q)$, *q a quadratic form over K and n is odd, or*
- $\mathrm{SU}_n(h)$, *h a hermitian form over a quadratic extension L/K.*

Then $\mathbf{gen}_K(G)$ *is finite.*

Theorem 6.

Let G be a simple algebraic group of type G_2.

(1) *If $K = k(x)$, where k is a number field, then $|\mathbf{gen}_K(G)| = 1$;*

(2) *If $K = k(x_1, \ldots, x_r)$ or $k(C)$, where k is a number field, then $\mathbf{gen}_K(G)$ is finite.*

- Adequate substitute for unramified algebras in case of algebraic groups is algebraic groups with good reduction.

- Adequate substitute for unramified algebras in case of algebraic groups is algebraic groups with good reduction.

Definition.

Let G be an absolutely almost simple algebraic group over K, v be a discrete valuation of K.

- Adequate substitute for unramified algebras in case of algebraic groups is algebraic groups with good reduction.

Definition.

Let G be an absolutely almost simple algebraic group over K, v be a discrete valuation of K.

G has good reduction at v if there exists a reductive group scheme $\mathcal{G}$ over valuation ring $\mathcal{O}_v$ with generic fibre G and special fiber (reduction)

$$\underline{G}^{(v)} = \mathcal{G} \otimes_{\mathcal{O}_v} K^{(v)}$$

a connected simple group of same type as G.

- $G = \mathrm{SL}_{1,D}$ has good reduction if there exists Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that $\mathcal{A} \otimes_{\mathcal{O}_v} K \simeq D$

- $G = \mathrm{SL}_{1,D}$ has good reduction if there exists Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that $\mathcal{A} \otimes_{\mathcal{O}_v} K \simeq D$ ($\Leftrightarrow$ D is unramified at v)

- $G = \mathrm{SL}_{1,D}$ has good reduction if there exists Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that $\mathcal{A} \otimes_{\mathcal{O}_v} K \simeq D$ ($\Leftrightarrow$ D is unramified at v)

- $G = \mathrm{Spin}_n(q)$ has good reduction if q is similar to a form over $\mathcal{O}_v$ with discriminant in $\mathcal{O}_v^{\times}$

- $G = \mathrm{SL}_{1,D}$ has good reduction if there exists Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that $\mathcal{A} \otimes_{\mathcal{O}_v} K \simeq D$ ($\Leftrightarrow$ D is unramified at v)

- $G = \mathrm{Spin}_n(q)$ has good reduction if q is similar to a form over $\mathcal{O}_v$ with discriminant in $\mathcal{O}_v^{\times}$

Theorem 7.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

- $G = \mathrm{SL}_{1,D}$ has good reduction if there exists Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that $\mathcal{A} \otimes_{\mathcal{O}_v} K \simeq D$ ($\Leftrightarrow$ D is unramified at v)

- $G = \mathrm{Spin}_n(q)$ has good reduction if q is similar to a form over $\mathcal{O}_v$ with discriminant in $\mathcal{O}_v^\times$

Theorem 7.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

Assume that $K^{(v)}$ is finitely generated, and G has good reduction at v.

- $G = \mathrm{SL}_{1,D}$ has good reduction if there exists Azumaya $\mathcal{O}_v$-algebra $\mathcal{A}$ such that $\mathcal{A} \otimes_{\mathcal{O}_v} K \simeq D$ ($\Leftrightarrow$ D is unramified at v)

- $G = \mathrm{Spin}_n(q)$ has good reduction if q is similar to a form over $\mathcal{O}_v$ with discriminant in $\mathcal{O}_v^\times$

Theorem 7.

Let G be an absolutely almost simple simply connected group over K, and v be a discrete valuation of K.

Assume that $K^{(v)}$ is finitely generated, and G has good reduction at v.

Then *every* $G' \in \mathbf{gen}_K(G)$ *has good reduction at v, and reduction* $\underline{G}'^{(v)} \in \mathbf{gen}_{K^{(v)}}(\underline{G}^{(v)})$.

Let K be a finitely generated field equipped with a set V of discrete valuations such that:

(I) for any $a \in K^{\times}$, set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite;

(II) for every $v \in V$, residue field $K^{(v)}$ is finitely generated.

Let K be a finitely generated field equipped with a set V of discrete valuations such that:

(I) for any $a \in K^{\times}$, set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite;

(II) for every $v \in V$, residue field $K^{(v)}$ is finitely generated.

Let K be a finitely generated field equipped with a set V of discrete valuations such that:

(I) for any $a \in K^{\times}$, set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite;

(II) for every $v \in V$, residue field $K^{(v)}$ is finitely generated.

Let K be a finitely generated field equipped with a set V of discrete valuations such that:

(I) for any $a \in K^{\times}$, set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite;

(II) for every $v \in V$, residue field $K^{(v)}$ is finitely generated.

Corollary.

Let G be an absolutely almost simple simply connected K-group. There exists a finite subset $S \subset V$ (depending on G) such that <u>every</u> $G' \in \mathbf{gen}_K(G)$ has good reduction at <u>all</u> $v \in V \setminus S$.

So, finiteness of $\mathbf{gen}_K(G)$ reduces to the following.

So, finiteness of $\mathbf{gen}_K(G)$ reduces to the following.

Question.

Can a finitely generated field K (of "good" characteristic etc.) be equipped with a set V of discrete valuations that satisfies (I) & (II) *and also*

So, finiteness of $\mathbf{gen}_K(G)$ reduces to the following.

Question.

Can a finitely generated field K (of "good" characteristic etc.) be equipped with a set V of discrete valuations that satisfies (I) & (II) *and also*

(Φ) *set of K-isomorphism classes of (inner) K-forms G' of G having good reduction at all $v \in V \setminus S$ is finite, for any finite $S \subset V$?*

So, finiteness of $\mathbf{gen}_K(G)$ reduces to the following.

Question.

Can a finitely generated field K *(of "good" characteristic etc.) be equipped with a set* V *of discrete valuations that satisfies* (I) & (II) *and also*

(Φ) *set of* K*-isomorphism classes of (inner)* K*-forms* G' *of* G *having good reduction at all* $v \in V \setminus S$ *is finite, for any finite* $S \subset V$*?*

- To prove above finiteness theorems for $\mathbf{gen}_K(G)$, we constructed such V in special situations.

For Spin_n over $K = k(C)$, k a number field:

For Spin_n over $K = k(C)$, k a number field:

- use Milnor's conjecture to reduces to finiteness of *unramified cohomology groups* $H^i(K, \mu_2)_V$;

For Spin_n over $K = k(C)$, k a number field:

- use Milnor's conjecture to reduces to finiteness of *unramified cohomology groups* $H^i(K, \mu_2)_V$;
- prove finiteness of $H^i(K, \mu_2)_V$ for all $i \geqslant 1$;

For Spin_n over $K = k(C)$, k a number field:

- use Milnor's conjecture to reduces to finiteness of *unramified cohomology groups* $H^i(K, \mu_2)_V$;
- prove finiteness of $H^i(K, \mu_2)_V$ for all $i \geqslant 1$;
- difficult case $i = 3$;

  when C is *projective* finiteness can be derived from results of Kato and Jannsen,

For Spin_n over $K = k(C)$, k a number field:

- use Milnor's conjecture to reduces to finiteness of *unramified cohomology groups* $H^i(K, \mu_2)_V$;
- prove finiteness of $H^i(K, \mu_2)_V$ for all $i \geqslant 1$;
- difficult case $i = 3$;

  when C is *projective* finiteness can be derived from results of Kato and Jannsen,

  but we need result for C affine.

For Spin_n over $K = k(C)$, k a number field:

- use Milnor's conjecture to reduces to finiteness of *unramified cohomology groups* $H^i(K, \mu_2)_V$;
- prove finiteness of $H^i(K, \mu_2)_V$ for all $i \geqslant 1$;
- difficult case $i = 3$;

  when C is *projective* finiteness can be derived from results of Kato and Jannsen,

  but we need result for C affine.

Presumably, V should be independent of type of G (divisorial set of places)

Theorem 8.

Suppose V satisfies (I) *&* (Φ). *Then map*

$$H^1(K, \overline{G}) \longrightarrow \prod_{v \in V} H^1(K_v, \overline{G})$$

for adjoint group $\overline{G}$ is proper. In particular, its kernel $Ш(\overline{G})$ is finite.

Theorem 8.

Suppose V satisfies (I) *&* (Φ). *Then map*

$$H^1(K, \overline{G}) \longrightarrow \prod_{v \in V} H^1(K_v, \overline{G})$$

for adjoint group $\overline{G}$ is proper. In particular, its kernel $Ш(\overline{G})$ is finite.

True for:

Theorem 8.

Suppose V satisfies (I) & (Φ). Then map

$$H^1(K, \overline{G}) \longrightarrow \prod_{v \in V} H^1(K_v, \overline{G})$$

for adjoint group $\overline{G}$ is proper. In particular, its kernel $\text{Ш}(\overline{G})$ is finite.

True for:

- PSL_n over a finitely generated field K, $(n, \mathrm{char}\, K) = 1$;

Theorem 8.

Suppose V satisfies (I) & (Φ). Then map

$$H^1(K, \overline{G}) \longrightarrow \prod_{v \in V} H^1(K_v, \overline{G})$$

for adjoint group $\overline{G}$ is proper. In particular, its kernel $Ш(\overline{G})$ is finite.

True for:

- PSL_n over a finitely generated field K, $(n, \mathrm{char}\, K) = 1$;
- $\mathrm{SO}_n(q)$ over $K = k(C)$, k a number field;

Theorem 8.

Suppose V satisfies (I) *&* (Φ). *Then map*

$$H^1(K, \overline{G}) \longrightarrow \prod_{v \in V} H^1(K_v, \overline{G})$$

for adjoint group $\overline{G}$ is proper. In particular, its kernel $Ш(\overline{G})$ is finite.

True for:

- PSL_n over a finitely generated field K, $(n, \mathrm{char}\, K) = 1$;
- $\mathrm{SO}_n(q)$ over $K = k(C)$, k a number field;
- G of type G_2 over $K = k(C)$, k a number field.

Theorem 9.

Let G_0 be an absolutely almost simple simply connected group over a finitely generated field k of characteristic zero.

Theorem 9.

Let G_0 be an absolutely almost simple simply connected group over a finitely generated field k of characteristic zero. Set $G = G_0 \otimes_k K$ where $K = k(x)$.

Theorem 9.

Let G_0 be an absolutely almost simple simply connected group over a finitely generated field k of characteristic zero. Set $G = G_0 \otimes_k K$ where $K = k(x)$.

Then any $H \in \mathbf{gen}_K(G)$ is of the form

$$H = H_0 \otimes_k K \quad \text{for some} \quad H_0 \in \mathbf{gen}_k(G_0).$$

Theorem 9.

Let G_0 be an absolutely almost simple simply connected group over a finitely generated field k of characteristic zero. Set $G = G_0 \otimes_k K$ where $K = k(x)$.

Then any $H \in \mathbf{gen}_K(G)$ is of the form

$$H = H_0 \otimes_k K \ \text{ for some } \ H_0 \in \mathbf{gen}_k(G_0).$$

Corollary.

In above notations, if $\mathbf{gen}_k(G_0)$ *is finite, then so is* $\mathbf{gen}_K(G)$. *In particular,* $\mathbf{gen}_K(G)$ *is finite if* k *is a number field.*

- Theorem and Corollary remain valid for $K = k(x_1, \ldots, x_r)$.

- Theorem and Corollary remain valid for $K = k(x_1, \ldots, x_r)$.

- We are **not** saying that

  if $H_0 \in \mathbf{gen}_k(G_0)$ then $H := H_0 \otimes_k K \in \mathbf{gen}_K(G)$.

  On the contrary:

- Theorem and Corollary remain valid for $K = k(x_1, \ldots, x_r)$.

- We are **not** saying that

  if $H_0 \in \mathbf{gen}_k(G_0)$ then $H := H_0 \otimes_k K \in \mathbf{gen}_K(G)$.

  On the contrary:

Corollary ("Killing the genus")

Let G be a group of type G_2 over a finitely generated field k of characteristic zero. Set $F = k(x_1, \ldots, x_6)$. Then $|\mathbf{gen}_F(G \otimes_k F)| = 1$.

- Theorem and Corollary remain valid for $K = k(x_1, \ldots, x_r)$.

- We are **not** saying that

$$\text{if} \quad H_0 \in \mathbf{gen}_k(G_0) \quad \text{then} \quad H := H_0 \otimes_k K \in \mathbf{gen}_K(G).$$

On the contrary:

Corollary ("Killing the genus")

Let G be a group of type G_2 over a finitely generated field k of characteristic zero. Set $F = k(x_1, \ldots, x_6)$. Then $|\mathbf{gen}_F(G \otimes_k F)| = 1$.

Proof uses properties of Pfister forms.

Analog for $G = \mathrm{SL}_{1,D}$, D a division algebra of degree n:

Analog for $G = \mathrm{SL}_{1,D}$, D a division algebra of degree n:

Set $F = k(x_1, \ldots, x_{n-1})$. *Then* $\mathbf{gen}_F(G \otimes_k F)$ *consists of*

$$G' = \mathrm{SL}_{1,D'} \quad \textit{where} \quad D' = D_1 \otimes_k F$$

and $\langle [D] \rangle = \langle [D_1] \rangle$ *in* $\mathrm{Br}(k)$*;*

Analog for $G = \mathrm{SL}_{1,D}$, D a division algebra of degree n:

Set $F = k(x_1, \ldots, x_{n-1})$. *Then* $\mathbf{gen}_F(G \otimes_k F)$ *consists of*

$$G' = \mathrm{SL}_{1,D'} \quad \text{where} \quad D' = D_1 \otimes_k F$$

and $\langle [D] \rangle = \langle [D_1] \rangle$ *in* $\mathrm{Br}(k)$; *in particular,* $|\mathbf{gen}_F(G \otimes_k F)| \leqslant n$.

Analog for $G = \mathrm{SL}_{1,D}$, D a division algebra of degree n:

Set $F = k(x_1, \ldots, x_{n-1})$. *Then* $\mathbf{gen}_F(G \otimes_k F)$ *consists of*

$$G' = \mathrm{SL}_{1,D'} \quad \textit{where} \quad D' = D_1 \otimes_k F$$

and $\langle [D] \rangle = \langle [D_1] \rangle$ *in* $\mathrm{Br}(k)$; *in particular,* $|\mathbf{gen}_F(G \otimes_k F)| \leqslant n$.

Proof uses:

Analog for $G = \mathrm{SL}_{1,D}$, D a division algebra of degree n:

Set $F = k(x_1, \ldots, x_{n-1})$. *Then* $\mathbf{gen}_F(G \otimes_k F)$ *consists of*

$$G' = \mathrm{SL}_{1,D'} \quad \textit{where} \quad D' = D_1 \otimes_k F$$

and $\langle [D] \rangle = \langle [D_1] \rangle$ *in* $\mathrm{Br}(k)$; *in particular,* $|\mathbf{gen}_F(G \otimes_k F)| \leqslant n$.

Proof uses:

- Amitsur's Theorem;

Analog for $G = \mathrm{SL}_{1,D}$, D a division algebra of degree n:

Set $F = k(x_1, \ldots, x_{n-1})$. *Then* $\mathbf{gen}_F(G \otimes_k F)$ *consists of*

$$G' = \mathrm{SL}_{1,D'} \quad \textit{where} \quad D' = D_1 \otimes_k F$$

and $\langle [D] \rangle = \langle [D_1] \rangle$ *in* $\mathrm{Br}(k)$; *in particular,* $|\mathbf{gen}_F(G \otimes_k F)| \leqslant n$.

Proof uses:

- Amitsur's Theorem;
- fact that function field of Severi-Brauer variety of algebra of degree n is a degree n extensions of $k(x_1, \ldots, x_{n-1})$.

Analog for $G = \mathrm{SL}_{1,D}$, D a division algebra of degree n:

Set $F = k(x_1, \ldots, x_{n-1})$. *Then* $\mathbf{gen}_F(G \otimes_k F)$ *consists of*

$$G' = \mathrm{SL}_{1,D'} \quad \textit{where} \quad D' = D_1 \otimes_k F$$

and $\langle [D] \rangle = \langle [D_1] \rangle$ *in* $\mathrm{Br}(k)$; *in particular,* $|\mathbf{gen}_F(G \otimes_k F)| \leqslant n$.

Proof uses:

- Amitsur's Theorem;
- fact that function field of Severi-Brauer variety of algebra of degree n is a degree n extensions of $k(x_1, \ldots, x_{n-1})$.

Corollary

Let D *be a quaternion algebra over a field* k *of* char $\neq 2$. *Then* $\mathbf{gen}(D \otimes_k k(x))$ *is trivial.*