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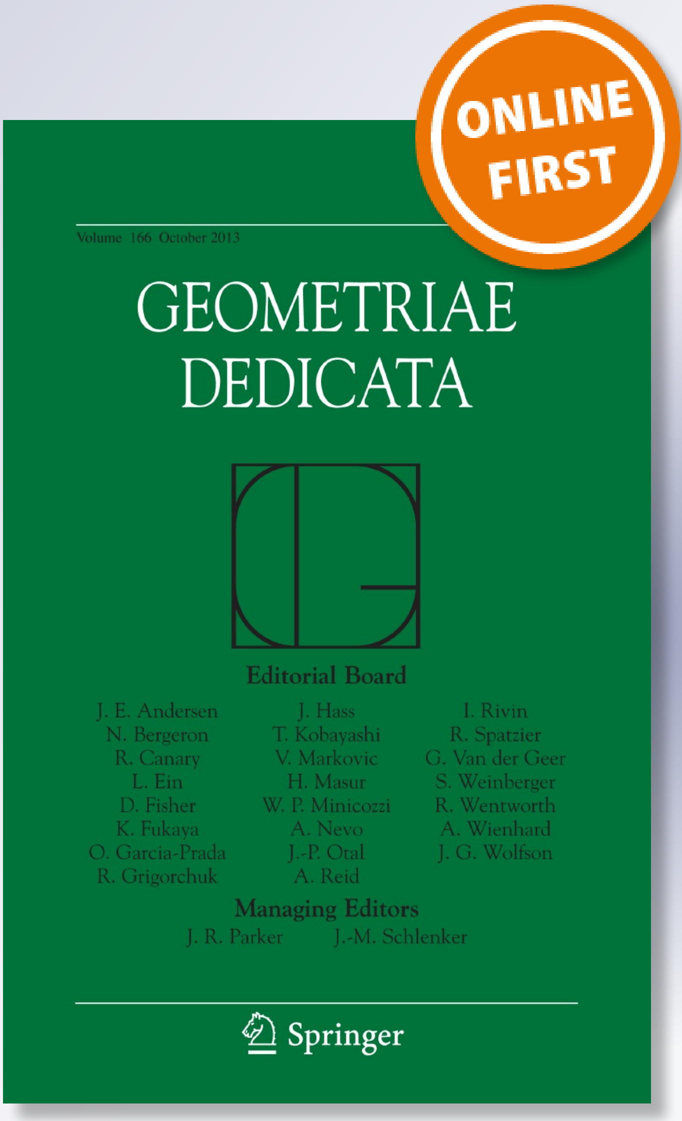

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ORIGINAL PAPER

On the fields generated by the lengths of closed geodesics in locally symmetric spaces

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Abstract This paper is the next installment of our analysis of length-commensurable locally symmetric spaces begun in Prasad and Rapinchuk (Publ Math IHES 109:113–184, 2009). For a Riemannian manifold M, we let $L(M)$ be the weak length spectrum of M, i.e. the set of lengths of all closed geodesics in M, and let $\mathscr{F}(M)$ denote the subfield of $\mathbb{R}$ generated by $L(M)$. Let now M_i be an arithmetically defined locally symmetric space associated with a simple algebraic $\mathbb{R}$-group G_i for $i = 1, 2$. Assuming Schanuel's conjecture from transcendental number theory, we prove (under some minor technical restrictions) the following dichotomy: either M_1 and M_2 are length-commensurable, i.e. $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$, or the compositum $\mathscr{F}(M_1)\mathscr{F}(M_2)$ has infinite transcendence degree over $\mathscr{F}(M_i)$ for at least one $i = 1$ or 2 (which means that the sets $L(M_1)$ and $L(M_2)$ are very different).

Keywords Locally symmetric spaces · Arithmetic groups

Mathematics Subject Classification 20G30 · 53C22 · 53C35

1 Introduction

This paper is a sequel to our work [14] on length-commensurable and isospectral locally symmetric spaces. Questions about length-commensurable and isospectral manifolds have received considerable attention in recent years (cf. [3,18]; a detailed survey is given in [13]). The question whether any two isospectral compact Riemannian manifolds are isometric was

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Dedicated to G. D. Mostow on his 90th birthday.

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reformulated by Mark Kac in a very appealing way as "Can you hear the shape of a drum?". In [14], we were able to resolve some of these questions for arithmetically defined locally symmetric spaces using the new notion of "weak commensurability" of Zariski-dense subgroups of semi-simple algebraic groups; the current paper expands and generalizes this work in several directions. More precisely, given a Riemannian manifold M, the (weak) *length spectrum* $L(M)$ is the set of lengths of all closed geodesics in M, and two Riemannian manifolds M_1 and M_2 are said to be *iso-length* if $L(M_1) = L(M_2)$, and *length-commensurable* if $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$. It was shown in [14] that length-commensurability has strong consequences, one of which is that length-commensurable arithmetically defined locally symmetric spaces of certain types are necessarily commensurable, i.e. they have a common finite-sheeted cover. In the current paper, we will study the following two interrelated questions: *Suppose that (locally symmetric spaces)* M_1 and M_2 *are not length-commensurable, i.e.* $\mathbb{Q} \cdot L(M_1) \neq \mathbb{Q} \cdot L(M_2)$. *Then*

(1) *How different are the sets* $L(M_1)$ *and* $L(M_2)$ (*or the sets* $\mathbb{Q} \cdot L(M_1)$ *and* $\mathbb{Q} \cdot L(M_2)$)?
(2) *Can* $L(M_1)$ *and* $L(M_2)$ *be related in* any *reasonable way?*

One can ask a variety of specific questions that fit the general framework provided by (1) and (2): for example, can $L(M_1)$ and $L(M_2)$ differ only in a finite number of elements, in other words, can the symmetric difference $L(M_1) \vartriangle L(M_2)$ be finite? Regarding (2), the relationship between $L(M_1)$ and $L(M_2)$ that makes most sense geometrically is that of *similarity*, requiring that there be a real number $\alpha > 0$ such that

$$L(M_2) = \alpha \cdot L(M_1) \ \ (\text{or } \mathbb{Q} \cdot L(M_2) = \alpha \cdot \mathbb{Q} \cdot L(M_1)\ ),$$

which geometrically means that M_1 and M_2 can be made iso-length (resp., length-commensurable) by scaling the metric on one of them. At the same time, one can consider more general relationships with less apparent geometric context like *polynomial equivalence* which means that there exist polynomials $p(x_1, \ldots, x_s)$ and $q(y_1, \ldots, y_t)$ with real coefficients such that for any $\lambda \in L(M_1)$ one can find $\mu_1, \ldots, \mu_s \in L(M_2)$ so that $\lambda = p(\mu_1, \ldots, \mu_s)$, and conversely, for any $\mu \in L(M_2)$ there exist $\lambda_1, \ldots, \lambda_t \in L(M_1)$ such that $\mu = q(\lambda_1, \ldots, \lambda_t)$. Our results show, in particular, that for most arithmetically defined locally symmetric spaces the fact that they are not length-commensurable implies that the sets $L(M_1)$ and $L(M_2)$ differ very significantly and in fact cannot be related by any generalized form of polynomial equivalence (cf. Sect. 7, particularly Corollary 7.3).

To formalize the idea of "polynomial relations" between the weak length spectra of Riemannian manifolds, we need to introduce some additional notations and definitions. For a Riemannian manifold M, we let $\mathscr{F}(M)$ denote the subfield of $\mathbb{R}$ generated by the set $L(M)$. Given two Riemannian manifolds M_1 and M_2, for $i \in \{1, 2\}$, we set $\mathscr{F}_i = \mathscr{F}(M_i)$ and consider the following condition

(T_i) the compositum $\mathscr{F}_1\mathscr{F}_2$ has infinite transcendence degree over the field $\mathscr{F}_{3-i}$.

In simple terms, the fact that condition (T_i) holds means that $L(M_i)$ contains "many" elements which are algebraically independent of all the elements of $L(M_{3-i})$. The goal of this paper is to prove that (T_i) indeed holds for at least one $i \in \{1, 2\}$ in various situations where M_1 and M_2 are pairwise non-length-commensurable locally symmetric spaces. These results can be used to prove a number of results on the nonexistence of nontrivial dependence between the weak length spectra along the lines indicated above—cf. Sect. 7. Here we only mention that (T_i) implies the following condition

(N_i) $L(M_i) \not\subset A \cdot \mathbb{Q} \cdot L(M_{3-i})$ for any finite set A of real numbers,

which informally means that the weak length spectrum of M_i is "very far"from being similar to the length spectrum of M_{3-i}.

To give the precise statements of our main results, we need to fix some notations most of which will be used throughout the paper. Let G_1 and G_2 be connected absolutely almost simple real algebraic groups such that $\mathcal{G}_i := G_i(\mathbb{R})$ is noncompact for both $i = 1$ and 2. (In Sects. 2–5 we will assume that both G_1 and G_2 are of adjoint type.) We fix a maximal compact subgroup $\mathcal{K}_i$ of $\mathcal{G}_i$, and let $\mathfrak{X}_i = \mathcal{K}_i \backslash \mathcal{G}_i$ denote the associated symmetric space. Furthermore, let $\Gamma_i \subset \mathcal{G}_i$ be a discrete torsion-free Zariski-dense subgroup, and let $\mathfrak{X}_{\Gamma_i} := \mathfrak{X}_i / \Gamma_i$ be the corresponding locally symmetric space. Set $M_i = \mathfrak{X}_{\Gamma_i}$ and $\mathscr{F}_i = \mathscr{F}(M_i)$. We also let K_{Γ_i} denote the subfield of $\mathbb{R}$ generated by the traces $\mathrm{TrAd}\gamma$ for $\gamma \in \Gamma_i$. Let w_i be the order of the (absolute) Weyl group of G_i.

Before formulating our results, we need to emphasize that the proofs *assume the validity of Schanuel's conjecture* in transcendental number theory (cf. Sect. 7), making the results *conditional.*

Theorem 1 (see Sect. 7; cf. Theorem 4.2) *Assume that the subgroups Γ_1 and Γ_2 are finitely generated (which is automatically the case if these subgroups are actually lattices).*

(1) If $w_1 > w_2$ then (T_1) holds;
(2) If $w_1 = w_2$ but $K_{\Gamma_1} \not\subset K_{\Gamma_2}$ then again (T_1) holds.

Thus, unless $w_1 = w_2$ and $K_{\Gamma_1} = K_{\Gamma_2}$, condition (T_i) holds for at least one $i \in \{1, 2\}$.

(We recall that $w_1 = w_2$ implies that either G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_n and the other of type C_n for some $n \geq 3$.)

Much more precise results are available when the groups Γ_1 and Γ_2 are arithmetic (cf. [14, Sect. 1] and Sect. 5 below regarding the notion of arithmeticity). As follows from Theorem 1, we only need to consider the case where $w_1 = w_2$ which we will assume. Then it is convenient to divide our results into three theorems, two of which treat the case where G_1 and G_2 are of the same Killing-Cartan type, and the third one the case where one of the groups is of type B_n and the other of type C_n for some $n \geq 3$ (we note that the combination of these three cases covers all possible situations where $w_1 = w_2$). When G_1 and G_2 are of the same type, we consider separately the cases where the common type is not one of the following: A_n, D_{2n+1} $(n > 1)$ and E_6 and where it is one of these types.

Theorem 2 (see Sect. 7; cf. Theorem 5.3) *Notations as above, assume that G_1 and G_2 are of the same Killing-Cartan type which is different from A_n, D_{2n+1} $(n > 1)$ and E_6 and that the subgroups Γ_1 and Γ_2 are arithmetic. Then either $M_1 := \mathfrak{X}_{\Gamma_1}$ and $M_2 := \mathfrak{X}_{\Gamma_2}$ are commensurable, hence $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$ and $\mathscr{F}_1 = \mathscr{F}_2$, or conditions (T_i) and (N_i) hold for at least one $i \in \{1, 2\}$.*

(We note that (T_i) and (N_i) may not hold for *both* $i = 1$ and 2; in fact it is possible that $L(M_1) \subset L(M_2)$, cf. Example 7.4.)

Theorem 3 (see Sect. 7; cf. Theorem 6.6) *Again, keep the above notations and assume that the common Killing-Cartan type of G_1 and G_2 is one of the following: A_n, $D_{2n+1} (n > 1)$ or E_6 and that the subgroups Γ_1 and Γ_2 are arithmetic. Assume in addition that $K_{\Gamma_i} \neq \mathbb{Q}$*

for at least one $i \in \{1, 2\}$. *Then either* $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$, *hence* $\mathscr{F}_1 = \mathscr{F}_2$ *(although* M_1 *and* M_2 *may not be commensurable), or conditions* (T_i) *and* (N_i) *hold for at least one* $i \in \{1, 2\}$.

These results can be used in various geometric situations. To illustrate the scope of possible applications, we will now give explicit statements for real hyperbolic manifolds (similar results are available for complex and quaternionic hyperbolic manifolds).

Corollary 1 (see Sect. 7) *Let* M_i $(i = 1, 2)$ *be the quotient of the real hyperbolic space* $\mathbb{H}^{d_i}$ *with* $d_i \neq 3$ *by a torsion-free Zariski-dense discrete subgroup* Γ_i *of* $G_i(\mathbb{R})$ *where* $G_i = \mathrm{PSO}(d_i, 1)$.

(i) If $d_1 > d_2$ *then conditions* (T_1) *and* (N_1) *hold.*
(ii) If $d_1 = d_2$ *but* $K_{\Gamma_1} \not\subset K_{\Gamma_2}$ *then again conditions* (T_1) *and* (N_1) *hold.*
Thus, unless $d_1 = d_2$ *and* $K_{\Gamma_1} = K_{\Gamma_2}$, *conditions* (T_i) *and* (N_i) *hold for at least one* $i \in \{1, 2\}$.
Assume now that $d_1 = d_2 =: d$ *and the subgroups* Γ_1 *and* Γ_2 *are arithmetic.*
(iii) If d *is either even or is congruent to* $3 (\mathrm{mod}\, 4)$, *then either* M_1 *and* M_2 *are commensurable, hence length-commensurable and* $\mathscr{F}_1 = \mathscr{F}_2$, *or* (T_i) *and* (N_i) *hold for at least one* $i \in \{1, 2\}$.
(iv) If $d \equiv 1 (\mathrm{mod}\, 4)$ *and in addition* $K_{\Gamma_i} \neq \mathbb{Q}$ *for at least one* $i \in \{1, 2\}$ *then either* M_1 *and* M_2 *are length-commensurable (although not necessarily commensurable), or conditions* (T_i) *and* (N_i) *hold for at least one* $i \in \{1, 2\}$.

The results of [5] enable us to consider the situation where one of the groups is of type B_n and the other is of type C_n.

Theorem 4 (see Sect. 8) *Notations as above, assume that* G_1 *is of type* B_n *and* G_2 *is of type* C_n *for some* $n \geq 3$ *and the subgroups* Γ_1 *and* Γ_2 *are arithmetic. Then either* (T_i) *and* (N_i) *hold for at least one* $i \in \{1, 2\}$, *or*

$$\mathbb{Q} \cdot L(M_2) = \lambda \cdot \mathbb{Q} \cdot L(M_1) \quad \text{where} \quad \lambda = \sqrt{\frac{2n+2}{2n-1}}.$$

The following interesting result holds for all types.

Theorem 5 (see Sect. 7; cf. Theorem 5.8) *For* $i = 1, 2$, *let* $M_i = \mathfrak{X}_{\Gamma_i}$ *be an arithmetically defined locally symmetric space, and assume that* $w_1 = w_2$. *If* M_2 *is compact and* M_1 *is not, then conditions* (T_1) *and* (N_1) *hold.*

Finally, we have the following statement which shows that the notion of "similarity" (or more precisely, "length-similarity") for arithmetically defined locally symmetric spaces is redundant.

Corollary 2 *Let* $M_i = \mathfrak{X}_{\Gamma_i}$ *for* $i = 1, 2$ *be arithmetically defined locally symmetric spaces. Assume that there exists* $\lambda \in \mathbb{R}_{>0}$ *such that*

$$\mathbb{Q} \cdot L(M_1) = \lambda \cdot \mathbb{Q} \cdot L(M_2).$$

Then

(i) if G_1 *and* G_2 *are of the same type which is different from* A_n, $D_{2n+1} (n > 1)$ *and* E_6, *then* M_1 *and* M_2 *are commensurable, hence length-commensurable;*

(ii) if G_1 and G_2 are of the same type which is one of the following: A_n, $D_{2n+1} (n > 1)$ or E_6 then, provided that $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$, the spaces M_1 and M_2 are length-commensurable (although not necessarily commensurable).

(See Corollary 7.6 for a more detailed statement.)

While the geometric results in [14] were derived from an analysis of the relationship between Zariski-dense subgroups of semi-simple algebraic groups called *weak commensurability*, the results described above require a more general and technical version of this notion which we call *weak containment*. We recall that given two semi-simple groups G_1 and G_2 over a field F and Zariski-dense subgroups $\Gamma_i \subset G_i(F)$ for $i = 1, 2$, two semi-simple elements $\gamma_i \in \Gamma_i$ are weakly commensurable if there exist maximal F-tori T_i of G_i such that $\gamma_i \in T_i(F)$, and for some characters χ_i of T_i (defined over an algebraic closure $\overline{F}$ of F), we have

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1.$$

Furthermore, Γ_1 and Γ_2 are weakly commensurable if every semi-simple element $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple element $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.

The following definition provides a generalization of the notion of weak commensurability which is adequate for our purposes.

Definition 1 Notations as above, semi-simple elements $\gamma_1^{(1)}, \ldots, \gamma_{m_1}^{(1)} \in \Gamma_1$ are *weakly contained* in Γ_2 if there are semi-simple elements $\gamma_1^{(2)}, \ldots, \gamma_{m_2}^{(2)} \in \Gamma_2$ such that

$$\chi_1^{(1)}(\gamma_1^{(1)}) \cdots \chi_{m_1}^{(1)}(\gamma_{m_1}^{(1)}) = \chi_1^{(2)}(\gamma_1^{(2)}) \cdots \chi_{m_2}^{(2)}(\gamma_{m_2}^{(2)}) \neq 1.$$

for some maximal F-tori $T_k^{(j)}$ of G_j containing $\gamma_k^{(j)}$ and some characters $\chi_k^{(j)}$ of $T_k^{(j)}$ for $j \in \{1, 2\}$ and $k \leq m_j$.

(It is easy to see that this property is independent of the choice of the maximal tori containing the elements in question.)

We also need the following.

Definition 2 (a) Let $T_1, \ldots, T_m$ be a finite collection of algebraic tori defined over a field K, and for each $i \leq m$, let $\gamma_i \in T_i(K)$. The elements $\gamma_1, \ldots, \gamma_m$ are called *multiplicatively independent* if a relation of the form

$$\chi_1(\gamma_1) \cdots \chi_m(\gamma_m) = 1,$$

where $\chi_j \in X(T_j)$, implies that

$$\chi_1(\gamma_1) = \cdots = \chi_m(\gamma_m) = 1.$$

(b) Let G be a semi-simple algebraic F-group. Semi-simple elements $\gamma_1, \ldots, \gamma_m \in G(F)$ are called *multiplicatively independent* if for some (equivalently, any) choice of maximal F-tori T_i of G such that $\gamma_i \in T_i(F)$ for $i \leq m$, these elements are multiplicatively independent in the sense of part (a).

We are now in a position to give a definition that plays the central role in the paper.

Definition 3 We say that Γ_1 and Γ_2 as above satisfy condition (C_i), where i =1 or 2, if for any $m \geq 1$ there exist semi-simple elements $\gamma_1, \ldots, \gamma_m \in \Gamma_i$ of infinite order that are multiplicatively independent and are *not* weakly contained in Γ_{3-i}.

Our main effort is focused on developing a series of conditions that guarantee the fact that Γ_1 and Γ_2 satisfy (C_i) for at least one $i \in \{1, 2\}$ (in fact, typically we are able to pin down the i). We note that in our situation one has an analog of lemma 2.4 of [14] for property (C_i), which allows us to assume where convenient that G_1 and G_2 are absolutely simple of adjoint type—we will make this additional assumption in Sects. 4, 5, and 8 where it helps to simplify some arguments. Before formulating a sample result, we would like to note that the notion of the trace subfield (field of definition) $K_{\Gamma_i} \subset F$ makes sense for *any* field F and not only for $F = \mathbb{R}$.

Theorem 6 (see Theorem 4.2) *Assume that* Γ_1 *and* Γ_2 *are finitely generated.*

(i) If $w_1 > w_2$ *then condition* (C_1) *holds;*
(ii) If $w_1 = w_2$ *but* $K_{\Gamma_1} \not\subset K_{\Gamma_2}$ *then again* (C_1) *holds.*

Thus, unless $w_1 = w_2$ *and* $K_{\Gamma_1} = K_{\Gamma_2}$, *condition* (C_i) *holds for at least one* $i \in \{1, 2\}$.

We prove much more precise results in the case where the Γ_i are arithmetic. The statements however are somewhat technical, and we refer the reader to Sect. 5 for their complete formulations.

The reader may have already noticed similarities in the statements of Theorem 1 and Theorem 6. The same similarities exist also between the "geometric" Theorems 2–4 and the corresponding "algebraic" results in Sect. 5. The precise connection between "algebra" and "geometry" is given by Proposition 7.1 which has the following consequence (Corollary 7.3):

If $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are locally symmetric spaces as above with finitely generated fundamental groups Γ_1 and Γ_2, then the fact that these groups satisfy condition (C_i) for some $i \in \{1, 2\}$ implies that the locally symmetric spaces satisfy conditions (T_i) and (N_i) for the same i.

It should be noted that the proof of Proposition 7.1 assumes the truth of Schanuel's conjecture, and in fact it is the only place in the paper where the latter is used. (So, since the "geometric" results in the paper, particularly those presented in Sect. 7, rely on Proposition 7.1, they depend on Schanuel's conjecture; at the same time, the results of Sects. 2–6 and the algebraic results of Sect. 8 are completely independent of it.) In conjunction with the results of Sect. 5, this provides a series of rather restrictive conditions on the arithmetic groups Γ_1 and Γ_2 in case (T_i) fails for both $i = 1$ and 2. Eventually, these conditions enable us to prove that if G_1 and G_2 are of the same type which is different from A_n, $D_{2n+1} (n > 1)$ or E_6 then $G_1 \simeq G_2$ over $K := K_{\Gamma_1} = K_{\Gamma_2}$ and hence the subgroups Γ_1 and Γ_2 are commensurable in the appropriate sense (viz., up to an isomorphism between G_1 and G_2), yielding the commensurability of the locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ (cf. Theorem 2). If G_1 and G_2 are of the same type which is one of the following A_n, $D_{2n+1} (n > 1)$ or E_6, then G_1 and G_2 may not be K-isomorphic, but using the results from [14], §9, and [15], we show that (under some minor restrictions) these groups necessarily have equivalent systems of maximal K-tori (see Sect. 6 for the precise definition) making the corresponding locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ length-commensurable, and thereby proving Theorem 3. To prove Theorem 4, we use the results of [5] that describe when two absolutely almost simple K-groups, one of type B_n and the other of type C_n $(n \geq 3)$, have the same isomorphism classes of maximal K-tori.

Notation For a field K, K_{sep} will denote a separable closure. Given a (finitely generated) field K of characteristic zero, we let V^K denote the set of (equivalence classes) of nontrivial valuations v of K with locally compact completion K_v. If $v \in V^K$ is nonarchimedean, then K_v is a finite extension of the p-adic field $\mathbb{Q}_p$ for some p; in the sequel this prime p will be denoted by p_v. Given a subset V of V^K consisting of nonarchimedean valuations, we set $\Pi_V = \{p_v | v \in V\}$.

2 Weak containment

The goal of this section is to derive several consequences of the relation of weak containment (see Definition 1 of the Introduction) that will be needed later. We begin with some definitions and results for algebraic tori. Given a torus T defined over a field K, we let K_T denote its (minimal) splitting field over K (contained in a fixed algebraic closure $\overline{K}$ of K). The following definition goes back to [10].

Definition 4 A K-torus T is called *K-irreducible* (or, *irreducible over* K) if it does not contain any proper K-subtori.

Recall that T is K-irreducible if and only if $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathrm{Gal}(K_T/K)$-module, cf. [10], Proposition 1. Now, let G be an absolutely almost simple algebraic K-group. For a maximal torus T of G, we let $\Phi = \Phi(G, T)$ denote the corresponding root system, and let $\mathrm{Aut}(\Phi)$ be the automorphism group of Φ. As usual, the Weyl group $W(\Phi) \subset \mathrm{Aut}(\Phi)$ will be identified with the Weyl group $W(G, T)$ of G relative to T. If T is defined over a field extension L of K, and L_T is the splitting field of T over L in an algebraic closure of the latter, then there is a natural injective homomorphism

$$\theta_T : \mathrm{Gal}(L_T/L) \to \mathrm{Aut}(\Phi).$$

Since $W(\Phi)$ acts absolutely irreducibly on $X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, we conclude that a maximal L-torus T of G such that $\theta_T(\mathrm{Gal}(L_T/L)) \supset W(G, T)$ is automatically L-irreducible. (We also recall for the convenience of further reference that if G is of inner type over L then $\theta_T(\mathrm{Gal}(L_T/L)) \subset W(G, T)$, cf. [14], Lemma 4.1.)

Definition 5 Let $T_1, \ldots, T_m$ be K-tori. We say that these tori are *independent* (over K) if their splitting fields $K_{T_1}, \ldots, K_{T_m}$ are linearly disjoint over K, i.e. the natural map

$$K_{T_1} \otimes_K \cdots \otimes_K K_{T_m} \longrightarrow K_{T_1} \cdots K_{T_m}$$

is an isomorphism.

Lemma 2.1 *Let $T_1, \ldots, T_m$ be K-tori, and for $i \leq m$, let $\gamma_i \in T_i(K)$ be an element of infinite order. Assume that $T_1, \ldots, T_m$ are independent, irreducible and nonsplit over some extension L of K. Then the elements $\gamma_1, \ldots, \gamma_m$ are multiplicatively independent (see Definition 2 in Sect. 1).*

Proof Suppose there exist characters $\chi_i \in X(T_i)$ such that

$$\chi_1(\gamma_1) \cdots \chi_m(\gamma_m) = 1.$$

Since $\chi_i(\gamma_i) \in L_{T_i}^{\times}$ and the tori $T_1, \ldots, T_m$ are independent over L, it follows that actually $\chi_i(\gamma_i) \in L^{\times}$ for all $i \leq m$. Then for any $\sigma \in \mathrm{Gal}(L_{T_i}/L)$ we have

$$(\sigma\chi_i - \chi_i)(\gamma_i) = 1. \tag{1}$$

Being a L-rational element of infinite order in an L-irreducible torus T_i, the element γ_i generates a Zariski-dense subgroup of the latter, so (1) implies that $\sigma\chi_i = \chi_i$. But $X(T_i)$ does not have nonzero $\mathrm{Gal}(L_{T_i}/L)$-fixed elements. Thus, $\chi_i = 0$ and $\chi_i(\gamma_i) = 1$. □

The following lemma is crucial for unscrambling relations of weak containment.

Lemma 2.2 *Let $T_1^{(1)}, \dots, T_{m_1}^{(1)}$ and $T_1^{(2)}, \dots, T_{m_2}^{(2)}$ be two finite families of algebraic K-tori, and suppose we are given a relation of the form*

$$\chi_1^{(1)}(\gamma_1^{(1)}) \cdots \chi_{m_1}^{(1)}(\gamma_{m_1}^{(1)}) = \chi_1^{(2)}(\gamma_1^{(2)}) \cdots \chi_{m_2}^{(2)}(\gamma_{m_2}^{(2)}), \tag{2}$$

where $\gamma_i^{(s)} \in T_i^{(s)}(K)$ and $\chi_i^{(s)} \in X(T_i^{(s)})$. Assume that $T_1^{(1)}, \dots, T_{m_1}^{(1)}$ are independent, irreducible and nonsplit over K. Then for every $i \leq m_1$ such that the corresponding character $\chi_i^{(1)}$ in (2) is nontrivial, there exists an integer $d_i > 0$ with the following property:

For any $\delta_i^{(1)} \in d_i X(T_i^{(1)})$ there are characters $\delta_j^{(2)} \in X(T_j^{(2)})$ for $j \leq m_2$ for which

$$\delta_i^{(1)}(\gamma_i^{(1)}) = \delta_1^{(2)}(\gamma_1^{(2)}) \cdots \delta_{m_2}^{(2)}(\gamma_{m_2}^{(2)}). \tag{3}$$

In addition, if $\gamma_i^{(1)}$ has infinite order and $\delta_i^{(1)} \neq 0$ then the common value in (3) is $\neq 1$.

Proof As the tori $T_1^{(1)}, \dots, T_{m_1}^{(1)}$ are independent over K, we have the natural isomorphism

$$\mathrm{Gal}(K_{T_1^{(1)}} \cdots K_{T_{m_1}^{(1)}}/K) \simeq \mathrm{Gal}(K_{T_1^{(1)}}/K) \times \cdots \times \mathrm{Gal}(K_{T_{m_1}^{(1)}}/K). \tag{4}$$

Since $T_i^{(1)}$ is K-irreducible and nonsplit, $X(T_i^{(1)})$ does not contain any nontrivial $\mathrm{Gal}(K_{T_i^{(1)}}/K)$-fixed elements. So, it follows from (4) that there exists $\sigma \in \mathrm{Gal}(\overline{K}/K)$ such that $\sigma\chi_i^{(1)} \neq \chi_i^{(1)}$ but $\sigma\chi_j^{(1)} = \chi_j^{(1)}$ for $j \neq i$. Applying $\sigma - 1$ to (2), we obtain

$$\mu_i^{(1)}(\gamma_i^{(1)}) = \mu_1^{(2)}(\gamma_1^{(2)}) \cdots \mu_{m_2}^{(2)}(\gamma_{m_2}^{(2)}), \tag{5}$$

where $\mu_j^{(s)} = \sigma\chi_j^{(s)} - \chi_j^{(s)}$, noting that $\mu_i^{(1)} \neq 0$. Again, since $T_i^{(1)}$ is K-irreducible and nonsplit, the $\mathrm{Gal}(\overline{K}/K)$-submodule of $X(T_i^{(1)})$ generated by $\mu_i^{(1)}$ has finite index, hence it contains $d_i X(T_i^{(1)})$ for some integer $d_i > 0$. Then any $\delta_i^{(1)} \in d_i X(T_i^{(1)})$ can be written as

$$\delta_i^{(1)} = \sum n_\sigma \sigma(\mu_i^{(1)}) \ \text{ for some } \ \sigma \in \mathrm{Gal}(\overline{K}/K) \ \text{ and } \ n_\sigma \in \mathbb{Z}.$$

So, using (5) we obtain that

$$\delta_i^{(1)}(\gamma_i^{(1)}) = \delta_1^{(2)}(\gamma_1^{(2)}) \cdots \delta_{m_2}^{(2)}(\gamma_{m_2}^{(2)})$$

with $\delta_j^{(2)} = \sum n_\sigma \sigma(\mu_j^{(2)})$ for $j \leq m_2$. Finally, if $\gamma_i^{(1)}$ is of infinite order then it generates a Zariski-dense subgroup of the K-irreducible torus $T_i^{(1)}$, and therefore $\delta_i^{(1)}(\gamma_i^{(1)}) \neq 1$ for any nonzero $\delta_i^{(1)} \in X(T_i^{(1)})$. □

The following theorem is an adaptation of a part of the Isogeny Theorem (Theorem 4.2) of [14] suitable for our purposes.

Theorem 2.3 *Let $T_1^{(1)}, \dots, T_{m_1}^{(1)}$ and $T_1^{(2)}, \dots, T_{m_2}^{(2)}$ be two finite families of algebraic K-tori, and suppose we are given a relation of the form*

$$\chi_1^{(1)}(\gamma_1^{(1)}) \cdots \chi_{m_1}^{(1)}(\gamma_{m_1}^{(1)}) = \chi_1^{(2)}(\gamma_1^{(2)}) \cdots \chi_{m_2}^{(2)}(\gamma_{m_2}^{(2)}), \tag{6}$$

where $\gamma_i^{(s)} \in T_i^{(s)}(K)$ and $\chi_i^{(s)} \in X(T_i^{(s)})$. Assume that the tori $T_1^{(1)}, \dots, T_{m_1}^{(1)}$ are independent, irreducible and nonsplit over K, and that the elements $\gamma_1^{(1)}, \dots, \gamma_{m_1}^{(1)}$ all have infinite order. Then for each $i \leq m_1$ such that the corresponding character $\chi_i^{(1)}$ in (6) is nontrivial, there exists a surjective K-homomorphism $T_j^{(2)} \to T_i^{(1)}$ for some $j \leq m_2$, hence, in *particular,* $K_{T_i^{(1)}} \subset K_{T_j^{(2)}}$. *Moreover, if all the tori are of the same dimension, the above homomorphism is an isogeny and* $K_{T_i^{(1)}} = K_{T_j^{(2)}}$.

Proof Fix $i \leq m_1$ such that $\chi_i^{(1)} \neq 0$. Applying Lemma 2.2, we see that there is a relation of the form

$$\delta_i^{(1)}(\gamma_i^{(1)}) = \delta_1^{(2)}(\gamma_1^{(2)}) \cdots \delta_{m_2}^{(2)}(\gamma_{m_2}^{(2)})$$

with $\delta_i^{(1)} \in X(T_i^{(1)})$, $\delta_i^{(1)} \neq 0$, and $\delta_j^{(2)} \in X(T_j^{(2)})$ for $j \leq m_2$. To simplify our notation, we set

$$T^{(1)} = T_i^{(1)}, \quad \gamma^{(1)} = \gamma_i^{(1)}, \quad \delta^{(1)} = \delta_i^{(1)}$$

and

$$T^{(2)} = T_1^{(2)} \times \cdots \times T_{m_2}^{(2)}, \quad \gamma^{(2)} = (\gamma_1^{(2)}, \ldots, \gamma_{m_2}^{(2)}), \quad \delta^{(2)} = (\delta_1^{(2)}, \ldots, \delta_{m_2}^{(2)}).$$

Then

$$\delta^{(1)}(\gamma^{(1)}) = \delta^{(2)}(\gamma^{(2)}) =: \lambda.$$

First, we will show that the Galois conjugates $\sigma(\lambda)$ for $\sigma \in \mathrm{Gal}(K_{T^{(1)}}/K)$ generate $K_{T^{(1)}}$ over K. Indeed, suppose $\tau \in \mathrm{Gal}(K_{T^{(1)}}/K)$ fixes all the $\sigma(\lambda)$'s. Then for any $\sigma \in \mathrm{Gal}(K_{T^{(1)}}/K)$ we have

$$(\tau\sigma(\delta^{(1)}))(\gamma^{(1)}) = \tau(\sigma(\lambda)) = \sigma(\lambda) = (\sigma(\delta^{(1)}))(\gamma^{(1)}).$$

Since $T^{(1)}$ is K-irreducible, the element $\gamma^{(1)} \in T^{(1)}(K)$, being of infinite order, generates a Zariski-dense subgroup of $T^{(1)}$. Hence, we conclude that $\tau(\sigma(\delta^{(1)})) = \sigma(\delta^{(1)})$ for all $\sigma \in \mathrm{Gal}(K_{T^{(1)}}/K)$. But the elements $\sigma(\delta^{(1)})$ span $X(T^{(1)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathbb{Q}$-vector space, so $\tau = \mathrm{id}$, and our claim follows.

Now, since all the elements $\sigma(\lambda)$ for $\sigma \in \mathrm{Gal}(K_{T^{(1)}}/K)$ belong to $K_{T^{(2)}}$, we obtain the inclusion $K_{T^{(1)}} \subset K_{T^{(2)}}$. So the restriction map

$$\mathscr{G} := \mathrm{Gal}(K_{T^{(2)}}/K) \longrightarrow \mathrm{Gal}(K_{T^{(1)}}/K)$$

is a surjective homomorphism. In the rest of the proof, we will view $X(T^{(1)})$ as a $\mathscr{G}$-module via this homomorphism. Define $\nu_i : \mathbb{Q}[\mathscr{G}] \to X(T^{(i)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$\sum_{\sigma \in \mathscr{G}} n_\sigma \sigma \mapsto \sum_{\sigma \in \mathscr{G}} n_\sigma \sigma(\delta^{(i)}).$$

We observe that $\delta^{(1)}(\gamma^{(1)}) = \delta^{(2)}(\gamma^{(2)})$ implies that for any $a = \sum n_\sigma \sigma \in \mathbb{Z}[\mathscr{G}]$, we have

$$\nu_2(a)(\gamma^{(2)}) = \prod \sigma(\delta^{(2)}(\gamma^{(2)}))^{n_\sigma} = \prod \sigma(\delta^{(1)}(\gamma^{(1)}))^{n_\sigma} = \nu_1(a)(\gamma^{(1)}). \tag{7}$$

It is now easy to show that

$$\mathrm{Ker}\nu_2 \subset \mathrm{Ker}\nu_1. \tag{8}$$

Indeed, let $a \in \mathbb{Z}[\mathscr{G}]$ be such that $\nu_2(a) = 0$. Then it follows from (7) that

$$\nu_2(a)(\gamma^{(2)}) = 1 = \nu_1(a)(\gamma^{(1)}).$$

As $\gamma^{(1)}$ generates a Zariski-dense subgroup of $T^{(1)}$, we conclude that $\nu_1(a) = 0$, and (8) follows.

Combining (8) with the fact that $\delta^{(1)}$ generates $X(T^{(1)}) \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}[\mathscr{G}]$-module, we get a surjective homomorphism

$$\alpha\colon \mathrm{Im}\nu_2 \longrightarrow \mathrm{Im}\nu_1 = X(T^{(1)}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

of $\mathbb{Q}[\mathscr{G}]$-modules. Because of semi-simplicity of $\mathbb{Q}[\mathscr{G}]$, there exists an injective $\mathbb{Z}[\mathscr{G}]$-module homomorphism $X(T^{(1)}) \to X(T^{(2)})$, hence a surjective K-homomorphism $\theta\colon T^{(2)} \to T^{(1)}$. Pick $j \leq m_2$ so that the restriction $\theta|_{T_j^{(2)}}$ is nontrivial. As $T^{(1)}$ is K-irreducible, we conclude that the resulting homomorphism $T_j^{(2)} \to T^{(1)} = T_i^{(1)}$ is surjective, hence the inclusion $K_{T_i^{(1)}} \subset K_{T_j^{(2)}}$. If $\dim T_j^{(2)} = \dim T_i^{(1)}$, then the above homomorphism is an isogeny implying that in fact $K_{T_i^{(1)}} = K_{T_j^{(2)}}$. □

3 Existence of independent irreducible tori

In order to apply Theorem 2.3 in our analysis of the weak containment relation, we need to provide an adequate supply of regular semi-simple elements in a given finitely generated Zariski-dense subgroup whose centralizers yield arbitrarily large families of independent irreducible tori. Such elements are constructed in this section using a suitable generalization, along the lines indicated in [12], of the result established in [11] (see also [14, §3] and [16]) guaranteeing the existence, in any Zariski-dense subgroup, of elements whose centralizers are irreducible tori.

Let G be a connected semi-simple algebraic group defined over a field K, and let T be a maximal torus of G defined over a field extension L of K. We will systematically use the notations introduced after Definition 4 in Sect. 2, particularly the natural homomorphism $\theta_T\colon \mathrm{Gal}(L_T/L) \to \mathrm{Aut}(\Phi(G,T))$. For the convenience of reference, we now quote Theorem 3.1 of [14].

Theorem 3.1 *Let G be a connected absolutely almost simple algebraic group defined over a finitely generated field K of characteristic zero, and L be a finitely generated field containing K. Let r be the number of nontrivial conjugacy classes in the (absolute) Weyl group of G, and suppose we are given r inequivalent nontrivial discrete valuations $v_1, \ldots, v_r$ of K such that the completion K_{v_i} is locally compact and contains L, and G splits over K_{v_i}, for each $i \leq r$. Then there exist maximal K_{v_i}-tori $T(v_i)$ of G, one for each $i \leq r$, with the property that for any maximal K-torus T of G which is conjugate to $T(v_i)$ by an element of $G(K_{v_i})$ for all $i \leq r$, we have*

$$\theta_T(\mathrm{Gal}(L_T/L)) \supset W(G,T). \tag{9}$$

The following corollary (see Corollary 3.2 in [14]) is derived from Theorem 3.1 using weak approximation property of the variety of maximal tori of G.

Corollary 3.2 *Let G, K and L be as in Theorem 3.1, and let V be a finite set of inequivalent nontrivial rank 1 valuations of K. Suppose that for each $v \in V$ we are given a maximal K_v-torus $T(v)$ of G. Then there exists a maximal K-torus T of G for which (9) holds and which is conjugate to $T(v)$ by an element of $G(K_v)$, for all $v \in V$.*

(In Corollary 3.2 of [14] it was assumed that for each $v \in V$, the completion K_v is locally compact. But as the Implicit Function Theorem holds over K_v for any rank 1 valuation v of K, the proof of Corollary 3.2 in [14] can be modified to prove the above more general result.)

We will now strengthen the above corollary to obtain arbitrarily large families of irreducible independent tori.

Theorem 3.3 *Let G be a connected absolutely almost simple algebraic group defined over a finitely generated field K of characteristic zero, and L be any finitely generated field extension of K over which G is of inner type. Furthermore, let V be a finite set of inequivalent nontrivial rank 1 valuations of K such that any $v \in V$ is either discrete or the corresponding completion K_v is locally compact. Fix $m \geq 1$, and suppose that for each $v \in V$ we are given m maximal K_v-tori $T_1(v), \ldots, T_m(v)$ of G. Then there exist maximal K-tori $T_1, \ldots, T_m$ of G such that*

(i) for each $j \leq m$, the torus T_j satisfies

$$\theta_{T_j}(\mathrm{Gal}(L_{T_j}/L)) \supset W(G, T_j), \tag{10}$$

in particular, T_j is L-irreducible;

(ii) T_j is conjugate to $T_j(v)$ by an element of $G(K_v)$ for all $v \in V$;

(iii) the tori $T_1, \ldots, T_m$ are independent over L.

Proof We will induct on m. If $m = 1$, then the existence of a maximal K-torus $T = T_1$ satisfying (i) and (ii) is established in Corollary 3.2, while condition (iii) is vacuous in this case. Now, let $m > 1$ and assume that the maximal tori $T_1, \ldots, T_{m-1}$ satisfying conditions (i), (ii), and independent over L, have already been found. Let L' denote the compositum of the fields $L_{T_1}, \ldots, L_{T_{m-1}}$. Applying Corollary 3.2 with L' in place of L, we find a maximal K-torus T_m which is conjugate to $T_m(v)$ by an element of $G(K_v)$ for all $v \in V$ and satisfies

$$\theta_{T_m}(\mathrm{Gal}(L'_{T_m}/L')) \supset W(G, T_m). \tag{11}$$

Then T_m obviously satisfies conditions (i) and (ii). To see that $T_1, \ldots, T_m$ satisfy condition (iii), we observe that as the group G is of inner type over L, according to [14], Lemma 4.1, we have

$$\theta_{T_j}(\mathrm{Gal}(L_{T_j}/L)) = W(G, T_j) \quad \text{for all} \quad j \leq m.$$

Since $L' = L_{T_1} \cdots L_{T_{m-1}}$, it follows from (11) that

$$[L_{T_1} \cdots L_{T_m} : L_{T_1} \cdots L_{T_{m-1}}] = |W(G, T_m)|.$$

By induction hypothesis, $T_1, \ldots, T_{m-1}$ are independent over L, hence

$$[L_{T_1} \cdots L_{T_{m-1}} : L] = \prod_{j=1}^{m-1} [L_{T_j} : L] = \prod_{j=1}^{m-1} |W(G, T_j)|.$$

Thus,

$$[L_{T_1} \cdots L_{T_m} : L] = \prod_{j=1}^{m} |W(G, T_j)| = \prod_{j=1}^{m} [L_{T_j} : L],$$

and therefore $T_1, \ldots, T_m$ are independent over L. □

Next, we will establish a variant of Theorem 3.3 which asserts the existence of regular semi-simple elements in a given Zariski-dense subgroup whose centralizers possess properties (i), (ii) and (iii) of the preceding theorem.

Theorem 3.4 *Let G, K and L be as in Theorem 3.3 and V be a finite set of inequivalent nontrivial discrete valuations of K such that for every $v \in V$, the completion K_v of K is locally compact. Again, fix $m \geq 1$, and suppose that for each $v \in V$ we are given m maximal K_v-tori $T_1(v), \ldots, T_m(v)$ of G. Let $\Gamma \subset G(K)$ be a finitely generated Zariski-dense subgroup such that the closure of the image of the diagonal map*

$$\Gamma \hookrightarrow \prod_{v \in V} G(K_v)$$

is open. Then there exist regular semi-simple elements $\gamma_1, \ldots, \gamma_m \in \Gamma$ of infinite order such that the maximal K-tori $T_j = Z_G(\gamma_j)^\circ$ for $j \leq m$, satisfy

(i) for each $j \leq m$ we have

$$\theta_{T_j}(\mathrm{Gal}(L_{T_j}/L)) \supset W(G, T_j) \tag{12}$$

(in particular, T_j is L-irreducible, hence γ_j generates a Zariski-dense subgroup of T_j);

(ii) T_j is conjugate to $T_j(v)$ by an element of $G(K_v)$ for all $v \in V$;

(iii) the tori $T_1, \ldots, T_m$ are independent over L.

We begin with the following lemma.

Lemma 3.5 *Let $\mathcal{G}$ be a connected absolutely almost simple algebraic group over a field $\mathcal{K}$ of characteristic zero, Γ be a Zariski-dense subgroup of $\mathcal{G}(\mathcal{K})$. Furthermore, let $\mathscr{V}$ be a finite set of nontrivial discrete valuations such that for each $v \in \mathscr{V}$, the completion $\mathcal{K}_v$ is locally compact, hence a finite extension of $\mathbb{Q}_{p_v}$ for some prime p_v. Assume that the closure of the image of the diagonal map*

$$\Gamma \longrightarrow \prod_{v \in \mathscr{V}} \mathcal{G}(\mathcal{K}_v) =: \mathcal{G}_{\mathscr{V}}$$

is open in $\mathcal{G}_{\mathscr{V}}$. Let now $\mathscr{W}$ be another finite set of nontrivial discrete valuations of $\mathcal{K}$ such that for each $w \in \mathscr{W}$ we have $\mathcal{K}_w = \mathbb{Q}_{p_w}$ for the corresponding prime p_w and that Γ is a nondiscrete subgroup of $\mathcal{G}(\mathcal{K}_w)$ (which is automatically the case if Γ is relatively compact in $\mathcal{G}(\mathcal{K}_w)$). If the primes p_w for $w \in \mathscr{W}$ are pairwise distinct and none of them is contained in $\Pi_{\mathscr{V}} = \{p_v | v \in \mathscr{V}\}$, then the closure $\overline{\Gamma}^{(\mathscr{V} \cup \mathscr{W})}$ of the image of the diagonal map

$$\Gamma \longrightarrow \prod_{v \in \mathscr{V} \cup \mathscr{W}} \mathcal{G}(\mathcal{K}_v) =: \mathcal{G}_{\mathscr{V} \cup \mathscr{W}}$$

is also open.

Proof Replacing Γ with $\Gamma \cap \Omega$ for a suitable open subgroup Ω of $\mathcal{G}_{\mathscr{V}}$, we can assume that the closure $\overline{\Gamma}^{(\mathscr{V})}$ of Γ in $\mathcal{G}_{\mathscr{V}}$ is of the form

$$\overline{\Gamma}^{(\mathscr{V})} = \prod_{v \in \mathscr{V}} \mathcal{U}_v$$

where $\mathcal{U}_v$ is an open pro-p_v subgroup of $\mathcal{G}(\mathcal{K}_v)$. (We notice that for any open subgroup $\Omega \subset \mathcal{G}_{\mathscr{V}}$, the intersection $\Gamma \cap \Omega$ is still Zariski-dense in G as its closure in $\mathcal{G}(\mathcal{K}_v)$ contains an open subgroup, for every $v \in \mathscr{V}$.) A standard argument (cf. [11], Lemma 2) shows that the closure $\overline{\Gamma}^{(w)}$ of Γ in $\mathcal{G}(\mathcal{K}_w)$ is open for any $w \in \mathscr{W}$. Moreover, as above, we can assume, after replacing Γ with a subgroup of finite index, that $\overline{\Gamma}^{(w)}$ is a pro-p_w group. It is enough to prove that

$$\overline{\Gamma}^{(\mathscr{V}\cup\mathscr{W})} = \overline{\Gamma}^{(\mathscr{V})} \times \prod_{w\in\mathscr{W}} \overline{\Gamma}^{(w)} =: \Theta. \tag{13}$$

Since the primes p_w, $w \in \mathscr{W}$, are pairwise distinct and none of them is contained in $\Pi_{\mathscr{V}}$, we conclude that $\overline{\Gamma}^{(w)}$ is the unique Sylow p_w-subgroup of Θ, for all $w \in \mathscr{W}$. As the projection $\overline{\Gamma}^{(\mathscr{V}\cup\mathscr{W})} \to \overline{\Gamma}^{(w)}$ is a surjective homomorphism of profinite groups, a Sylow pro-p_w subgroup of $\overline{\Gamma}^{(\mathscr{V}\cup\mathscr{W})}$ must map onto $\overline{\Gamma}^{(w)}$. This implies that $\overline{\Gamma}^{(w)} \subset \overline{\Gamma}^{(\mathscr{V}\cup\mathscr{W})}$ for each $w \in \mathscr{W}$, and (13) follows. □

Proof of Theorem 3.4 We fix a matrix realization of G as a K-subgroup of GL_n, and pick a finitely generated subring R of K such that $\Gamma \subset \mathrm{GL}_n(R)$. We will now argue by induction on m. Let r be the number of nontrivial conjugacy classes in the (absolute) Weyl group of G. For $m = 1$ the argument basically mimics the proof of Theorem 2 in [11]. More precisely, by Proposition 1 of [11], we can choose r distinct primes $p_1, \ldots, p_r \notin \Pi_V$ such that for each $i \in \{1, \ldots, r\}$ there exists an embedding $\iota_{p_i} : L \hookrightarrow \mathbb{Q}_{p_i}$ such that $\iota_{p_i}(R) \subset \mathbb{Z}_{p_i}$ and G splits over $\mathbb{Q}_{p_i}$. For a nontrivial discrete valuation v of K and a given maximal K_v-torus T of G, we let $\mathscr{U}(T, v)$ denote the set of elements of the form gtg^{-1}, with $t \in T(K_v)$ regular and $g \in G(K_v)$. It is known that $\mathscr{U}(T, v)$ is a solid[1] open subset of $G(K_v)$ (cf. [14], Lemma 3.4). Let v_i be pullback to L of the p_i-adic valuation on $\mathbb{Q}_{p_i}$ under ι_{p_i} (so that $L_{v_i} = \mathbb{Q}_{p_i}$). Let $T(v_1), \ldots, T(v_r)$ be the tori given by Theorem 3.1. By our construction, for each $i \le r$, the group Γ is contained in $G(\mathbb{Z}_{p_i})$, hence is relatively compact. Thus Lemma 3.5 applies, and since for any $v \in V \cup \{v_1, \ldots, v_r\}$, the group $G(K_v)$ contains a torsion-free open subgroup, it follows from Lemma 3.5 that there exists an element of infinite order

$$\gamma_1 \in \Gamma \bigcap \left(\prod_{v\in V} \mathscr{U}(T_1(v), v) \times \prod_{i\le r} \mathscr{U}(T(v_i), v_i) \right),$$

and this element is as required. For $m > 1$, we proceed as in the proof of Theorem 3.3. Suppose that the elements $\gamma_1, \ldots, \gamma_{m-1}$ for which the corresponding $T_1, \ldots, T_{m-1}$ satisfy (i) and (ii), and are independent over L, have already been found. Let L' denote the compositum of the fields $L_{T_1}, \ldots, L_{T_{m-1}}$. We then again use Proposition 1 of [11] to find r distinct primes $p'_1, \ldots, p'_r \notin \Pi_V$ such that for each $i \le r$, there exists an embedding $\iota'_{p'_i} : L' \hookrightarrow \mathbb{Q}_{p'_i}$ with the property $\iota'_{p'_i}(R) \subset \mathbb{Z}_{p'_i}$. As G splits over L', it splits over $\mathbb{Q}_{p'_i}$. Let v'_i be the pullback of the p'_i-adic valuation on $\mathbb{Q}_{p'_i}$ under $\iota'_{p'_i}$ (and then $L'_{v'_i} = \mathbb{Q}_{p'_i}$). We use Theorem 3.1 to find, for each $i \le r$, an $L'_{v'_i}$-torus $T'(v'_i)$ of G such that for any maximal K-torus T' of G which is conjugate to $T'(v'_i)$ by an element of $G(L'_{v'_i})$ for all $i \le r$, we have

$$\theta_{T'}(\mathrm{Gal}(L'_{T'}/L')) \supset W(G, T').$$

As above, there exists an element of infinite order

$$\gamma_m \in \Gamma \bigcap \left(\prod_{v\in V} \mathscr{U}(T_m(v), v) \times \prod_{i\le r} \mathscr{U}(T'(v'_i), v'_i) \right)$$

[1] We recall that a subset of a topological group is called *solid* if it meets every open subgroup of that group.

Then γ_m clearly satisfies (i) and (ii), and the fact that $T_1, \ldots, T_m$ are independent over L is established just as in the proof of Theorem 3.3. □

4 Field of definition

Let G_1 and G_2 be connected absolutely simple algebraic groups of adjoint type defined over a field F of characteristic zero. As before, we let w_i denote the order of the (absolute) Weyl group of G_i for $i = 1, 2$. Suppose that for each $i \in \{1, 2\}$ we are given a finitely generated Zariski-dense subgroup Γ_i of $G_i(F)$. Our goal in Sects. 4–5 is to develop a series of conditions which must hold in order to prevent the subgroups Γ_1 and Γ_2 from satisfying condition (C_i) (see Definition 3 in Sect. 1) for at least one $i \in \{1, 2\}$. Here is our first, rather straightforward, result in this direction.

Theorem 4.1 (i) *If every regular semi-simple element* $\gamma \in \Gamma_1$ *of infinite order is weakly contained in* Γ_2 *then* $\mathrm{rk} G_1 \leq \mathrm{rk} G_2$ *and* w_1 *divides* w_2.

(ii) *If* $w_1 > w_2$, *then property* (C_1) *holds.*

Proof (i) We fix a finitely generated subfield K of F such that for $i = 1$ and 2, the group G_i is defined and of inner type over K and $\Gamma_i \subset G_i(K)$. By Theorem 3.4, there exists a regular semi-simple element $\gamma \in \Gamma_1$ of infinite order such that for the corresponding torus $T = Z_{G_1}(\gamma)^\circ$ we have

$$\theta_T(\mathrm{Gal}(K_T/K)) \supset W(G_1, T);$$

we notice that since G_1 is of inner type over K, this inclusion is actually an equality, cf. Lemma 4.1 of [14]. The fact that γ is weakly contained in Γ_2 means that one can find semi-simple elements $\gamma_1^{(2)}, \ldots, \gamma_{m_2}^{(2)} \in \Gamma_2$ so that for some characters $\chi \in X(T)$ and $\chi_j^{(2)} \in X(T_j^{(2)})$, where $T_j^{(2)}$ is a maximal K-torus of G_2 containing $\gamma_j^{(2)}$, there is a relation of the form

$$\chi(\gamma) = \chi_1^{(2)}(\gamma_1^{(2)}) \cdots \chi_{m_2}^{(2)}(\gamma_{m_2}^{(2)}) \neq 1.$$

Then it follows from Theorem 2.3 that for some $j \leq m_2$, there exists a surjective K-homomorphism $T_j^{(2)} \to T$. Then $\mathrm{rk} G_1 \leq \mathrm{rk} G_2$ and there exists a surjective homomorphism $\mathrm{Gal}(K_{T_j^{(2)}}/K) \to \mathrm{Gal}(K_T/K)$. Since

$$\theta_{T_j^{(2)}}(\mathrm{Gal}(K_{T_j^{(2)}}/K)) \subset W(G_2, T_j^{(2)})$$

(Lemma 4.1 of [14]), our assertion follows.

(ii) The argument here basically repeats the argument given above with minor modifications. Let K be chosen as in the proof of (i). To verify property (C_1), we use Theorem 3.4 to find, for any given $m \geq 1$, regular semi-simple elements $\gamma_1, \ldots, \gamma_m \in \Gamma_1$ of infinite order such that for the corresponding maximal K-tori $T_i = Z_{G_1}(\gamma_i)^\circ$ of G_1 we have

$$\theta_{T_i}(\mathrm{Gal}(K_{T_i}/K)) \supset W(G_1, T_i) \quad \text{for all} \quad i \leq m,$$

and the tori $T_1, \ldots, T_m$ are independent over K. Then the elements $\gamma_1, \ldots, \gamma_m$ are multiplicatively independent by Lemma 2.1, and we only need to show that they are not weakly contained in Γ_2 given that $w_1 > w_2$. Otherwise, we would have a relation of the form

$$\chi_1(\gamma_1) \cdots \chi_m(\gamma_m) = \chi_1^{(2)}(\gamma_1^{(2)}) \cdots \chi_{m_2}^{(2)}(\gamma_{m_2}^{(2)}) \neq 1$$

with $\chi_j \in X(T_j)$ and the other objects as in the proof of (i). Invoking again Theorem 2.3, we see that for some $i \leq m$ and $j \leq m_2$, there exists a surjective K-homomorphism $T_j^{(2)} \to T_i$. As above, this implies that w_1 divides w_2, contradicting the fact that by our assumption $w_1 > w_2$. □

Now, let $K_i = K_{\Gamma_i}$ denote the field of definition of Γ_i, i.e. the subfield of F generated by the traces $\mathrm{TrAd}_{G_i}(\gamma)$ for all $\gamma \in \Gamma_i$ (cf. [22]). Since Γ_i is finitely generated, $\mathrm{Ad}_{G_i}(\Gamma_i)$ is contained in $\mathrm{GL}_{n_i}(F_i)$ for some finitely generated subfield F_i of F. Then K_i is a subfield of F_i, hence it is finitely generated. Since G_i is adjoint, according to the results of Vinberg [22], it is defined over K_i and $\Gamma_i \subset G_i(K_i)$.

The following theorem (see Theorem 6 of the introduction) is the main result of this section.

Theorem 4.2 *(i) If $w_1 > w_2$ then condition (C_1) holds;*
(ii) If $w_1 = w_2$ but $K_1 \not\subset K_2$ then again (C_1) holds.
Thus, unless $w_1 = w_2$ and $K_1 = K_2$, condition (C_i) holds for at least one $i \in \{1, 2\}$.

Proof Assertion (i) has already been established in Theorem 4.1. For $i = 1, 2$, as the group G_i has been assumed to be of adjoint type, it is defined over K_i and $\Gamma_i \subset G_i(K_i)$. Set $K = K_1K_2$, and pick a finite extension L of K so that G_i splits over L for both $i \in \{1, 2\}$; clearly, L is finitely generated. Fix a matrix realization of G_1 as a K_1-subgroup of GL_n, and pick a finitely generated subring R of K_1 so that $\Gamma \subset G_1(R)$.

Since by our assumption $K_1 \not\subset K_2$, we have $K_2 \subsetneqq K \subset L$. So, using Proposition 5.1 of [14], we can find a prime q such that there exists a pair of embeddings

$$\iota^{(1)}, \iota^{(2)} : L \hookrightarrow \mathbb{Q}_q$$

which have the same restrictions to K_2 but different restrictions to K, hence to K_1, and which satisfy the condition $\iota^{(j)}(R) \subset \mathbb{Z}_q$ for $j = 1, 2$. Let $v^{(j)}$ be the pullback to K_1 of the q-adic valuation of $\mathbb{Q}_q$ under $\iota^{(j)}|_{K_1}$. The group $G_1((K_1)_{v^{(j)}})$ can be naturally identified with $G_1^{(j)}(\mathbb{Q}_q)$, where $G_1^{(j)}$ denotes the algebraic $\mathbb{Q}_q$-group obtained from the K_1-group G_1 by the extension of scalars $\iota^{(j)}|_{K_1} : K_1 \to \mathbb{Q}_q$, for $j = 1, 2$. Since $\iota^{(1)}$ and $\iota^{(2)}$ have different restrictions to K_1, it follows from Proposition 5.2 of [14] that the closure of the image of Γ_1 under the diagonal embedding

$$\Gamma_1 \longrightarrow G_1((K_1)_{v^{(1)}}) \times G_1((K_1)_{v^{(2)}}) \tag{14}$$

is open. By our construction, G_1 splits over $(K_1)_{v^{(1)}} = \mathbb{Q}_q$ (recall that $\iota^{(1)}(L) \subset \mathbb{Q}_q$ and G_1 splits over L), so we can pick a $(K_1)_{v^{(1)}}$-split torus $T^{(v^{(1)})}$ of G_1. Furthermore, by Theorem 6.21 of [7] there exists a maximal $(K_1)_{v^{(2)}}$-torus $T^{(v^{(2)})}$ of G_1 which is anisotropic over $(K_1)_{v^{(2)}}$.

Set $V = \{v^{(1)}, v^{(2)}\}$. It follows from Theorem 3.4 that for any $m \geq 1$ there exist regular semi-simple elements $\gamma_1, \ldots, \gamma_m \in \Gamma_1$ of infinite order such that the maximal tori $T_i = Z_{G_1}(\gamma_i)^\circ$ for $i \leq m$ are independent over L and satisfy the following conditions for all $i \leq m$:

- $\theta_{T_i}(\mathrm{Gal}(L_{T_i}/L)) \supset W(G_1, T_i)$;
- T_i is conjugate to $T^{(v)}$ for $v \in V$.

We claim that these elements allow us to check the property (C_1). Indeed, it follows from Lemma 2.1 that these elements are multiplicatively independent, and we only need to show that they are not weekly contained in Γ_2. Assume the contrary. As $w_1 = w_2$, we conclude that $\mathrm{rk} G_1 = \mathrm{rk} G_2$, and there exists a maximal K_2-torus T' of G_2 that admits an L-isogeny $\kappa\colon T' \to T$ onto $T = T_i$ for some $i \leq m$ (see the proof of Theorem 4.1(ii)), and then

$$L_T = L_{T'} =: \mathscr{F}.$$

Observe that

$$\mathscr{F} = L \cdot K_{1T} = L \cdot K_{2T'}. \tag{15}$$

Fix some extensions

$$\tilde{\iota}^{(1)}, \tilde{\iota}^{(2)}\colon \mathscr{F} \to \overline{\mathbb{Q}}_q \qquad (\overline{\mathbb{Q}}_q \text{ is the algebraic closure of } \mathbb{Q}_q)$$

of $\iota^{(1)}$ and $\iota^{(2)}$ respectively. Let u be the pullback to K_2 of the q-adic valuation of $\mathbb{Q}_q$ under $\iota^{(1)}|_{K_2} = \iota^{(2)}|_{K_2}$. Furthermore, let $\tilde{v}^{(1)}$, $\tilde{v}^{(2)}$ (resp., $\tilde{u}^{(1)}$, $\tilde{u}^{(2)}$) be the valuations of K_{1T} (resp., of $K_{2T'}$) obtained as pullbacks of the valuation of $\overline{\mathbb{Q}}_q$ under appropriate restrictions of $\tilde{\iota}^{(1)}$ and $\tilde{\iota}^{(2)}$. Then $\tilde{u}^{(1)}$ and $\tilde{u}^{(2)}$ are two extensions of u to the Galois extension $K_{2T'}/K_2$, and therefore

$$\big[(K_{2T'})_{\tilde{u}^{(1)}} : (K_2)_u\big] = \big[(K_{2T'})_{\tilde{u}^{(2)}} : (K_2)_u\big]. \tag{16}$$

On the other hand, since $\iota^{(j)}(L) \subset \mathbb{Q}_q$ for $j = 1, 2$, we have

$$(K_2)_u = \mathbb{Q}_q \quad \text{and} \quad (K_1)_{v^{(1)}} = \mathbb{Q}_q = (K_1)_{v^{(2)}}.$$

Moreover, it follows from (15) that

$$(K_{2T'})_{\tilde{u}^{(j)}} = (K_{1T})_{\tilde{v}^{(j)}} \quad \text{for} \quad j = 1, 2. \tag{17}$$

But, by our construction, T is $(K_1)_{v^{(1)}}$-split and $(K_1)_{v^{(2)}}$-anisotropic. So,

$$\big[(K_{1T})_{\tilde{v}^{(1)}} : (K_1)_{v^{(1)}}\big] = 1 \quad \text{and} \quad \big[(K_{1T})_{\tilde{v}^{(2)}} : (K_1)_{v^{(2)}}\big] \neq 1$$

This, in view of (17), contradicts (16). So, the elements $\gamma_1, \ldots, \gamma_m$ are not weakly contained in Γ_2, verifying condition (C_1). □

5 Arithmetic groups

In this section, we will treat the case where the Zariski-dense subgroups $\Gamma_i \subset G_i(F)$ are S-arithmetic. For our purposes, it is convenient to use the description of these subgroups introduced in [14], Sect. 1, and for the reader's convenience we briefly recall here the relevant definitions and results. So, let G be a connected absolutely almost simple algebraic group defined over a field F of characteristic zero, let $\overline{G}$ be the corresponding adjoint group, and let $\pi: G \to \overline{G}$ be the natural isogeny. Suppose we are given:

- a number field K together with a *fixed* embedding $K \hookrightarrow F$;
- an F/K-form $\mathscr{G}$ of $\overline{G}$ (which means that the group ${}_F\mathscr{G}$ obtained by the base change $K \hookrightarrow F$ is F-isomorphic to $\overline{G}$);
- a finite set S of places of K that contains V_K^∞ but does not contain any nonarchimedean places where $\mathscr{G}$ is anisotropic.

We then have an embedding $\iota\colon \mathscr{G}(K) \hookrightarrow \overline{G}(F)$, which is well-defined up to an F-automorphism of $\overline{G}$. Now, let $\mathscr{O}_K(S)$ be the ring of S-integers in K (with $\mathscr{O}_K = \mathscr{O}_K(V_K^\infty)$ denoting the ring of algebraic integers in K). Fix a K-embedding $\mathscr{G} \hookrightarrow \mathrm{GL}_n$, and set

$\mathscr{G}(\mathscr{O}_K(S)) = \mathscr{G}(K) \cap \mathrm{GL}_n(\mathscr{O}_K(S))$. A subgroup $\Gamma \subset G(F)$ is called $(\mathscr{G}, K, S)$-*arithmetic* if $\pi(\Gamma)$ is commensurable with $\sigma(\iota(\mathscr{G}(\mathscr{O}_K(S))))$ for some F-automorphism σ of $\overline{G}$. As usual, $(\mathscr{G}, K, V_K^{\infty})$-arithmetic subgroups will simply be called $(\mathscr{G}, K)$-arithmetic. We recall (Lemma 2.6 of [14]) that if $\Gamma \subset G(F)$ is a Zariski-dense $(\mathscr{G}, K, S)$-arithmetic subgroup then the trace field K_{Γ} coincides with K.

Now, for $i = 1, 2$, let G_i be a connected absolutely simple F-group of adjoint type. We will say that the subgroups $\Gamma_i \subset G_i(F)$ are *commensurable up to an F-isomorphism* between G_1 and G_2 if there exists an F-isomorphism $\sigma\colon G_1 \to G_2$ such that $\sigma(\Gamma_1)$ is commensurable with Γ_2 in the usual sense, i.e. their intersection is of finite index in both of them. According to Proposition 2.5 of [14], if Γ_i is a Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroup of $G_i(F)$ for $i = 1, 2$, then Γ_1 and Γ_2 are commensurable up to an F-isomorphism between G_1 and G_2 if and only if $K_1 = K_2 =: K$, $S_1 = S_2$ and $\mathscr{G}_1$ and $\mathscr{G}_2$ are K-isomorphic.

In this section, unless stated otherwise, we will assume that the absolute Weyl groups of G_1 and G_2 are of equal order.

Theorem 5.1 *Let G_1 and G_2 be connected absolutely simple algebraic groups of adjoint type defined over a field F of characteristic zero such that $w_1 = w_2$, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroup for $i = 1, 2$. Furthermore, let L_i be the minimal Galois extension of K_i over which $\mathscr{G}_i$ becomes an inner form. Then, unless* **all** *of the following conditions are satisfied:*

(a) $K_1 = K_2 =: K$,
(b) $\mathrm{rk}_{K_v} \mathscr{G}_1 = \mathrm{rk}_{K_v} \mathscr{G}_2$ *for all* $v \in V^K$,
(c) $L_1 = L_2$,
(d) $S_1 = S_2$,

condition (C_i) holds for at least one $i \in \{1, 2\}$.

Proof (a): Since the trace field K_{Γ_i} coincides with K_i, our assertion in case (a) fails to hold follows from Theorem 4.2. So, in the rest of the proof we may (and we will) assume that $K_1 = K_2 =: K$. Then $\Gamma_i \subset \mathscr{G}_i(K)$ for $i = 1, 2$.

(b): Suppose that for some $v_0 \in V^K$ we have

$$\mathrm{rk}_{K_{v_0}} \mathscr{G}_1 > \mathrm{rk}_{K_{v_0}} \mathscr{G}_2. \tag{18}$$

We will now show that condition (C_1) holds. Set $V = S_1 \cup \{v_0\}$, and for each $v \in V$ pick a maximal K_v-torus $T^{(v)}$ of $\mathscr{G}_1$ satisfying $\mathrm{rk}_{K_v} T^{(v)} = \mathrm{rk}_{K_v} \mathscr{G}_1$. Given $m \geq 1$, we can use Theorem 3.3 to find maximal K-tori $T_1, \ldots, T_m$ of $\mathscr{G}_1$ that are independent over L_1 and satisfy the following properties for each $i \leq m$:

- $\theta_{T_i}(\mathrm{Gal}(L_{1T_i}/L_1)) = W(\mathscr{G}_1, T_i)$;
- T_i is conjugate to $T^{(v)}$ by an element of $\mathscr{G}_1(K_v)$ for all $v \in V$.

We recall that by Dirichlet's Theorem (cf. [7], Theorem 5.12), for a K-torus T and a finite subset S of V^K containing V_{∞}^K we have

$$T(\mathscr{O}_K(S)) \simeq H \times \mathbb{Z}^{d_T(S) - \mathrm{rk}_K T},$$

where H is a finite group and $d_T(S) = \sum_{v \in S} \mathrm{rk}_{K_v} T$. Since Γ_1 has been assumed to be Zariski-dense in $\mathscr{G}_1$, it is infinite, and hence, $\sum_{v \in S_1} \mathrm{rk}_{K_v} \mathscr{G}_1 > 0$. Now we have

$$d_{T_i}(S_1) := \sum_{v \in S_1} \mathrm{rk}_{K_v} T_i = \sum_{v \in S_1} \mathrm{rk}_{K_v} \mathscr{G}_1 > 0.$$

As T_i is clearly K-anisotropic, we conclude from the above that the group $T_i(\mathscr{O}_K(S_1))$ contains a subgroup isomorphic to $\mathbb{Z}^{d_{T_i}(S_1)}$, and so, in particular, one can find an element $\gamma_i \in \Gamma_1 \cap T_i(K)$ of infinite order. We will use the elements $\gamma_1, \ldots, \gamma_m$ to verify property (C_1). Indeed, these elements are multiplicatively independent by Lemma 2.1, and it remains to show that they are not weakly contained in Γ_2. Otherwise, there would exist a relation of the form

$$\chi_1(\gamma_1)\cdots\chi_m(\gamma_m) = \chi_1^{(2)}(\gamma_1^{(2)})\cdots\chi_{m_2}^{(2)}(\gamma_{m_2}^{(2)}) \neq 1 \tag{19}$$

for some semi-simple elements $\gamma_1^{(2)}, \ldots, \gamma_{m_2}^{(2)} \in \Gamma_2 \subset \mathscr{G}_2(K)$, some characters $\chi_i \in X(T_i)$, some tori $T_j^{(2)} \subset \mathscr{G}_2$ such that $\gamma_j^{(2)} \in T_j^{(2)}(K)$ and some characters $\chi_j^{(2)} \in X(T_j^{(2)})$. Since $w_1 = w_2$ and therefore G_1 and G_2 have the same absolute rank, it would follow from Theorem 2.3 that for some $i \leq m$ and $j \leq m_2$ there is a K-isogeny $T_j^{(2)} \to T_i$, and therefore

$$\mathrm{rk}_{K_{v_0}} T_i = \mathrm{rk}_{K_{v_0}} T_j^{(2)}.$$

Since by our choice

$$\mathrm{rk}_{K_{v_0}} T_i = \mathrm{rk}_{K_{v_0}} \mathscr{G}_1 \quad \text{and} \quad \mathrm{rk}_{K_{v_0}} T_j^{(2)} \leq \mathrm{rk}_{K_{v_0}} \mathscr{G}_2,$$

this would contradict (18).

(c): Let us show that $L_1 = L_2$ automatically follows from the fact that

$$\mathrm{rk}_{K_v} \mathscr{G}_1 = \mathrm{rk}_{K_v} \mathscr{G}_2 \quad \text{for all} \quad v \in V^K \tag{20}$$

(which we may assume in view of (b)). By symmetry, it is enough to establish the inclusion $L_1 \subset L_2$. Assume the contrary. Then for the finite Galois extension $L := L_1 L_2$ of K we can find a nontrivial element $\sigma \in \mathrm{Gal}(L/L_2) \subset \mathrm{Gal}(L/K)$. According to Theorem 6.7 of [7], there exists a finite subset S of V^K such that for any $v \in V^K \setminus S$, the group $\mathscr{G}_2$ is quasi-split over K_v. Furthermore, by Chebotarev's Density Theorem, there exists a nonarchimedean place $v \in V^K \setminus S$ with the property that for its extension $\bar{v}$ to L, the field extension $L_{\bar{v}}/K_v$ is unramified and its Frobenius automorphism $\mathrm{Fr}(L_{\bar{v}}|K_v)$ is σ. Then $L_2 \subset K_v$, and therefore $\mathscr{G}_2$ is K_v-split. On the other hand, $L_1 \not\subset K_v$, implying that $\mathscr{G}_1$ is not K_v-split. Since G_1 and G_2 have the same absolute rank (as $w_1 = w_2$), this contradicts (20).

(d): If $S_1 \neq S_2$ then, by symmetry, we can assume that there exists $v_0 \in S_1 \setminus S_2$ (any such v_0 is automatically nonarchimedean). We will show that then condition (C_1) holds. As in part (b), for a given $m \geq 1$, we can pick maximal K-tori $T_1, \ldots, T_m$ of $\mathscr{G}_1$ so that they are independent over L_1 and satisfy the following conditions for each $i \leq m$:

- $\theta_{T_i}(\mathrm{Gal}(L_{1\,T_i}/L_1)) = W(\mathscr{G}_1, T_i)$;
- $\mathrm{rk}_{K_{v_0}} T_i = \mathrm{rk}_{K_{v_0}} \mathscr{G}_1$.

Due to our convention that S_1 does not contain any nonarchimedean anisotropic places for $\mathscr{G}_1$, we have $\mathrm{rk}_{K_{v_0}}\, T_i = \mathrm{rk}_{K_{v_0}}\, \mathscr{G}_1 > 0$, hence

$$d_{T_i}(S_1 \setminus \{v_0\}) < d_{T_i}(S_1).$$

Consequently, it follows from Dirichlet's Theorem (cf. (b)) that one can pick $\gamma_i \in \Gamma_1 \cap T_i(\mathscr{O}_K(S_1))$ so that its image in $T_i(\mathscr{O}_K(S_1))/T_i(\mathscr{O}_K(S_1 \setminus \{v_0\}))$ has infinite order for $i = 1, \ldots, m$. We claim that the elements $\gamma_1, \ldots, \gamma_m$ verify property (C_1).

As in (b), these elements are multiplicatively independent by Lemma 2.1, and we only need to show that they are not weakly contained in Γ_2. Assume the contrary. Then there exists a relation of the form (19) as in (b). Invoking Lemma 2.2, we see that there exist $i \leq m$ and $d_i > 0$ such that for any $\lambda_i \in d_i X(T_i)$ there is a relation of the form

$$\lambda_i(\gamma_i) = \prod_{j=1}^{m_2} \lambda_j^{(2)}(\gamma_j^{(2)}) \tag{21}$$

with $\lambda_j^{(2)} \in X(T_j^{(2)})$. On the other hand, by our construction the image of γ_i in $T_i(\mathscr{O}_K(S_1))/T_i(\mathscr{O}_K(S_1 \setminus \{v_0\}))$ has infinite order, and therefore the subgroup $\langle\gamma_i\rangle$ is unbounded in $T_i(K_{v_0})$. It follows that there exists $\lambda_i \in d_i X(T_i)$ for which $\lambda_i(\gamma_i) \in \overline{K_{v_0}}$ is not a unit (with respect to the extension of v_0). Pick for this λ_i the corresponding expression (21). Since $v_0 \notin S_2$, for each $j \leq m_2$, the subgroup $\langle\gamma_j^{(2)}\rangle$ is bounded in $T_j^{(2)}(K_{v_0})$. Hence, the value $\lambda_j^{(2)}(\gamma_j^{(2)}) \in \overline{K_{v_0}}$ is a unit. Then (21) leads to a contradiction. □

Remark 5.2 The argument used in parts (b) and (d) actually proves the following: Let G_1 and G_2 be absolutely simple algebraic groups defined over a field F of characteristic zero such that $w_1 = w_2$, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(\mathscr{G}_i, K, S_i)$-arithmetic subgroup for $i = 1, 2$. Furthermore, let V be a finite subset of V^K containing S_1 and let L be a finite extension of K. If condition (C_1) does not hold then there exists a maximal K-torus T_1 of $\mathscr{G}_1$ satisfying $\theta_{T_1}(\mathrm{Gal}(L_{T_1}/L)) \supset W(\mathscr{G}_1, T_1)$ and $\mathrm{rk}_{K_v}\, T_1 = \mathrm{rk}_{K_v}\, \mathscr{G}_1$ for all $v \in V$ such that for some maximal K-torus T_2 of $\mathscr{G}_2$ there is a K-isogeny $T_2 \to T_1$. We will use this statement below.

Here is an algebraic counterpart of Theorem 2 of the introduction.

Theorem 5.3 *Let G_1 and G_2 be two connected absolutely simple algebraic groups of the same Killing-Cartan type different from A_n, D_{2n+1} $(n > 1)$ and E_6, defined over a field F of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroup for $i = 1, 2$. If Γ_1 and Γ_2 are not commensurable (up to an F-isomorphism between G_1 and G_2) then condition (C_i) holds for at least one $i \in \{1, 2\}$.*

Proof If either $K_1 \neq K_2$ or $S_1 \neq S_2$, condition (C_i) for some $i \in \{1, 2\}$ holds by Theorem 5.1. So, we may assume that

$$K_1 = K_2 =: K \quad \text{and} \quad S_1 = S_2 = S. \tag{22}$$

We first treat the case where the common type of G_1 and G_2 is not D_{2n} $(n \geq 2)$, i.e. it is one of the following: $A_1,\ B_n,\ C_n\ (n \geq 2),\ E_7,\ E_8,\ F_4,\ G_2$. According to Theorem 5.1(b), if $\mathrm{rk}_{K_v}\mathscr{G}_1 \neq \mathrm{rk}_{K_v}\mathscr{G}_2$ for at least one $v \in V^K$, then condition (C_i) again holds for at least one $i \in \{1, 2\}$. Thus, we may assume that

$$\mathrm{rk}_{K_v}\mathscr{G}_1 = \mathrm{rk}_{K_v}\mathscr{G}_2 \quad \text{for all} \quad v \in V^K. \tag{23}$$

As we discussed in ([14], Sect. 6, proof of Theorem 4), for the types under consideration (23) implies that $\mathscr{G}_1 \simeq \mathscr{G}_2$ over K, combining which with (22), we obtain that Γ_1 and Γ_2 are commensurable (cf. [14], Proposition 2.5).

Consideration of groups of type D_{2n} relies on some additional results. In an earlier version of this paper, these were derived from [15] for $n > 2$ (and then Theorem 5.3 was also formulated for type D_{2n} with $n > 2$). Recently, Skip Garibaldi [4] gave an alternate proof of the required fact which works for all $n \geq 2$ (including triality forms of type D_4). This led to the current (complete) form of Theorem 5.3 and also showed that groups of type D_4 do not need to be excluded in Theorem 4 of [14] and its (geometric) consequences (such as Theorem 8.16 of [14]). Here is the precise formulation of Garibaldi's result.

Theorem 5.4 *([4], Theorem 14) Let G_1 and G_2 be connected absolutely simple adjoint groups of type D_{2n} for some $n \geq 2$ over a global field K such that G_1 and G_2 have the same quasi-split inner form—i.e., the smallest Galois extension of K over which G_1 is of inner type is the same as for G_2. If there exists a maximal torus T_i in G_i for $i = 1$ and 2 such that*

(1) there exists a K_{sep}-isomorphism $\phi\colon G_1 \to G_2$ whose restriction to T_1 is a K-isomorphism $T_1 \to T_2$; and
(2) there is a finite set $\mathcal{V}$ of places of K such that:

(a) For all $v \notin \mathcal{V}$, G_1 and G_2 are quasi-split over K_v,
(b) For all $v \in \mathcal{V}$, $(T_i)_{K_v}$ contains a maximal K_v-split subtorus in $(G_i)_{K_v}$;

then G_1 and G_2 are isomorphic over K.

We will actually use the following consequence of the preceding theorem.

Theorem 5.5 *Let G_1 and G_2 be connected absolutely simple algebraic groups of type D_{2n} over a number field K such that*

(a) $\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2$ for all $v \in V^K$;
(b) $L_1 = L_2$ where L_i is the minimal Galois extension of K over which G_i becomes an inner form.

Let $\mathcal{V} \subset V^K$ be a finite set of places such that G_1 is quasi-split over K_v for $v \in V^K \setminus \mathcal{V}$. Let T_1 be a maximal K-torus of G_1 satisfying

(α) $\theta_{T_1}(\mathrm{Gal}(K_{T_1}/K)) \supset W(G_1, T_1)$,
(β) $\mathrm{rk}_{K_v} T_1 = \mathrm{rk}_{K_v} G_1$ *for all* $v \in \mathcal{V}$.

If there exists a K-isogeny $\varphi\colon T_2 \to T_1$ from a maximal K-torus T_2 of G_2, then G_1 and G_2 are isogenous over K.

To derive Theorem 5.5 from Theorem 5.4, we can assume that both G_1 and G_2 are adjoint. Now note that it follows from Lemma 4.3 in [14] that, due to condition (α), one can assume without any loss of generality that the comorphism $\varphi^*\colon X(T_1) \to X(T_2)$ satisfies $\varphi^*(\Phi(G_1, T_1)) = \Phi(G_2, T_2)$. Then φ is actually a K-isomorphism of tori that extends to a $\overline{K}$-isomorphism $\phi\colon G_2 \to G_1$. So, we can use Theorem 5.4 to obtain Theorem 5.5.

To complete the proof of Theorem 5.3, we observe that if neither (C_1) nor (C_2) holds, then according to Theorem 5.1, conditions (a) and (b) of Theorem 5.5 are satisfied for $\mathscr{G}_1$ and $\mathscr{G}_2$. Fix a finite set of places $V \subset V^K$ that contains S_1 and is big enough so that $\mathscr{G}_1$ and $\mathscr{G}_2$ are quasi-split over K_v for all $v \in V^K \setminus V$. Using Remark 5.2, we can find a maximal K-torus T_1 of $\mathscr{G}_1$ that satisfies conditions (α) and (β) of Theorem 5.5 and a maximal K-torus T_2 of G_2 which is isogeneous to T_1 over K. Then $\mathscr{G}_1 \simeq \mathscr{G}_2$ over K by Theorem 5.5, making Γ_1 and Γ_2 commensurable as above.

Our next result imposes further restrictions on the arithmetic groups Γ_1 and Γ_2 given the fact that both the conditions (C_1) and (C_2) fail to hold.

Theorem 5.6 *Let G_1 and G_2 be two connected absolutely simple algebraic groups over a field F of characteristic zero such that $w_1 = w_2$, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(\mathscr{G}_i, K, S)$-arithmetic subgroup for $i = 1, 2$. If both (C_1) and (C_2) fail to hold, then $\mathrm{rk}_K \mathscr{G}_1 = \mathrm{rk}_K \mathscr{G}_2$. Moreover, if G_1 and G_2 are of the same Killing-Cartan type, then the Tits indices of $\mathscr{G}_1/K_v$ and $\mathscr{G}_2/K_v$ are isomorphic for all $v \in V^K$, and the Tits indices of $\mathscr{G}_1/K$ and $\mathscr{G}_2/K$ are isomorphic.*

The proof relies on the following theorem which was actually established in [14], Sect. 7 (although it was not stated there explicitly).

Theorem 5.7 *Let G_1 and G_2 be two connected absolutely simple algebraic K-groups, let L_i be the minimal Galois extension of K over which G_i are of inner type, and let $\mathcal{V}$ be a finite subset of V^K such that both G_1 and G_2 are K_v-quasi-split for all $v \notin \mathcal{V}$. Furthermore, let T_i be a maximal K-torus of G_i, where $i = 1, 2$, such that*

(1) $\theta_{T_i}(\mathrm{Gal}(K_{T_i}/K)) \supset W(G_i, T_i)$;
(2) $\mathrm{rk}_{K_v} T_i = \mathrm{rk}_{K_v} G_i$ *for all* $v \in \mathcal{V}$.

If $L_1 = L_2$ and there exists a K-isogeny $T_1 \to T_2$, then $\mathrm{rk}_K G_1 = \mathrm{rk}_K G_2$. Moreover, if G_1 and G_2 are of the same Killing-Cartan type then the Tits indices of G_1/K_v and G_2/K_v are isomorphic for all $v \in V^K$, and the Tits indices of G_1/K and G_2/K are isomorphic.

For the reader's convenience, we will give a proof of this theorem in the Appendix.

Proof of Theorem 5.6. To derive the required fact from Theorem 5.7, we basically mimic the argument used to consider type D_{2n} in Theorem 5.3. More precisely, we pick a finite set V of places of K containing S_1 so that the groups $\mathscr{G}_1$ and $\mathscr{G}_2$ are quasi-split over K_v for all $v \in V^K \setminus V$. Since by our assumption both (C_1) and (C_2) fail to hold, we can use Remark 5.2 to find of maximal K-torus T_1 of $\mathscr{G}_1$ that satisfies conditions (1) and (2) of Theorem 5.7 for $i = 1$, and a maximal K-torus T_2 of $\mathscr{G}_2$ which is isogeneous to T_1 over K. Since $\mathrm{rk}_{K_v} \mathscr{G}_1 = \mathrm{rk}_{K_v} \mathscr{G}_2$, we obtain that condition (2) holds also for $i = 2$. Furthermore, condition (1) for $i = 1$ combined with the fact that $L_1 = L_2$, by order consideration, yields that the inclusion $\theta_{T_2}(\mathrm{Gal}(L_{2T_2}/L_2)) \subset W(\mathscr{G}_2, T_2)$ is in fact an equality, so (2) holds for $i = 2$ as well. Now, applying Theorem 5.7 we obtain Theorem 5.6. $\square$

We conclude this section with a variant of Theorem 5.6 which has an interesting geometric application (see Theorem 5 in the Introduction; this theorem will be proved in Sect. 7). Let Γ_i is a Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroup of G_i, and assume that $\mathscr{G}_1$ is K_1-isotropic and $\mathscr{G}_2$ is K_2-anisotropic. It follows from Theorem 5.1 (for $K_1 \neq K_2$) and Theorem 5.6 (for $K_1 = K_2$) that then condition (C_i) holds for at least one $i \in \{1, 2\}$. In fact, assuming that $w_1 = w_2$, one can always guarantee that condition (C_1) holds:

Theorem 5.8 *Let G_1 and G_2 be two connected absolutely simple algebraic groups with $w_1 = w_2$. Let Γ_i be a Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroup of G_i for $i = 1, 2$, and assume that $\mathscr{G}_1$ is K_1-isotropic and $\mathscr{G}_2$ is K_2-anisotropic. Then property (C_1) holds.*

The proof relies on the following version of Theorem 5.7 which treats the case where the fields of definitions of Γ_1 and Γ_2 are not necessarily the same.

Theorem 5.7′ *For $i = 1, 2$, let G_i be a connected absolutely simple algebraic group over a number field K_i, and let L_i be the minimal Galois extension of K_i over which G_i is of inner type. Assume that $K_1 \subset K_2$, $L_2 \subset K_2L_1$, $w_1 = w_2$ and $\mathrm{rk}_{K_1} G_1 > 0$. Furthermore, let $\mathcal{V}_1 \subset V^{K_1}$ be a finite subset such that G_2 is quasi-split over K_{2v} for all $v \notin \mathcal{V}_2$, where $\mathcal{V}_2$ consists of all extensions of places contained in $\mathcal{V}_1$ to K_2, and let T_1 be a maximal K_1-torus of G_1 such that*

(1) $\theta_{T_1}(\mathrm{Gal}(K_{1T_1}/K_1)) \supset W(G_1, T_1)$;
(2) $\mathrm{rk}_{K_{1v}} T_1 = \mathrm{rk}_{K_{1v}} G_1$ for all $v \in \mathcal{V}_1$.

If there exists a maximal torus T_2 of G_2 and a K_2-isogeny $T_1 \to T_2$, then $\mathrm{rk}_{K_2} G_2 > 0$.

This result is also proved in the Appendix along with Theorem 5.7.

Proof of Theorem 5.8 If $K_1 \not\subset K_2$ then the fact that (C_1) holds follows from Theorem 4.2 (cf. the proof of Theorem 5.1(a)). So, in the rest of the argument we may assume that $K_1 \subset K_2$.

Next, suppose that $L_2 \not\subset K_2L_1$. In this case, the argument imitates the proof of Theorem 5.1(c). More precisely, we have $K_2L_1 \subsetneqq L_1L_2$. So, if $\mathfrak{L}$ is the normal closure of L_1L_2 over K_1, then there exists $\sigma \in \mathrm{Gal}(\mathfrak{L}/K_1)$ that restricts trivially to K_2L_1 and nontrivially to L_1L_2. By Chebotarev's Density Theorem, we can find $v_0 \in V^{K_1} \setminus S_1$ which is unramified in $\mathfrak{L}/K_1$ and for which the Frobenius automorphism $\mathrm{Fr}(\tilde{v}_0|v_0)$ equals σ for an appropriate extension $\tilde{v}_0|v_0$, and in addition the group $\mathscr{G}_1$ is quasi-split over K_{1v_0}. Let u_0 be the restriction of $\tilde{v}_0$ to K_2. By construction, we have $L_1 \subset K_{1v_0}$, which means that $\mathscr{G}_1$ is actually split over K_{1v_0}; at the same time, $L_2 \not\subset K_{2u_0}$, and therefore $\mathscr{G}_2$ is not split over K_{2u_0}. Set $L = L_1L_2$ and $\mathcal{V}_1 = S_1 \cup \{v_0\}$. Fix $m \geq 1$, and using Theorem 3.3 pick maximal K_1-tori $T_1, \ldots, T_m$ of $\mathscr{G}_1$ that are independent over L and satisfy the following two conditions for each $j \leq m$:

- $\theta_{T_j}(\mathrm{Gal}(L_{T_j}/L)) = W(G_1, T_j)$;
- $\mathrm{rk}_{K_{1v}} T_j = \mathrm{rk}_{K_{1v}} \mathscr{G}_1$ for all $v \in \mathcal{V}_1$.

As in the proof of Theorem 5.1(b), it follows from Dirichlet's Theorem that one can pick elements $\gamma_j \in \Gamma_1 \cap T_j(K_1)$ for $j \leq m$ of infinite order. By Lemma 2.1, the elements $\gamma_1, \ldots, \gamma_m$ are multiplicatively independent, so to establish property (C_1) in the case at hand, it remains to show that these elements are not weakly contained in Γ_2. Assume the contrary. Then according to Theorem 2.3 (with $K = K_2$), there exists a maximal K_2-torus $T^{(2)}$ of $\mathscr{G}_2$ and a K_2-isogeny $T^{(2)} \to T_j$ for some $j \leq m$. Clearly, T_j is split over K_{1v_0}, hence also over K_{2u_0}. We conclude that $T^{(2)}$ is also split over K_{2u_0}, which is impossible as $\mathscr{G}_2$ is not K_{2u_0}-split. This verifies property (C_1) in this case. (We note that so far we have not used the assumption that $\mathscr{G}_1$ is K_1-isotropic and $\mathscr{G}_2$ is K_2-anisotropic.)

It remains to consider the case where $K_1 \subset K_2$ and $L_2 \subset K_2L_1$. Here the argument is very similar to the one given above but uses a different choice of the set $\mathcal{V}_1$ and relies on Theorem 5.7′. More precisely, pick a finite subset $\mathcal{V}_1 \subset V^{K_1}$ containing S_1 so that $\mathscr{G}_2$ is quasi-split over K_{2v} for all $v \in \mathcal{V}_2$, where $\mathcal{V}_2$ consists of all extensions of places in $\mathcal{V}_1$ to K_2. Assume that (C_1) does not hold, i.e., there exists $m \geq 1$ such that any m multiplicatively independent semi-simple elements of Γ_1 of infinite order are necessarily weakly contained in Γ_2. Fix such an m, and using the same L as above, pick maximal K_1-tori $T_1, \ldots, T_m$ of $\mathscr{G}_1$ that are independent over L and satisfy the above bulleted conditions for this new choice of $\mathcal{V}_1$. Arguing as in the previous paragraph, we see that again, there exists a maximal K_2-torus $T^{(2)}$ of $\mathscr{G}_2$ and a K_2-isogeny $T^{(2)} \to T_j$ for some $j \leq m$. Then it follows from Theorem 5.7′ that $\mathscr{G}_2$ is K_2-isotropic, a contradiction. $\square$

It would be interesting to determine if the assumption that $w_1 = w_2$ in Theorem 5.8 can be omitted. In this connection, we would like to ask the following

Question *Is it possible to construct K_1-isotropic $\mathscr{G}_1$ and K_2-anisotropic $\mathscr{G}_2$ over number fields $K_1 \subset K_2$ so that every K_1-anisotropic torus of $\mathscr{G}_1$ is K_2-isomorphic to a K_2-torus of $\mathscr{G}_2$?*

(Obviously, an affirmative answer to this question with $K_1 = \mathbb{Q}$ would lead to an example where every semi-simple element of infinite order in Γ_1 would be weakly contained in Γ_2 and therefore (C_1) would not hold.)

It was observed by Skip Garibaldi that the above question has an affirmative answer over fields other than finite extensions of $\mathbb{Q}$. Of course, the trivial example is $\mathscr{G}_1 = \mathrm{SL}_2$ and $\mathscr{G}_2 = \mathrm{SL}_{1,D}$ where D is the algebra of Hamiltonian quaternions over $\mathbb{R}$, but a similar construction can be implemented with K_2/K_1 being a nontrivial finite extension of (infinite) algebraic extensions of $\mathbb{Q}$. More precisely, we recall that a field K is called euclidean if it is real and has a unique quadratic extension which is then necessarily $K(\sqrt{-1})$ (E. Becker). Furthermore, K is hereditarily euclidean if it is euclidean and every finite real extension of it is also euclidean. Given two distinct real closed subfields R_1 and R_2 of a fixed algebraic closure of $\mathbb{Q}$, their intersection $K_1 = R_1 \cap R_2$ is a hereditarily euclidean field ([6,17]); clearly, K_1 has a nontrivial finite real extension K_2, which can be found, for example, inside R_2. Then the groups $\mathscr{G}_1 = \mathrm{SL}_2$ over K_1 and $\mathscr{G}_2 = \mathrm{SL}_{1,D}$ over K_2, where again D is the algebra of Hamiltonian quaternions over K_2, provide a required example.

6 Groups of types A_n, D_n and E_6

It is known that the assertion of Theorem 5.3 may fail if the common Killing-Cartan type of the groups G_1 and G_2 is one of the following: A_n, D_{2n+1} $(n > 1)$ or E_6 (cf. Examples 6.5, 6.6, 6.7 and §9 in [14]). Nevertheless, a suitable analog of Theorem 5.3 with interesting geometric consequences can still be given (cf. Theorem 6.6). It is based on the following notion.

Definition Let G_1 and G_2 be connected absolutely almost simple algebraic groups defined over a field K. We say that G_1 and G_2 have *equivalent systems of maximal K-tori* if for every maximal K-torus T_1 of G_1 there exists a $\overline{K}$-isomorphism $\varphi\colon G_1 \to G_2$ such that the restriction $\varphi|_{T_1}$ is defined over K, and conversely, for every maximal K-torus T_2 of G_2 there exists a $\overline{K}$-isomorphism $\psi\colon G_2 \to G_1$ such that the restriction $\psi|_{T_2}$ is defined over K.

We note that given a $\overline{K}$-isomorphism $\varphi\colon G_1 \to G_2$ as in the definition, the torus $T_2 = \varphi(T_1)$ is defined over K and the corresponding map $X(T_2) \to X(T_1)$ induces a bijection $\Phi(G_2, T_2) \to \Phi(G_1, T_1)$. This observation implies that if G_i is a connected absolutely almost simple real algebraic group, $\Gamma_i \subset G_i(\mathbb{R})$ is a torsion-free $(\mathscr{G}_i, K)$-arithmetic subgroup and $\mathfrak{X}_{\Gamma_i}$ is the associated locally symmetric space, where $i = 1, 2$, then the fact that $\mathscr{G}_1$ and $\mathscr{G}_2$ have equivalent systems of maximal K-tori entails that $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable (see Proposition 9.14 of [14]). For technical reasons, in this section it is more convenient for us to deal with simply connected groups rather than with adjoint ones which are more natural from the geometric standpoint. So, we observe in this regard that if simply connected K-groups G_1 and G_2 have equivalent systems of maximal K-tori then so do the corresponding adjoint groups $\overline{G}_1$ and $\overline{G}_2$ (and vice versa).

We will now describe fairly general conditions guaranteeing that two forms over a number field K, of an absolutely almost simple simply connected group of one of types A_n, $D_{2n+1} (n > 1)$, or E_6, have equivalent systems of maximal K-tori.

Theorem 6.1 *Let G_1 and G_2 be two connected absolutely almost simple simply connected algebraic groups of one of the following types: A_n, $D_{2n+1}(n > 1)$ or E_6, defined over a number field K, and let L_i be the minimal Galois extension of K over which G_i is of inner type. Assume that*

$$\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2 \quad \text{for all} \quad v \in V^K, \tag{24}$$

hence[2] *$L_1 = L_2 =: L$. Moreover, if G_1 and G_2 are of type D_{2n+1} we assume that for each real place v of K, we can find maximal K_v-tori T_i^v of G_i for $i = 1, 2$, such that $\mathrm{rk}_{K_v} T_i^v = \mathrm{rk}_{K_v} G_i$ and there exists a K_v-isomorphism $T_1^v \to T_2^v$ that extends to a $\overline{K_v}$-isomorphism $G_1 \to G_2$. If*

(1) one can pick maximal K-tori T_i^0 of G_i for $i = 1, 2$ with a K-isomorphism $T_1^0 \to T_2^0$ that extends to a $\overline{K}$-isomorphism $G_1 \to G_2$, and

(2) there exists a place v_0 of K such that one of the groups G_i is K_{v_0}-anisotropic (and then both are such due to (24)),

[2] As we have seen in the proof of Theorem 5.1, the former condition automatically implies the latter.

then G_1 and G_2 have equivalent systems of maximal K-tori.

Proof We begin by establishing first the corresponding local assertion.

Lemma 6.2 *Let $\mathscr{G}_1$ and $\mathscr{G}_2$ be two connected absolutely almost simple simply connected algebraic groups of one of the following types: A_ℓ ($\ell \geq 1$), D_ℓ ($\ell \geq 5$) or E_6, over a nondiscrete locally compact field $\mathscr{K}$ of characteristic zero, and let $\mathscr{L}_i$ be the minimal Galois extension of $\mathscr{K}$ over which $\mathscr{G}_i$ is of inner type. Assume that*

$$\mathscr{L}_1 = \mathscr{L}_2 =: \mathscr{L} \quad and \quad \mathrm{rk}_{\mathscr{K}}\, \mathscr{G}_1 = \mathrm{rk}_{\mathscr{K}}\, \mathscr{G}_2,$$

and moreover, in case $\mathscr{G}_1$ and $\mathscr{G}_2$ are of type D_ℓ and $\mathscr{K} = \mathbb{R}$, there exist maximal $\mathscr{K}$-tori $\mathscr{T}_i$ of $\mathscr{G}_i$ such that $\mathrm{rk}_{\mathscr{K}}\, \mathscr{T}_i = \mathrm{rk}_{\mathscr{K}}\, \mathscr{G}_i$ for $i = 1, 2$, with a $\mathscr{K}$-isomorphism $\mathscr{T}_1 \to \mathscr{T}_2$ that extends to a $\overline{\mathscr{K}}$-isomorphism $\mathscr{G}_1 \to \mathscr{G}_2$. Then

- *(i) except in the case where $\mathscr{G}_1$ and $\mathscr{G}_2$ are inner K-forms of a split group of type A_ℓ with $\ell > 1$, we have $\mathscr{G}_1 \simeq \mathscr{G}_2$ over $\mathscr{K}$;*
- *(ii) in all cases, $\mathscr{G}_1$ and $\mathscr{G}_2$ have equivalent systems of maximal $\mathscr{K}$-tori.*

Proof (i): First, let $\mathscr{G}_1$ and $\mathscr{G}_2$ be *outer* $\mathscr{K}$-forms of a split group of type A_ℓ associated with a quadratic extension $\mathscr{L}$ of $\mathscr{K}$. Then $\mathscr{G}_i = \mathrm{SU}(\mathscr{L}, h_i)$ where h_i is a nondegenerate Hermitian form on $\mathscr{L}^n$, $n = \ell + 1$, with respect to the nontrivial automorphism of $\mathscr{L}/\mathscr{K}$. Since $\mathrm{rk}_{\mathscr{K}}\mathscr{G}_i$ coincides with the Witt index of the Hermitian form h_i, the forms h_1 and h_2 have equal Witt index. On the other hand, it is well-known, and easy to see, that the similarity class of an *anisotropic* Hermitian form over $\mathscr{L}$ is determined by its dimension (which for nonarchimedean v is necessarily ≤ 2). So, the fact that h_1 and h_2 have equal Witt index implies that h_1 and h_2 are similar, hence $\mathscr{G}_1 \simeq \mathscr{G}_2$, as required.

Now, suppose $\mathscr{G}_1$ and $\mathscr{G}_2$ are of type D_ℓ with $\ell \geq 5$. If $\mathscr{K} = \mathbb{C}$ then there is nothing to prove; otherwise there is a unique quaternion central division algebra $\mathscr{D}$ over $\mathscr{K}$. For each $i \in \{1, 2\}$, we have two possibilities: either $\mathscr{G}_i = \mathrm{Spin}_n(q_i)$ where q_i is a nondegenerate quadratic form over $\mathscr{K}$ of dimension $n = 2\ell$ (orthogonal type), or $\mathscr{G}_i$ is the universal cover of $\mathrm{SU}(\mathscr{D}, h_i)$ where h_i is a nondegenerate skew-Hermitian form on $\mathscr{D}^\ell$ with respect to the canonical involution on $\mathscr{D}$ (quaternionic type). We will now show that in our situation, $\mathscr{G}_1$ and $\mathscr{G}_2$ are both of the same, orthogonal or quaternionic, type. First, we treat the case where $\mathscr{K}$ is nonarchimedean. Assume that $\mathscr{G}_1$ is of orthogonal, and $\mathscr{G}_2$ is of quaternionic, type. Then $\mathrm{rk}_{\mathscr{K}}\mathscr{G}_1 \geq (2\ell - 4)/2 = \ell - 2$, while $\mathrm{rk}_{\mathscr{K}}\mathscr{G}_2 \leq \ell/2$. So, $\mathrm{rk}_{\mathscr{K}}\mathscr{G}_1 = \mathrm{rk}_{\mathscr{K}}\mathscr{G}_2$ is impossible as $\ell \geq 5$, a contradiction. Over $\mathscr{K} = \mathbb{R}$, however, one can have $\mathscr{G}_1$ of orthogonal type and $\mathscr{G}_2$ of quaternionic type with the same $\mathscr{K}$-rank, so to prove our assertion in this case we need to use the hypothesis that there exist maximal $\mathscr{K}$-tori $\mathscr{T}_i$ of $\mathscr{G}_i$ such that $\mathrm{rk}_{\mathscr{K}}\, \mathscr{T}_i = \mathrm{rk}_{\mathscr{K}}\, \mathscr{G}_i$, with a $\mathscr{K}$-isomorphism $\mathscr{T}_1 \to \mathscr{T}_2$ that extends to a $\overline{\mathscr{K}}$-isomorphism $\mathscr{G}_1 \to \mathscr{G}_2$. Such an isomorphism induces an isomorphism between the Tits indices of $\mathscr{G}_1/\mathscr{K}$ and $\mathscr{G}_2/\mathscr{K}$ (cf. the discussion in §7.1 of [14]). However, if $\mathscr{G}_1$ is of orthogonal type, and $\mathscr{G}_2$ of quaternionic, the corresponding Tits indices are not isomorphic, and our assertion follows.

Now, let $\mathscr{G}_1$ and $\mathscr{G}_2$ be of quaternionic type. It is known that two nondegenerate skew-Hermitian forms over $\mathscr{D}$ are equivalent if they have the same dimension and in addition the same discriminant in the nonarchimedean case (cf. [19], Ch. 10, Theorem 3.6 in the nonarchimedean case, and Theorem 3.7 in the archimedean case). If h_1 and h_2 are the skew-Hermitian forms defining $\mathscr{G}_1$ and $\mathscr{G}_2$ respectively, then the condition that h_1 and h_2 have the same discriminant is equivalent to the fact that $\mathscr{L}_1 = \mathscr{L}_2$, and therefore holds in our situation. Thus, h_1 and h_2 are equivalent, hence $\mathscr{G}_1$ and $\mathscr{G}_2$ are $\mathscr{K}$-isomorphic.

Next, let $\mathscr{G}_1$ and $\mathscr{G}_2$ be of orthogonal type, $\mathscr{G}_i = \mathrm{Spin}(q_i)$. To show that $\mathscr{G}_1 \simeq \mathscr{G}_2$, it is enough to show that q_1 and q_2 are similar. Let q_i^a be a maximal anisotropic subform of q_i for $i = 1, 2$. The condition $\mathrm{rk}_{\mathscr{K}}\, \mathscr{G}_1 = \mathrm{rk}_{\mathscr{K}}\, \mathscr{G}_2$ yields that q_1 and q_2 have the same Witt index, so we just need to show that q_1^a and q_2^a are similar. If $\mathscr{K} = \mathbb{R}$, then any two anisotropic forms of the same dimension are similar, and there is nothing to prove. Now, let $\mathscr{K}$ be nonarchimedean. Our claim is obvious if $q_1^a = q_2^a = 0$; in the two remaining cases the common dimension of q_1^a and q_2^a can only be 2 or 4. To treat binary forms, we observe that q_1 and q_2, hence also q_1^a and q_2^a, have the same discriminant, and two binary forms of the same discriminant are similar. The claim for quaternary forms follows from the fact that there exists a single equivalence class of such anisotropic forms (this equivalence class is represented by the reduced-norm form of $\mathscr{D}$).

Finally, we consider groups of type E_6. If $\mathscr{K} = \mathbb{R}$ then by inspecting the tables in [21] we find that there are two possible indices for the inner forms with the corresponding groups having $\mathbb{R}$-ranks 2 and 6, and there are three possible indices for outer forms for which the $\mathbb{R}$-ranks are 0, 2 and 4. Thus, since G_1 and G_2 are simultaneously either inner or outer forms and have the same $\mathbb{R}$-rank, they are $\mathbb{R}$-isomorphic. To establish the same conclusion in the nonarchimedean case, we recall that then an outer form of type E_6 is always quasi-split (cf. [7], Proposition 6.15), so for outer forms our assumption that $\mathscr{L}_1 = \mathscr{L}_2$ implies that $G_1 \simeq G_2$. Since there exists only one nonsplit inner form of type E_6 (this follows, for example, from the proof of Lemma 9.9(ii) in [14]), our assertion holds in this case as well.

(ii): It remains to be shown that if $\mathscr{G}_1$ and $\mathscr{G}_2$ are inner forms of type A_ℓ over $\mathscr{K}$ such that $\mathrm{rk}_{\mathscr{K}}\, \mathscr{G}_1 = \mathrm{rk}_{\mathscr{K}}\, \mathscr{G}_2$, then $\mathscr{G}_1$ and $\mathscr{G}_2$ have equivalent systems of maximal $\mathscr{K}$-tori. We have $\mathscr{G}_i = \mathrm{SL}_{d_i, \mathscr{D}_i}$ where $\mathscr{D}_i$ is a central division algebra over $\mathscr{K}$ of degree n_i and

$$\mathrm{rk}_{\mathscr{K}}\, \mathscr{G}_i = d_i - 1 \quad \text{and} \quad d_i m_i = \ell + 1 =: n.$$

Thus, in our situation $d_1 = d_2$ and $n_1 = n_2$. Furthermore, it is well-known (cf. [15], Proposition 2.6) that a commutative étale n-dimensional $\mathscr{K}$-algebra $\mathscr{E} = \prod_{j=1}^{s} \mathscr{E}^{(j)}$, where $\mathscr{E}^{(j)}/\mathscr{K}$ is a finite (separable) field extension, embeds in $\mathscr{A}_i := M_{d_i}(\mathscr{D}_i)$ if and only if each degree $[\mathscr{E}^{(j)} : \mathscr{K}]$ is divisible by n_i. So, we conclude that $\mathscr{E}$ embeds in $\mathscr{A}_1$ if and only if it embeds in $\mathscr{A}_2$. On the other hand, any maximal $\mathscr{K}$-torus $\mathscr{T}_1$ of $\mathscr{G}_1$ coincides with the torus $\mathrm{R}^{(1)}_{\mathscr{E}_1/\mathscr{K}}(\mathrm{GL}_1)$ associated with the group of norm one elements in some n-dimensional commutative étale subalgebra $\mathscr{E}_1$ of $\mathscr{A}_1$. As we noted above, $\mathscr{E}_1$ embeds in $\mathscr{A}_2$, and then using the Skolem–Noether Theorem (see Footnote 1 on p. 592 in [15]) one can construct an isomorphism $\mathscr{A}_1 \otimes_{\mathscr{K}} \overline{\mathscr{K}} \simeq \mathscr{A}_2 \otimes_{\mathscr{K}} \overline{\mathscr{K}}$ that maps $\mathscr{E}_1$ to a subalgebra $\mathscr{E}_2 \subset \mathscr{A}_2$. This isomorphism gives rise to a $\overline{K}$-isomorphism $\mathscr{G}_1 \simeq \mathscr{G}_2$ that induces a $\mathscr{K}$-isomorphism between $\mathscr{T}_1$ and $\mathscr{T}_2 := \mathrm{R}_{\mathscr{E}_2/\mathscr{K}}(\mathrm{GL}_1)$. By symmetry, $\mathscr{G}_1$ and $\mathscr{G}_2$ have equivalent systems of maximal $\mathscr{K}$-tori. $\square$

To complete the proof of Theorem 6.1, we fix a $\overline{K}$-isomorphism $\varphi_0 \colon G_1 \to G_2$ such that the restriction $\varphi_0|_{T_1^0}$ is a K-isomorphism between T_1^0 and T_2^0. Let T_1 be an arbitrary maximal K-torus of G_1. Then by Lemma 6.2, for any $v \in V^K$, there exists a $\overline{K_v}$-isomorphism $\varphi_v : G_1 \to G_2$ whose restriction to T_1 is defined over K_v. Then $\varphi_v = \alpha \cdot \varphi_0$ for some $\alpha \in \mathrm{Aut}G_2$. There exists an automorphism of G_2 that acts as $t \mapsto t^{-1}$ on $T_2 := \varphi_v(T_1)$. Moreover, for groups of the types listed in the theorem, this automorphism represents the only nontrivial element of $\mathrm{Aut}G_2/\mathrm{Inn}G_2$. So, if necessary, we can replace φ_v by the composite of φ_v with this automorphism to ensure that α is inner (and the restriction of φ_v to T_1 is still defined over K, cf. the proof of Lemma 9.7 in [14]). This shows that T_1 admits a *coherent* (relative to φ_0) K_v-embedding in G_2 (in the terminology introduced in [14], §9), for every $v \in V^K$. Since T_1 is K_{v_0}-anisotropic, $\text{Ш}^2(T_1)$ is trivial (cf. [7], Proposition 6.12). So, by Theorem 9.6 of [14], T_1 admits a coherent K-defined embedding in G_2 which in particular is a K-embedding $T_1 \to G_2$ which extends to a $\overline{K}$-isomorphism $G_1 \to G_2$. By symmetry, G_1 and G_2 have equivalent systems of maximal K-tori.

The following proposition complements Theorem 6.1 for groups of type A_n in that it does not assume the existence of a place $v_0 \in V^K$ where the groups are anisotropic.

Proposition 6.3 *Let G_1 and G_2 be two connected absolutely almost simple simply connected algebraic groups of type A_n over a number field K, and let L_i be the minimal Galois extension of K over which G_i is of inner type. Assume that*

$$\mathrm{rk}_{K_v} G_1 = \mathrm{rk}_{K_v} G_2 \quad \textit{for all} \;\; v \in V^K, \tag{25}$$

hence $L_1 = L_2 =: L$. In each of the following situations:

(1) G_1 and G_2 are inner forms,
(2) G_1 and G_2 are outer forms, and one of them is represented by $\mathrm{SU}(D, \tau)$*, where D is a central* division *algebra over L with an involution τ of the second kind that restricts to the nontrivial automorphism σ of L/K (then both groups are of this form),*

the groups G_1 and G_2 have equivalent systems of maximal K-tori.

Proof (1): We have $G_i = \mathrm{SL}_{1,A_i}$ where A_i is a central simple algebra over K of dimension $(n+1)^2$, and as in the proof of Lemma 6.2, it is enough to show that a commutative étale $(n+1)$-dimensional K-algebra E embeds in A_1 if and only if it embeds in A_2. For $v \in V^K$, we can write

$$A_i \otimes_K K_v = M_{d_i^{(v)}}(\Delta_i^{(v)})$$

where $\Delta_i^{(v)}$ is a central division algebra over K_v, of degree $m_i^{(v)}$. As in the proof of Lemma 6.2, we conclude that (25) implies $m_1^{(v)} = m_2^{(v)}$. On the other hand, it is well-known (cf. [15], Propositions 2.6 and 2.7) that an $(n+1)$-dimensional commutative étale K-algebra $E = \prod_{j=1}^{s} E^{(j)}$, where $E^{(j)}/K$ is a finite (separable) field extension, embeds in A_i if and only if for each $j \le s$ and all $v \in V^K$, the local degree $[E_w^{(j)} : K_v]$ is divisible by $m_i^{(v)}$ for all extensions $w|v$, and the required fact follows.

(2): We have $G_i = \mathrm{SU}(D_i, \tau_i)$, where D_i is a central simple algebra of degree $m = n+1$ over L with an involution τ_i such that $\tau_i|L = \sigma$. Assume that D_1 is a division algebra. Then it follows from the Albert–Hasse–Brauer–Noether Theorem that $m = \mathrm{lcm}_{w \in V^L}(m_1^{(w)})$, where for $w \in V^L$, $D_i \otimes_L L_w = M_{d_i^{(w)}}(\Delta_i^{(w)})$ with $\Delta_i^{(w)}$ a central division algebra over L_w of degree $m_i^{(w)}$. For $j = 1, 2$, set

$$V_j^L = \{w \in V^L | [L_w : K_v] = j \;\text{ where }\; w|v\}.$$

It is well-known that $m_i^{(w)} = 1$ for $w \in V_2^L$, so

$$m = \mathrm{lcm}_{w \in V_1^L}(m_1^{(w)}).$$

On the other hand, for $w \in V_1^L$ we have $G_i \simeq \mathrm{SL}_{d_i^{(w)}, \Delta_i^{(w)}}$ over $K_v = L_w$, hence $\mathrm{rk}_{K_v} G_i = d_i^{(w)} - 1$. Thus, (25) implies that $m_1^{(w)} = m_2^{(w)}$ for all $w \in V_1^L$, and therefore

$$m = \mathrm{lcm}_{w \in V_1^L}(m_2^{(w)}).$$

It follows that D_2 is a division algebra, as required.

Next, since any maximal K-torus of G_i is of the form $\mathrm{R}_{E/K}(\mathrm{GL}_1) \cap G_i$ for some m-dimensional commutative étale L-algebra invariant under τ_i (cf. [15], Proposition 2.3), it is enough to show that for an m-dimensional commutative étale L-algebra E with an involutive automorphism τ such that $\tau|L = \sigma$, the existence of an embedding $\iota_1 \colon (E, \tau) \hookrightarrow (D_1, \tau_1)$ as L-algebras with involutions is equivalent to the existence of an embedding $\iota_2 \colon (E, \tau) \hookrightarrow (D_2, \tau_2)$. Since D_1 is a division algebra, the existence of ι_1 implies that E/L is a field extension, and then by Theorem 4.1 of [15], the existence of ι_2 is equivalent to the existence of an $(L \otimes_K K_v)$-embedding

$$\iota_2^{(v)} \colon (E \otimes_K K_v, \tau \otimes \mathrm{id}_{K_v}) \hookrightarrow (D_2 \otimes_K K_v, \tau_2 \otimes \mathrm{id}_{K_v})$$

for all $v \in V^K$. If $v \in V^K$ has two extensions $w', w'' \in V_1^L$, then $m_i^{(w')} = m_i^{(w'')} =: m_i^{(v)}$ and a necessary and sufficient condition for the existence of $\iota_i^{(v)}$ is that for any extension u of v to E, the local degree $[E_u : K_v]$ is divisible by $m_i^{(v)}$ (cf. Proposition A.3 in [9]). Therefore, since $m_1^{(v)} = m_2^{(v)}$, the existence of $\iota_1^{(v)}$ implies that of $\iota_2^{(v)}$. If v has only one extension w to L, then $w \in V_2^L$ and

$$(D_i \otimes_K K_v, \tau_i \otimes \mathrm{id}_{K_v}) \simeq (M_m(L_w), \theta_i)$$

with θ_i given by $\theta((x_{st})) = a_i^{-1}(\overline{x}_{ts})a_i$ where $x \mapsto \overline{x}$ denotes the nontrivial automorphism of L_w/K_v and a_i is a Hermitian matrix. Furthermore, $\mathrm{rk}_{K_v}\, G_i$ equals the Witt index $i(h_i)$ of the Hermitian form h_i with matrix a_i. Then (25) yields that $i(h_1) = i(h_2)$ which as we have seen in the proof of Lemma 6.2(i) implies that h_1 and h_2 are similar. Hence,

$$(D_1 \otimes_K K_v, \tau_1 \otimes \mathrm{id}_{K_v}) \simeq (D_2 \otimes_K K_v, \tau_2 \otimes \mathrm{id}_{K_v}),$$

and therefore again the existence of $\iota_1^{(v)}$ implies the existence of $\iota_2^{(v)}$.

Finally, since D_2 is also a division algebra, we can use the above argument to conclude that (D_1, τ_1) and (D_2, τ_2) in fact have the same m-dimensional commutative étale L-subalgebras invariant under the involutions as claimed. □

Remark 6.4 (1) We have already noted prior to Proposition 6.3 that the assumption (2) of Theorem 6.1 is not needed in the statement of the proposition. So, it is worth mentioning that assumption (1) in this situation is in fact satisfied automatically: for groups of outer type A_n this follows from Corollary 4.5 in [15], while for groups of inner type A_n it is much simpler, viz. in the notation used in the proof of Proposition 6.3(1), one shows that the algebras A_1 and A_2 contain a common field extension of K of degree $(n+1)$. This can also be established for groups of type D_n with n odd using Proposition A of [15].

(2) We would like to clarify that assumption (2) of Theorem 6.1 is only needed to conclude that $Ш^2(T_1)$ is trivial for any maximal K-torus T_1 of G_1. However, this fact holds for any maximal K-torus in a connected absolutely almost simple simply connected algebraic K-group of inner type A_n *unconditionally*, cf. Remark 9.13 in [14]. So, the proof of Theorem 6.1 actually yields part (1) of Proposition 6.3.

Corollary 6.5 *Let G_1 and G_2 be two connected absolutely almost simple simply connected algebraic groups of type A_{p-1}, where p is a prime, over a number field K. Assume that (25) holds and that $L_1 = L_2 =: L$. Then G_1 and G_2 have equivalent systems of maximal K-tori.*

Indeed, if G_1 and G_2 are inner forms (in particular, if $p = 2$) then our assertion immediately follows from Proposition 6.3(1). Furthermore, if one of the groups is of the form $\mathrm{SU}(D, \tau)$ where D is a central *division* algebra over L of degree p then we can use Proposition 6.3(2). It remains to consider the case where $G_i = \mathrm{SU}(L, h_i)$ with h_i a nondegenerate hermitian form on L^p for $i = 1, 2$. Then the proof of Lemma 6.2(i) shows that h_1 and h_2 are similar over L_w for all $w \in V_2^L$. But then h_1 and h_2 are similar, i.e., $G_1 \simeq G_2$ over K and there is nothing to prove.

Here is a companion to Theorem 5.3 for groups of types A, D and E_6.

Theorem 6.6 *Let G_1 and G_2 be two connected absolutely almost simple algebraic groups of the same Killing-Cartan type which is one of the following: A_n, D_{2n+1} $(n > 1)$ or E_6 defined over a field F of characteristic zero, and let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(\mathscr{G}_i, K_i, S_i)$-arithmetic subgroup. Assume that for at least one $i \in \{1, 2\}$ there exists $v_0^{(i)} \in V^{K_i}$ such that $\mathscr{G}_i$ is anisotropic over $(K_i)_{v_0^{(i)}}$. Then either condition (C_i) holds for some $i \in \{1, 2\}$, or $K_1 = K_2 =: K$ and the groups $\mathscr{G}_1$ and $\mathscr{G}_2$ have equivalent systems of maximal K-tori.*

(We note that if $\mathscr{G}_1$ and $\mathscr{G}_2$ have equivalent systems of maximal K-tori then (C_i) can hold only if $S_1 \neq S_2$.)

Proof We can obviously assume that for $i = 1, 2$, the group G_i is adjoint and $\Gamma_i \subset \mathscr{G}_i(K_i)$. According to Theorem 5.1, if neither (C_1) nor (C_2) hold, then we have

$$K_1 = K_2 =: K, \quad L_1 = L_2 =: L, \quad S_1 = S_2 =: S$$

and

$$\mathrm{rk}_{K_v}\mathscr{G}_1 = \mathrm{rk}_{K_v}\mathscr{G}_2 \quad \text{for all} \quad v \in V^K.$$

Furthermore, there exists $m \geq 1$ such that any m multiplicatively independent semi-simple elements $\gamma_1, \ldots, \gamma_m \in \Gamma_1$ are necessarily weakly contained in Γ_2. Arguing as in the proof of Theorem 5.1, we can find m multiplicatively independent elements $\gamma_1, \ldots, \gamma_m \in \Gamma_1$ so that the corresponding tori $T_i = Z_{\mathscr{G}_1}(\gamma_i)^\circ$ satisfy the following:

- $\theta_{T_i}(\mathrm{Gal}(L_{T_i}/L)) = W(\mathscr{G}_1, T_i)$;
- $\mathrm{rk}_{K_v} T_i = \mathrm{rk}_{K_v}\mathscr{G}_1$ for all $v \in S$.

Then the fact that $\gamma_1, \ldots, \gamma_m$ are weakly contained in Γ_2 would imply that there exists a maximal K-torus T_2^0 of $\mathscr{G}_2$ and an $i \leq m$ such that there is a K-isogeny $T_2^0 \to T_1^0 := T_i$. Since the common type of $\mathscr{G}_1$ and $\mathscr{G}_2$ is different from $B_2 = C_2$, F_4 and G_2, it follows from Lemma 4.3 and Remark 4.4 in [14] that one can scale the isogeny so that it induces an isomorphism between the root systems $\Phi(\mathscr{G}_1, T_1^0)$ and $\Phi(\mathscr{G}_2, T_2^0)$, and therefore extends to a $\overline{K}$-isomorphism $\mathscr{G}_1 \to \mathscr{G}_2$ as these groups are adjoint. Passing to the simply connected groups $\widetilde{\mathscr{G}_1}$ and $\widetilde{\mathscr{G}_2}$ and the corresponding tori $\widetilde{T}_1^0$ and $\widetilde{T}_2^0$, we see that there exists a K-isomorphism $\widetilde{T}_1^0 \to \widetilde{T}_2^0$ that extends to a $\overline{K}$-isomorphism $\widetilde{\mathscr{G}_1} \to \widetilde{\mathscr{G}_2}$. Note that by our construction we have $\mathrm{rk}_{K_v} T_i^0 = \mathrm{rk}_{K_v} \mathscr{G}_i$ for $i = 1, 2$ and all real places v of K. In view of our assumptions, we can invoke Theorem 6.1 to conclude that $\widetilde{\mathscr{G}_1}$ and $\widetilde{\mathscr{G}_2}$ have equivalent systems of maximal K-tori, and then the same remains true for $\mathscr{G}_1$ and $\mathscr{G}_2$. □

It follows from Proposition 6.3 and Corollary 6.5 that the assertion of Theorem 6.6 remains valid without the assumption that there be $v_0^{(i)} \in V^{K_i}$ such that $\mathscr{G}_i$ is $(K_i)_{v_0^{(i)}}$-anisotropic for groups of type A_n in the following three situations: (1) one of the $\mathscr{G}_i$'s is an inner form; (2) the simply connected cover of one of the $\mathscr{G}_i$'s is isomorphic to $\mathrm{SU}(D, \tau)$ where D is a central *division* algebra over L with an involution τ of the second kind that restricts to the nontrivial automorphism of L/K; (3) $n = p - 1$ where p is a prime.

7 Fields generated by the lengths of closed geodesics: Proof of Theorems 1, 2, 3 and 5

Let G be an absolutely simple adjoint algebraic $\mathbb{R}$-group such that $\mathcal{G} := G(\mathbb{R})$ is noncompact. Pick a maximal compact subgroup $\mathcal{K}$ of $\mathcal{G}$, and let $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$ denote the corresponding symmetric space considered as a Riemannian manifold with the metric induced by the Killing form. Given a discrete torsion-free subgroup $\Gamma \subset \mathcal{G}$, we consider the associated locally symmetric space $\mathfrak{X}_\Gamma := \mathfrak{X}/\Gamma$. It was shown in [14], 8.4, that every (nontrivial) semi-simple element $\gamma \in \Gamma$ gives rise to a closed geodesic c_γ in $\mathfrak{X}_\Gamma$, and conversely, every closed geodesic can be obtained that way. Moreover, the length $\ell(c_\gamma)$ can be written in the form $(1/n_\gamma)\cdot\lambda_\Gamma(\gamma)$ where $n_\gamma \geq 1$ is an integer and

$$\lambda_\Gamma(\gamma) = \left(\sum_\alpha (\log|\alpha(\gamma)|)^2\right)^{1/2} \tag{26}$$

where the summation is over all roots α of G with respect to an arbitrary maximal $\mathbb{R}$-torus T containing γ (Proposition 8.5 of [14]). In particular, for the set $L(\mathfrak{X}_\Gamma)$ of lengths of all closed geodesics in $\mathfrak{X}_\Gamma$ we have

$$\mathbb{Q}\cdot L(\mathfrak{X}_\Gamma) = \mathbb{Q}\cdot\{\lambda_\Gamma(\gamma) | \gamma \in \Gamma \text{ nontrivial semi-simple}\},$$

and the subfield of $\mathbb{R}$ generated by $L(\mathfrak{X}_\Gamma)$ coincides with the subfield generated by the values $\lambda_\Gamma(\gamma)$ for all semi-simple $\gamma \in \Gamma$.

Now, let G_1 and G_2 be two absolutely simple adjoint algebraic $\mathbb{R}$-groups such that the group $\mathcal{G}_i := G_i(\mathbb{R})$ is noncompact for both $i = 1, 2$. For each $i \in \{1, 2\}$, we pick a maximal compact subgroup $\mathcal{K}_i$ of $\mathcal{G}_i := G_i(\mathbb{R})$ and consider the symmetric space $\mathfrak{X}_i = \mathcal{K}_i\backslash\mathcal{G}_i$. Furthermore, given a discrete torsion-free Zariski-dense subgroup Γ_i of $\mathcal{G}_i$, we let $\mathfrak{X}_{\Gamma_i} := \mathfrak{X}_i/\Gamma_i$ denote the associated locally symmetric space. As above, for $i = 1, 2$, we let w_i denote the order of the Weyl group of G_i with respect to a maximal torus, and let K_{Γ_i} be the field of definition of Γ_i, i.e. the subfield of $\mathbb{R}$ generated by the traces $\mathrm{Tr}\mathrm{Ad}\gamma$ for $\gamma \in \Gamma_i$. In this section, we will focus our attention on the fields $\mathscr{F}_i$ generated by the set $L(\mathfrak{X}_{\Gamma_i})$, for $i = 1, 2$.

The results of this section depend on the truth of Schanuel's conjecture from transcendental number theory (hence they are *conditional*). For the reader's convenience we recall its statement (cf. [1,2], p. 120).

Schanuel's conjecture. *If $z_1, \ldots, z_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$, then the transcendence degree (over $\mathbb{Q}$) of the field generated by*

$$z_1, \ldots, z_n;\ e^{z_1}, \ldots, e^{z_n}$$

is $\geq n$.

Assuming Schanuel's conjecture and developing the techniques of [12], we prove the following proposition which enables us to connect the results of the previous sections to some geometric problems involving the sets $L(\mathfrak{X}_{\Gamma_i})$ and the fields $\mathscr{F}_i$.

Proposition 7.1 *Let $\mathscr{K} \subset \mathbb{R}$ be a subfield of finite transcendence degree d over $\mathbb{Q}$, let G_1 and G_2 be semi-simple $\mathscr{K}$-groups, and for $i \in \{1, 2\}$, let $\Gamma_i \subset G_i(\mathscr{K}) \subset G_i(\mathbb{R})$ be a discrete Zariski-dense torsion-free subgroup. As above, for $i = 1, 2$, let $\mathscr{F}_i$ be the subfield of $\mathbb{R}$ generated by the $\lambda_{\Gamma_i}(\gamma)$ for all nontrivial semi-simple $\gamma \in \Gamma_i$, where $\lambda_{\Gamma_i}(\gamma)$ is given by equation (26) for $G = G_i$. If nontrivial semi-simple elements $\gamma_1, \ldots, \gamma_m \in \Gamma_1$ are multiplicatively independent and are not weakly contained in Γ_2, then the transcendence degree of $\mathscr{F}_2(\lambda_{\Gamma_i}(\gamma_1), \ldots, \lambda_{\Gamma_i}(\gamma_m))$ over $\mathscr{F}_2$ is $\geq m - d$.*

Proof We can assume that $m > d$ as otherwise there is nothing to prove. It was shown in [14] (see the remark after Proposition 8.5) that for $i = 1, 2$ and any nontrivial semi-simple element $\gamma \in \Gamma_i$, the value $\lambda_{\Gamma_i}(\gamma)^2$, where $\lambda_{\Gamma_i}(\gamma)$ is provided by (26), can be written in the form

$$\lambda_{\Gamma_i}(\gamma)^2 = \sum_{k=1}^{p} s_k \left(\log \chi_k(\gamma)\right)^2, \tag{27}$$

where $\chi_1, \ldots, \chi_p$ are some *positive* characters (see [14, Sect. 8.1] for the definition) of a maximal $\mathbb{R}$-torus T of G_i containing γ, and $s_1, \ldots, s_p$ are some positive rational numbers. Furthermore, we note that if $\gamma \in \Gamma_i$ is a semi-simple element $\neq 1$ and T is a maximal $\mathbb{R}$-torus of G_i containing γ then the condition $|\alpha(\gamma)| = 1$ for all roots α of G_i with respect to T would imply that the nontrivial subgroup $\langle\gamma\rangle$ is discrete and relatively compact, hence finite. This is impossible as Γ_i is torsion-free, so we conclude from (26) that $\lambda_{\Gamma_i}(\gamma) > 0$ for any nontrivial $\gamma \in \Gamma_i$. Thus, assuming that $\gamma \in \Gamma_i$ is nontrivial and renumbering the characters in (27), we can arrange so that

$$a_{\gamma,1} = \log \chi_1(\gamma), \ldots, a_{\gamma,d_\gamma} = \log \chi_{d_\gamma}(\gamma) \ \text{ with } d_\gamma \geq 1,$$

form a basis of the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by $\log \chi_1(\gamma), \ldots, \log \chi_p(\gamma)$. Then we can write $\lambda_{\Gamma_i}(\gamma)^2 = q_\gamma(a_{\gamma,1}, \ldots, a_{\gamma,d_\gamma})$ where $q_\gamma(t_1, \ldots, t_{d_\gamma})$ is a nontrivial rational quadratic form. Thus, for any nontrivial semi-simple $\gamma \in \Gamma_i$ there exists a finite set $A_\gamma = \{a_{\gamma,1}, \ldots, a_{\gamma,d_\gamma}\}$, with $d_\gamma \geq 1$, of real numbers linearly independent over $\mathbb{Q}$, each of which is the logarithm of the value of a positive character on γ, such that

$$\lambda_{\Gamma_i}(\gamma)^2 = q_\gamma(a_{\gamma,1}, \ldots, a_{\gamma,d_\gamma}),$$

where $q_\gamma(t_1, \ldots, t_{d_\gamma})$ is a nonzero rational quadratic form. We fix such A_γ and q_γ for each nontrivial semi-simple $\gamma \in \Gamma_i$, where $i = 1, 2$, for the remainder of the argument. Let $\mathscr{M}_i$ be the subfield of $\mathbb{R}$ generated by the values $\lambda_{\Gamma_i}(\gamma)^2 = q_\gamma(a_{\gamma,1}, \ldots, a_{\gamma,d_\gamma})$ for all nontrivial semi-simple $\gamma \in \Gamma_i$.

Now, suppose $\gamma_1, \ldots, \gamma_m \in \Gamma_1$ are as in the statement of the proposition. It is enough to show that for any *finitely generated* subfield $\mathscr{M}_2' \subset \mathscr{M}_2$, we have

$$\text{tr.deg}_{\mathscr{M}_2'} \mathscr{M}_2'(\lambda_{\Gamma_i}(\gamma_1)^2, \ldots, \lambda_{\Gamma_i}(\gamma_m)^2) \geq m - d.$$

Indeed, this would imply that $\text{tr.deg}_{\mathscr{M}_2} \mathscr{M}_2(\lambda_{\Gamma_i}(\gamma_1)^2, \ldots, \lambda_{\Gamma_i}(\gamma_m)^2)$, and hence (as $\mathscr{F}_2/\mathscr{M}_2$ is algebraic) $\text{tr.deg}_{\mathscr{F}_2} \mathscr{F}_2(\lambda_{\Gamma_i}(\gamma_1)^2, \ldots, \lambda_{\Gamma_i}(\gamma_m)^2)$ is $\geq m - d$, yielding the proposition. We now note that any finitely generated subfield $\mathscr{M}_2' \subset \mathscr{M}_2$ is contained in a subfield of the form $\mathscr{P}_{\Theta_2}$ for some finite set $\Theta_2 = \{\gamma_1^{(2)}, \ldots, \gamma_{m_2}^{(2)}\}$ of nontrivial semi-simple elements of Γ_2, which by definition is generated by $\bigcup_{k=1}^{m_2} A_{\gamma_k^{(2)}}$. So, it is enough to prove that if $\gamma_1, \ldots, \gamma_m \in \Gamma_1$ are as in the statement of the proposition then for any finite set Θ_2 of nontrivial semi-simple elements of Γ_2 we have

$$\text{tr.deg}_{\mathscr{P}_{\Theta_2}} \mathscr{P}_{\Theta_2}(\lambda_{\Gamma_i}(\gamma_1), \ldots, \lambda_{\Gamma_i}(\gamma_m)) \geq m - d. \tag{28}$$

Since the elements $\gamma_1, \ldots, \gamma_m$ are multiplicatively independent, the elements of

$$A = \bigcup_{j=1}^{m} A_{\gamma_j}$$

are linearly independent (over $\mathbb{Q}$). Let B be a maximal linearly independent (over $\mathbb{Q}$) subset of $\bigcup_{k=1}^{m_2} A_{\gamma_k^{(2)}}$. Since $\gamma_1, \ldots, \gamma_m$ are not weakly contained in Γ_2, the elements of $A \cup B$ are linearly independent over $\mathbb{Q}$. Let $\alpha = |A|$ and $\beta = |B|$. Then by Schanuel's conjecture, the transcendence degree over $\mathbb{Q}$ of the field generated by

$$A \cup B \cup \widetilde{A} \cup \widetilde{B}, \quad \text{where } \widetilde{A} = \{e^s | s \in A\} \text{ and } \widetilde{B} = \{e^s | s \in B\},$$

is $\geq \alpha + \beta$. But the set $\widetilde{A} \cup \widetilde{B}$ consists of the values of certain characters on certain semi-simple elements lying in $\Gamma_i \subset G_i(\mathscr{K})$, and therefore is contained in $\overline{\mathscr{K}}$. It follows that the transcendence degree over $\mathbb{Q}$ of the field generated by $\widetilde{A} \cup \widetilde{B}$ is $\leq d$, and therefore the transcendence degree of the field generated by $A \cup B$ is $\geq \alpha + \beta - d$. So,

$$\begin{aligned} \text{tr.deg}_{\mathbb{Q}(B)} \mathbb{Q}(A \cup B) &= \text{tr.deg}_{\mathbb{Q}} \mathbb{Q}(A \cup B) - \text{tr.deg}_{\mathbb{Q}} \mathbb{Q}(B) \\ &\geq (\alpha + \beta - d) - \beta = \alpha - d. \end{aligned}$$

Thus, there exists a subset $C \subset A$ of cardinality $\leq d$ such that the elements of $A \setminus C$ are algebraically independent over $\mathbb{Q}(B)$. Since C intersects at most d of the sets A_{γ_j}, $j \leq m$, we see that after renumbering, we can assume that the elements of

$$D = \bigcup_{j=1}^{m-d} A_{\gamma_j}$$

are algebraically independent over $\mathbb{Q}(B)$. Since $\mathbb{Q}(B)$ coincides with $\mathscr{P}_{\Theta_2}$, (28) follows from the following simple lemma. □

Lemma 7.2 *Let F be a field, and let $E = F(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are algebraically independent over F. Let*

$$\{1, 2, \ldots, n\} = I_1 \cup \cdots \cup I_s$$

be an arbitrary partition, and let E_j be the field generated over F by the t_i for $i \in I_j$. For each $j \in \{1, \ldots, s\}$, pick $f_j \in E_j \setminus F$. Then

$$\text{tr.deg}_F\, F(f_1, \ldots, f_s) = s.$$

Now if property (C_i) holds for $i = 1$ or 2, then Proposition 7.1 implies the following at once.

Corollary 7.3 *Notations and assumptions are as in Proposition 7.1, assume that condition (C_i) holds for either $i = 1$ or 2. Then the transcendence degree of $\mathscr{F}_1\mathscr{F}_2$ over $\mathscr{F}_{3-i}$ is infinite, i.e. condition (T_i) (of the introduction) holds.*

Combining this corollary respectively with Theorems 4.2 and 5.3 we obtain Theorems 1 and 2 of the introduction.

It follows from ([7], Theorem 5.7) that given a discrete torsion-free $(\mathscr{G}_i, K_i)$-arithmetic subgroup of $\mathcal{G}_i$, the compactness of the locally symmetric space $\mathfrak{X}_{\Gamma_i}$ is equivalent to the fact that $\mathscr{G}_i$ is K_i-anisotropic. Combining this with Theorem 5.8 and Corollary 7.3, we obtain Theorem 5.

Theorem 1 has the following important consequence. In [14], Sect. 8, we had to single out the following exceptional case

($\mathcal{E}$) One of the locally symmetric spaces, say, $\mathfrak{X}_{\Gamma_1}$, is 2-dimensional and the corresponding discrete subgroup Γ_1 cannot be conjugated into $\text{PGL}_2(K)$, for any number field $K \subset \mathbb{R}$, and the other space, $\mathfrak{X}_{\Gamma_2}$, has dimension > 2, which was then excluded in some of our results. Theorem 1(1) shows that the locally symmetric spaces as in ($\mathcal{E}$) can *never* be length-commensurable (assuming Schanuel's conjecture), and therefore all our results are in fact valid without the exclusion of case ($\mathcal{E}$).

It should be noted that while Theorem 2 asserts that conditions (T_i) and (N_i) hold for *at least one* $i \in \{1, 2\}$, these may not hold for both i as the following example demonstrates.

Example 7.4 Let D_1 and D_2 be the quaternion algebras over $\mathbb{Q}$ with the sets of ramified places $\{2, 3\}$ and $\{2, 3, 5, 7\}$, respectively. Set $G_i = \mathrm{PSL}_{1,D_i}$, and let Γ_i be a torsion-free subgroup of $G_i(\mathbb{Q})$, for $i = 1, 2$. Over $\mathbb{R}$, both G_1 and G_2 are isomorphic to $G = \mathrm{PSL}_2$, so Γ_1 and Γ_2 can be viewed as arithmetic subgroups of $\mathcal{G} = G(\mathbb{R})$. The symmetric space $\mathfrak{X}$ associated with $\mathcal{G}$ is the hyperbolic plane $\mathbb{H}^2$, so the corresponding locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are arithmetically defined hyperbolic 2-manifolds that are not commensurable as the groups G_1 and G_2 are not $\mathbb{Q}$-isomorphic. At the same time, our choice of D_1 and D_2 implies that every maximal subfield of D_2 is isomorphic to a maximal subfield of D_1 which entails that $\mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_2}) \subset \mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_1})$, hence $\mathscr{F}_2 \subset \mathscr{F}_1$. Thus, $\mathscr{F}_1\mathscr{F}_2 = \mathscr{F}_1$, so ($T_1$) does not hold (although (T_2) does hold).

Next, we will derive Theorem 3 from Theorem 6.6. Let Γ_i be $(\mathscr{G}_i, K_i)$-arithmetic. Assume that (T_i), hence (C_i), does not hold for either $i = 1$ or 2. Then by Theorem 6.6 we necessarily have $K_1 = K_2 =: K$, and the groups $\mathscr{G}_1$, $\mathscr{G}_2$ have equivalent systems of maximal K-tori. By the assumption made in Theorem 3, $K \neq \mathbb{Q}$. The field K has the real place associated with the identity embedding $K \hookrightarrow \mathbb{R}$ but since $K \neq \mathbb{Q}$, it necessarily has another archimedean place v_0, and the discreteness of Γ_i implies that $\mathscr{G}_i$ is K_{v_0}-anisotropic. Thus, Theorem 6.6 applies to the effect that the groups $\mathscr{G}_1$ and $\mathscr{G}_2$ have equivalent systems of maximal K-tori. Then the fact that $\mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_1}) = \mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_2})$ follows from the following.

Proposition 7.5 (cf. [14], Proposition 9.14) *Let G_1 and G_2 be connected absolutely simple algebraic groups such that $\mathcal{G}_i = G_i(\mathbb{R})$ is noncompact for both $i = 1, 2$, and let $\mathfrak{X}_i$ be the symmetric space associated with $\mathcal{G}_i$. Furthermore, let $\Gamma_i \subset \mathcal{G}_i$ be a discrete torsion-free $(\mathscr{G}_i, K)$-arithmetic subgroup (where $K \subset \mathbb{R}$ is a number field), and $\mathfrak{X}_{\Gamma_i} = \mathfrak{X}/\Gamma_i$ be the corresponding locally symmetric space for $i = 1, 2$. If $\mathscr{G}_1$ and $\mathscr{G}_2$ have equivalent systems of maximal K-tori, then $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable.*

This is essentially Proposition 9.14 of [14] except that here we require that the groups $\mathscr{G}_1$ and $\mathscr{G}_2$ have equivalent systems of maximal K-tori instead of the more technical requirement of having *coherently* equivalent systems of maximal K-tori used in [14]; this change however does not affect the proof.

The analysis of our argument in conjunction with Proposition 6.3 and Corollary 6.5 shows that the assertion of Theorem 3 remains valid without the assumption that $K_{\Gamma_i} \neq \mathbb{Q}$ at least in the following situations where $\mathscr{G}_1$ and $\mathscr{G}_2$ are of type A_n: (1) one of the $\mathscr{G}_i$'s is an inner form; (2) one of the $\mathscr{G}_i$'s is represented by $\mathrm{SU}(D, \tau)$ where D is a central *division* algebra over L with an involution τ of the second kind that restricts to the nontrivial automorphism of L/K; (3) $n = p - 1$, where p is a prime.

To illustrate our general results in a concrete geometric situation, we will now prove Corollary 1 of the introduction. The hyperbolic d-space $\mathbb{H}^d$ is the symmetric space of the group $G(d) = \mathrm{PSO}(d, 1)$. For $d \geq 2$, set $\ell = \left[\dfrac{d+1}{2}\right]$. Then for $d \neq 3$, $G(d)$ is an absolutely simple group of type B_ℓ if d is even, and of type D_ℓ if d is odd. Furthermore, the order $w(d)$ of the Weyl group of $G(d)$ is given by:

$$w(d) = \begin{cases} 2^\ell \cdot \ell!, & d \text{ is even}, \\ 2^{\ell-1} \cdot \ell!, & d \text{ is odd}. \end{cases}$$

One easily checks that $w(d) < w(d+1)$ for any $d \geq 2$, implying that $w(d_1) > w(d_2)$ whenever $d_1 > d_2$. With these remarks, assertions (i) and (ii) follow from Theorem 1. Furthermore, using the above description of the Killing-Cartan type of $G(d)$ one easily derives assertions (iii) and (iv) from Theorems 2 and 3, respectively.

Generalizing the notion of length-commensurability, one can define two Riemannian manifolds M_1 and M_2 to be "length-similar" if there exists a real number $\lambda > 0$ such that

$$\mathbb{Q} \cdot L(M_2) = \lambda \cdot \mathbb{Q} \cdot L(M_1).$$

One can show, however, that for arithmetically defined locally symmetric space, in most cases, this notion is redundant, viz. it coincides with the notion of length commensurability.

Corollary 7.6 *Let $\Gamma_i \subset G_i(\mathbb{R})$ be a finitely generated Zariski-dense torsion-free subgroup. Assume that there exists $\lambda \in \mathbb{R}_{>0}$ such that*

$$\mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_1}) = \lambda \cdot \mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_2}). \tag{29}$$

Then

(i) $w_1 = w_2$ (hence either G_1 and G_2 are of the same type, or one of them is of type B_n and the other of type C_n for some $n \geq 3$) and $K_{\Gamma_1} = K_{\Gamma_2} =: K$.
Assume now that Γ_1 and Γ_2 are arithmetic. Then

(ii) $\mathrm{rk}_{\mathbb{R}} G_1 = \mathrm{rk}_{\mathbb{R}} G_2$, and either $G_1 \simeq G_2$ over $\mathbb{R}$, or one of the groups is of type B_n and the other is of type C_n;

(iii) if Γ_i is $(\mathcal{G}_i, K)$-arithmetic then $\mathrm{rk}_K \mathcal{G}_1 = \mathrm{rk}_K \mathcal{G}_2$, and consequently, if one of the spaces is compact, the other must also be compact;

(iv) if G_1 and G_2 are of the same type which is different from A_n, D_{2n+1} $(n > 1)$ or E_6 then $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable, hence length-commensurable;

(v) if G_1 and G_2 are of the same type which is one of the following: A_n, D_{2n+1} $(n > 1)$ or E_6, then provided that $K_{\Gamma_i} \neq \mathbb{Q}$ for at least one $i \in \{1, 2\}$, the spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are length-commensurable (although not necessarily commensurable).

Proof If (29) holds then obviously (N_i) cannot possibly hold for either $i = 1$ or 2. So, assertion (i) immediately follows from Theorem 1. Now, if Γ_i is $(\mathscr{G}_i, K)$-arithmetic, then neither (N_1) nor (N_2) holds, so neither (C_1) nor (C_2) can hold (cf. Corollary 7.3). So by Theorem 5.1 we have $\mathrm{rk}_{K_v} \mathscr{G}_1 = \mathrm{rk}_{K_v} \mathscr{G}_2$ for all $v \in V^K$; in particular, $\mathrm{rk}_{\mathbb{R}} G_1 = \mathrm{rk}_{\mathbb{R}} G_2$. Moreover, if G_1 and G_2 are of the same type then by Theorem 5.6, the Tits indices over $\mathbb{R}$ of G_1 and G_2 are isomorphic, and therefore $G_1 \simeq G_2$, so assertion (ii) follows. Regarding (iii), the fact that $\mathrm{rk}_K \mathscr{G}_1 = \mathrm{rk}_K \mathscr{G}_2$ is again a consequence of Theorem 5.6 in conjunction with Corollary 7.3; to relate this to the compactness of the corresponding locally symmetric spaces one argues as in the proof of Theorem 5 above. Finally, assertions (iv) and (v) follow from Theorems 2 and 3 respectively. $\square$

We note that assertions (iv) and (v) of the above corollary imply that if two arithmetically defined locally symmetric spaces of the same group are not length-commensurable then they can rarely be made length-commensurable by scaling the metric on one of them (cf. however, Theorem 4).

8 Groups of types B_n and C_n: Proof of Theorem 4

The goal of this section is to prove Theorem 4. Our argument will heavily rely on the results of [5]. Here is one of the main results of [5].

Theorem 8.1 ([5], Theorem 1.2) *Let G_1 and G_2 be connected absolutely simple adjoint groups of types B_n and C_n ($n \geq 3$) respectively over a field F of characteristic zero, and let Γ_i be a Zariski-dense $(\mathscr{G}_i, K, S)$-arithmetic subgroup. Then Γ_1 and Γ_2 are weakly commensurable if and only if the following conditions hold:*

(1) $\mathrm{rk}_{K_v} \mathscr{G}_1 = \mathrm{rk}_{K_v} \mathscr{G}_2 = n$ *(in other words, $\mathscr{G}_1$ and $\mathscr{G}_2$ are split over K_v) for all nonarchimedean $v \in V^K$, and*
(2) $\mathrm{rk}_{K_v} \mathscr{G}_1 = \mathrm{rk}_{K_v} \mathscr{G}_2 = 0$ *or n (i.e., both $\mathscr{G}_1$ and $\mathscr{G}_2$ are simultaneously either anisotropic or split) for every archimedean $v \in V^K$.*

Furthermore, it has been shown in [5] that the same two conditions precisely characterize the situations where $\mathscr{G}_1$ and $\mathscr{G}_2$ have the same isogeny classes of maximal K-tori, or, equivalently, $\mathscr{G}_1$ and $\widetilde{\mathscr{G}_2}$ (the universal cover of $\mathscr{G}_2$) have the same isomorphism classes of maximal K-tori. We need the following proposition which has actually been established in the course of the proof of Theorem 8.1 in [5].

Proposition 8.2 (cf. Corollary 5.8 in [5]) *Notations and conventions be as in Theorem 8.1. Assume that $v_0 \in V^K$ is such that the corresponding condition (1) or (2) fails. Then for at least one $i \in \{1, 2\}$ there exists a K_{v_0}-isotropic maximal torus $T_i(v_0)$ of $\mathscr{G}_i$ such that no maximal K-torus T_i of $\mathscr{G}_i$ satisfying*

(i) $\theta_{T_i}(\mathrm{Gal}(K_{T_i}/K)) = W(\mathscr{G}_i, T_i)$,
(ii) *T_i is conjugate to $T_i(v_0)$ by an element of $\mathscr{G}_i(K_{v_0})$*

is K-isogeneous to a maximal K-torus of $\mathscr{G}_{3-i}$.

We will now use this proposition to prove the following.

Proposition 8.3 *Notations and conventions be as in Theorem 8.1. Assume that there exists $v_0 \in V^K$ such that the corresponding condition (1) or (2) fails. Then condition (C_i) holds for at least one $i \in \{1, 2\}$.*

Proof As G_1 and G_2 are adjoint, $\Gamma_i \subset \mathscr{G}_i(K)$ for $i = 1, 2$. Pick $i \in \{1, 2\}$ and a maximal K_{v_0}-torus $T_i(v_0)$ of $\mathscr{G}_i$ as in Proposition 8.2; we will show that property (C_i) holds for this i. Fix $m \geq 1$. Using Theorem 3.3, we can find maximal K-tori $T_1, \ldots, T_m$ of $\mathscr{G}_i$ that are independent over K and satisfy the following conditions for each $j \leq m$:

- $\theta_{T_j}(\mathrm{Gal}(K_{T_j}/K)) = W(\mathscr{G}_i, T_j)$,
- T_j is conjugate to $T_i(v_0)$ by an element of $\mathscr{G}_i(K_{v_0})$, and

$\mathrm{rk}_{K_v} T_j = \mathrm{rk}_{K_v} \mathscr{G}_i$ for all $v \in S \setminus \{v_0\}$.

Since $T_i(v_0)$ is K_v-isotropic, we have $d_{T_j}(S) := \sum_{v \in S} \mathrm{rk}_{K_v} T_j > 0$ no matter whether or not v_0 belongs to S. Besides, T_j is automatically K-anisotropic, so it follows from Dirichlet's Theorem that for each $j \leq m$, $\Gamma_i \cap T_j(K)$ contains an element, say γ_j, of infinite order. These elements γ_j are multiplicatively independent by Lemma 2.1, so we only need to show that they are not weakly contained in Γ_{3-i}. However, by Theorem 2.3, a relation of weak containment would imply that T_j for some $j \leq m$ would admit a K-isogeny onto a maximal K-torus of $\mathscr{G}_{3-i}$. But this is impossible by Proposition 8.2. □

Corollary 8.4 *Let M_1 be an arithmetic quotient of the real hyperbolic space $\mathbb{H}^{2n}$ with $n \geq 3$, and M_2 be an arithmetic quotient of the quaternionic hyperbolic space $\mathbb{H}^m_{\mathbf{H}}$. Then M_1 and M_2 satisfy (T_i) and (N_i) for at least one $i \in \{1, 2\}$; in particular, M_1 and M_2 are not length-commensurable.*

Proof We recall that $\mathbb{H}^{2n}$ is the symmetric space of the real rank-1 form of type B_n, and $\mathbb{H}^m_{\mathbf{H}}$ is the symmetric space of the real rank-1 form of type C_m. So, if $m \neq n$ then our claim directly follows from Theorem 1 of the introduction. If $n = m$, the assertion is obtained by combining Proposition 8.3 with Corollary 7.3. □

Remark 8.5 If $m = n$ in Corollary 8.4 then one can actually guarantee that properties (T_1) and (N_1) hold, and this in fact does not require any assumptions of arithmeticity (so, in effect, the arithmeticity assumption in the corollary is not needed in all cases as Theorem 1 used to treat the case $m \neq n$ does not rely on it). More generally, one can prove the following. Let G_1 and G_2 be absolutely simple adjoint real algebraic groups of types B_n and $C_n (n \geq 3)$ respectively, with G_1 isotropic and G_2 isotropic but nonsplit (over $\mathbb{R}$), and let $\mathfrak{X}_i$ be the symmetric space of $\mathcal{G}_i = G_i(\mathbb{R})$ for $i = 1, 2$. Furthermore, let M_i be the quotient of $\mathfrak{X}_i$ by a finitely generated torsion-free discrete Zariski-dense subgroup Γ_i of $\mathcal{G}_i$. Then properties (T_1) and (N_1) hold. Indeed, it is enough to show that (C_1) holds. Pick a finitely generated subfield K of $\mathbb{R}$ such that both G_1 and G_2 are defined over K and $\Gamma_i \subset G_i(K)$. By combining the ideas developed in the proof of Theorem 3.3 with those from [11] one can find, for a given $m \geq 1$, regular and $\mathbb{R}$-regular semi-simple elements of infinite order $\gamma_1, \ldots, \gamma_m \in \Gamma_1$ such that the corresponding maximal K-tori $T_j = Z_{G_1}(\gamma_j)^\circ$ of G_1 satisfy the following conditions: (1) $\theta_{T_j}(\mathrm{Gal}(K_{T_j}/K)) = W(G_1, T_j)$ for all $j = 1, \ldots, m$, and (2) $T_1, \ldots, T_m$ are independent over K. We recall that an element $x \in G(\mathbb{R})$, where G is a connected semi-simple real algebraic group, is called $\mathbb{R}$-*regular* if the number of eigenvalues, counted with multiplicities, of modulus 1 of Adx is minimum possible; the fact we need is that if $x \in G(\mathbb{R})$ is regular and $\mathbb{R}$-regular then the torus $T = Z_G(x)^\circ$ contains a maximal $\mathbb{R}$-split torus of G (cf. [8], Lemma 1.5). The elements $\gamma_1, \ldots, \gamma_m$ are multiplicatively independent, and we only need to prove that they are not weakly contained in Γ_2. If $\gamma_1, \ldots, \gamma_m$ are weakly contained in Γ_2, there will exist $j \leq m$ such that T_j is K-isogenous to a maximal K-torus of G_2. Then according to Proposition 5.6 in [5], the torus T_j is actually K-isomorphic to a maximal K-torus of the simply connected group $\widetilde{G}_2$. But this is impossible since γ_j is $\mathbb{R}$-regular and hence T_j contains a maximal $\mathbb{R}$-split torus of G_1 and G_2 is not $\mathbb{R}$-split—cf. Remark 3.5 in [5].

Proof of Theorem 4 Let G_1 and G_2 be connected absolutely simple adjoint algebraic $\mathbb{R}$-groups of type B_n and C_n $(n \geq 3)$ respectively, and let Γ_i be a discrete torsion-free $(\mathscr{G}_i, K_i)$-arithmetic subgroup of $\mathcal{G}_i = G_i(\mathbb{R})$, for $i = 1, 2$. If $K_1 \neq K_2$, then either condition (T_1) or (T_2) holds for the locally symmetric spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ by Theorem 1. So, let us assume that $K_1 = K_2 =: K$. If there exists $v_0 \in V^K$ such that the corresponding condition (1) or (2) of Theorem 8.1 fails, then by Proposition 8.3 the groups Γ_1 and Γ_2 satisfy (C_i) for at least one $i \in \{1, 2\}$, and then (T_i) holds for the same i (cf. Corollary 7.3). So, to complete the proof of Theorem 4, it only remains to show that if conditions (1) and (2) of Theorem 8.1 hold for all $v \in V^K$, then

$$\mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_2}) = \lambda \cdot \mathbb{Q} \cdot L(\mathfrak{X}_{\Gamma_1}) \quad \text{where } \lambda = \sqrt{\frac{2n+2}{2n-1}}. \tag{30}$$

We will show that provided the conditions (1) and (2) of Theorem 8.1 hold, given a maximal K-torus T_1 of $\mathscr{G}_1$, there exists a maximal K-torus T_2 of $\widetilde{\mathscr{G}}_2$ and a K-isomorphism $T_1 \to T_2$ such that for any $\gamma_1 \in T_1(K)$, and the corresponding $\gamma_2 \in T_2(K)$, one can *relate* the following sets

$$\{\alpha(\gamma_1) \mid \alpha \in \Phi(\mathscr{G}_1, T_1)\} \quad \text{and} \quad \{\alpha(\gamma_2) \mid \alpha \in \Phi(\widetilde{\mathscr{G}}_2, T_2)\},$$

and derive information about the ratio of the lengths of the closed geodesics associated to γ_1 and γ_2. The easiest way to do this is to use the description of maximal K-tori of $\mathscr{G}_1$ and $\widetilde{\mathscr{G}}_2$ in terms of commutative étale algebras.

The group $\mathscr{G}_1$ can be realize as the special unitary group $\mathrm{SU}(A_1, \tau_1)$ where $A_1 = M_{2n+1}(K)$ and τ_1 is an involution of A_1 of orthogonal type (which means that $\dim_K A_1^{\tau_1} = (2n+1)(n+1)$). Any maximal K-torus T_1 of $\mathscr{G}_1$ corresponds to a maximal commutative étale τ_1-invariant subalgebra E_1 of A_1 such that $\dim_K E_1^{\tau_1} = n+1$; more precisely, $T_1 = \left(\mathrm{R}_{E_1/K}(\mathrm{GL}_1) \cap \mathscr{G}_1\right)^{\circ}$. It is more convenient for our purposes to think that T_1 corresponds to an embedding $\iota_1 \colon (E_1, \sigma_1) \hookrightarrow (A_1, \tau_1)$ of algebras with involution, where E_1 is a commutative étale K-algebra of dimension $(2n+1)$ equipped with an involution σ_1 such that $\dim_K E_1^{\sigma_1} = n+1$.

Similarly, the group $\widetilde{\mathscr{G}}_2$ can be realized as the special unitary group $\mathrm{SU}(A_2, \tau_2)$, where A_2 is a central simple algebra over K of dimension $4n^2$, and τ_2 is an involution of A_2 of symplectic type (i.e., $\dim_K A_2^{\tau_2} = (2n-1)n$). Furthermore, any maximal K-torus T_2 corresponds to an embedding $\iota_2 \colon (E_2, \sigma_2) \hookrightarrow (A_2, \tau_2)$ of algebras with involution where E_2 is a commutative étale K-algebra of dimension $2n$ equipped with an involution σ_2 such that $\dim_K E_2^{\sigma_2} = n$.

Now, any involutory commutative étale algebra (E_1, σ_1) as above admits a decomposition

$$(E_1, \sigma_1) = (\widetilde{E}_1, \tilde{\sigma}_1) \oplus (K, \mathrm{id}_K)$$

where $\widetilde{E}_1 \subset E_1$ is a $2n$-dimensional σ_1-invariant subalgebra and $\tilde{\sigma}_1 = \sigma_1 | \widetilde{E}_1$; note that $\dim_K \widetilde{E}_1^{\tilde{\sigma}_1} = n$. It has been shown in [5] using Theorem 7.3 of [15] that if conditions (1) and (2) of Theorem 8.1 hold then (E_1, σ_1) as above admits an embedding $\iota_1 \colon (E_1, \sigma_1) \hookrightarrow (A_1, \tau_1)$ if and only if $(E_2, \sigma_2) := (\widetilde{E}_1, \tilde{\sigma}_1)$ admits an embedding $\iota_2 \colon (E_2, \sigma_2) \hookrightarrow (A_2, \tau_2)$. This implies that for any maximal K-torus T_1 of $\mathscr{G}_1$ there exists a K-isomorphism $\varphi \colon T_1 \to T_2$ onto a maximal K-torus T_2 of $\widetilde{\mathscr{G}}_2$ that arises from the above correspondence between the associated algebras (E_1, σ_1) and (E_2, σ_2), and vice versa. Fix the tori T_1, T_2, the K-isomorphism φ, the algebras (E_1, σ_1), (E_2, σ_2) and the embeddings ι_1, ι_2 for the remainder of this section. We assume henceforth that the discrete torsion-free subgroups $\Gamma_i \subset \mathcal{G}_i$ are $(\mathscr{G}_i, K)$-arithmetic. Given $\gamma_1 \in T_1(K) \cap \Gamma_1$, set $\gamma_2 = \varphi(\gamma_1) \in T_2(K)$. Then there exists $n_2 \geq 1$ such that $\gamma_2^{n_2} \in \Gamma_2$. It follows from the discussion at the beginning of Sect. 7 that the ratio $\ell_{\Gamma_2}(c_{\gamma_2^{n_2}})/\ell_{\Gamma_1}(c_{\gamma_1})$ of the lengths of the corresponding geodesics is a rational multiple of the ratio $\lambda_{\Gamma_2}(\gamma_2)/\lambda_{\Gamma_1}(\gamma_1)$. Let us show that in fact

$$\lambda_{\Gamma_2}(\gamma_2)/\lambda_{\Gamma_1}(\gamma_1) = \sqrt{\frac{2n+2}{2n-1}}. \tag{31}$$

Indeed, let $x \in E_1$ such that $\iota_1(x) = \gamma_1$. The roots of the characteristic polynomial of x are of the form

$$\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}, 1$$

for some complex numbers $\lambda_1, \ldots, \lambda_n$. Then

$$\{\alpha(\gamma_1) | \alpha \in \Phi(\mathscr{G}_1, T_1)\} = \{\lambda_i^{\pm 1}\} \cup \{\lambda_i^{\pm 1} \cdot \lambda_j^{\pm 1} | i < j\}. \tag{32}$$

For the corresponding "truncated" element $\tilde{x} \in \widetilde{E}_1 = E_2$, the roots of the characteristic polynomial are

$$\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1},$$

and

$$\{\alpha(\gamma_2)|\alpha \in \Phi(\widetilde{\mathscr{G}}_2, T_2)\} = \{\lambda_i^{\pm 2}\} \cup \{\lambda_i^{\pm 1} \cdot \lambda_j^{\pm 1}|i < j\}. \tag{33}$$

Set $\mu_i = \log|\lambda_i|$. Then it follows from (32) that

$$\lambda_{\Gamma_1}(\gamma_1)^2 = \sum_{i=1}^{n}(\pm\mu_i)^2 + \sum_{1\leq i<j\leq n}(\pm\mu_i \pm \mu_j)^2 = (4n-2)\cdot\sum_{i=1}^{n}\mu_i^2.$$

Similarly, we derive from (33) that

$$\lambda_{\Gamma_2}(\gamma_2)^2 = \sum_{i=1}^{n}(\pm 2\mu_i)^2 + \sum_{1\leq i<j\leq n}(\pm\mu_i \pm \mu_j)^2 = 4(n+1)\cdot\sum_{i=1}^{n}\mu_i^2.$$

Comparing these equations, we obtain (31). Then the inclusion $\supset$ in (30) follows immediately, and the opposite inclusion is established by a symmetric argument, completing the proof of Theorem 4. ◻

Remark 8.6 Using Theorem 4, one can construct compact locally symmetric spaces with isometry groups of types B_n and C_n ($n \geq 3$), respectively, that are length-similar—so, these spaces can be made length-commensurable by scaling the metric on one of them. According to the results of Yeung [23], however, scaling will never make these spaces (or their finite-sheeted covers) isospectral.

Appendix: Proof of Theorems 5.7 and 5.7′

First, we need to review some notions pertaining to the Tits index and recall some of the results established in [14]. Let G be a semi-simple algebraic K-group. Pick a maximal K-torus T_0 of G that contains a maximal K-split torus S_0 and choose compatible orderings (also called "coherent orderings") on $X(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(S_0) \otimes_{\mathbb{Z}} \mathbb{R}$ (this means that the linear map between these vector spaces induced by the restriction $X(T_0) \to X(S_0)$ takes nonnegative elements to nonnegative elements). Let $\Delta_0 \subset \Phi(G, T_0)$ denote the system of simple roots corresponding to the chosen ordering on $X(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$. Then a root $\alpha \in \Delta_0$ (or the corresponding vertex in the Dynkin diagram) is *distinguished* in the Tits index of G/K if its restriction to S_0 is nontrivial. Let $\Delta_0^{(d)}$ be the set of distinguished roots in Δ_0 and P be the minimal parabolic K-subgroup containing S_0 determined by the above ordering on $\Phi(G, T_0)$ $(\subset X(T_0))$. Then $Z_G(S_0)$ is the unique Levi subgroup of P containing T_0, and $\Delta_0 \setminus \Delta_0^{(d)}$ is a basis of its root system with respect to T_0. Moreover, the set $\Phi(P, T_0)$ of roots of P with respect to T_0 is the union of positive roots in $\Phi(G, T_0)$ (positive with respect to the ordering fixed above) and the roots $\Phi(Z_G(S_0), T_0)$ of the subgroup $Z_G(S_0)$; hence, $\Delta_0 \setminus \Delta_0^{(d)} = \Delta_0 \cap (-\Phi(P, T_0))$. The set of roots of the unipotent radical of P is the set of all positive roots except the roots which are nonnegative integral linear combination of the roots in $\Delta_0 \setminus \Delta_0^{(d)}$.

The notion of a distinguished vertex is invariant in the following sense: choose another compatible orderings on $X(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(S_0) \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\Delta_0' \subset \Phi(G, T_0)$ be the system of simple roots corresponding to this new ordering and ${\Delta_0'}^{(d)}$ be the set of distinguished simple roots. Then there exists a unique element w in the Weyl group $W(G, T_0)$ such that $\Delta_0' = w(\Delta_0)$ and we call the identification of Δ_0 with Δ_0' using w the *canonical identification*. We assert that the canonical identification identifies distinguished roots with distinguished roots.

To see this, note that if P' is the minimal parabolic k-subgroup containing S_0 determined by the new ordering, then there exists $n \in N_G(S_0)(K)$ such that $P' = nPn^{-1}$. As $nT_0n^{-1} \subset Z_G(S_0)$, we can find $z \in Z_G(S_0)(K_{\mathrm{sep}})$ such that $znT_0n^{-1}z^{-1} = T_0$, i.e., zn normalizes T_0, and $zn(\Delta_0 \setminus \Delta_0^{(d)}) = \Delta_0' \setminus \Delta_0'^{(d)}$. It is obvious that $znPn^{-1}z^{-1} = P'$ and that zn carries the set of roots which are positive with respect to the first ordering into the set of roots which are positive with respect to the second ordering. Therefore, zn carries Δ_0 into Δ_0', and hence w is its image in the Weyl group. From this we conclude that $w(\Delta_0 \setminus \Delta_0^{(d)}) = \Delta_0' \setminus \Delta_0'^{(d)}$, which implies that $w(\Delta_0^{(d)}) = \Delta_0'^{(d)}$. This proves our assertion.

We recall that G is K-isotropic if and only if the Tits index of G/K has a distinguished vertex; more generally, $\mathrm{rk}_K\, G$ equals the number of distinguished orbits in Δ_0 under the $*$-action. We refer the reader to [21, §2.3] or [14, §4] for the definition and properties of the $*$-action; recall only that the property of being a distinguished vertex is preserved by the $*$-action.

Let now T be an arbitrary maximal K-torus of G. Fix a system of simple roots $\Delta \subset \Phi(G,T)$. Let $\mathscr{K}$ be a field extension of K over which both T and T_0 split. Then there exists $g \in G(\mathscr{K})$ such that the inner automorphism $i_g\colon x \mapsto gxg^{-1}$ carries T_0 onto T and $i_g^*(\Delta) = \Delta_0$. Moreover, such a g is unique up to right multiplication by an element of $T_0(\mathscr{K})$, implying that the identification of Δ with Δ^0 provided by i_g^* does not depend on the choice of g, and we call it the *canonical identification*. (Note that this agrees with our earlier notion if $T = T_0$.) A vertex $\alpha \in \Delta$ is said to *correspond to a distinguished vertex* in the Tits index of G/K if the vertex $\alpha_0 \in \Delta_0$ corresponding to α in the canonical identification is distinguished; the set of all such vertices in Δ will be denoted by $\Delta^{(d)}(K)$. Clearly, the group G is quasi-split over K if and only if $\Delta^{(d)}(K) = \Delta$. The notion of canonical identification can be extended in the obvious way to the situation where we are given two maximal K-tori T_1, T_2 of G and the systems of simple roots $\Delta_i \in \Phi(G, T_i)$ for $i = 1, 2$; under the canonical identification $\Delta_1^{(d)}(K)$ is mapped onto $\Delta_2^{(d)}(K)$. The $*$-action of the absolute Galois group $\mathrm{Gal}(K_{sep}/K)$ on Δ_1 and Δ_2 commutes with the canonical identification of Δ_1 with Δ_2, see Lemma 4.1(a) of [14]. The set $\Delta^{(d)}(K)$ is invariant under the $*$-action, so it makes sense to talk about distinguished orbits.

Let now K be a number field. We say that an orbit of the $*$-action in Δ is *distinguished everywhere* if it is contained in $\Delta^{(d)}(K_v)$ for all $v \in V^K$. The following was established in [14], Proposition 7.2:

- An orbit of the $*$-action in Δ is distinguished (i.e., is contained in $\Delta^{(d)}(K)$) if and only if it is distinguished everywhere.
  This implies the following (Corollary 7.4 in [14]):
- Let G be an absolutely almost simple group of one of the following types: $B_n (n \geq 2)$, $C_n (n \geq 2)$, E_7, E_8, F_4 or G_2. If G is isotropic over K_v for all real $v \in V_\infty^K$, then G is isotropic over K. Additionally, if G is as above, but not of type E_7, then $\mathrm{rk}_K\, G = \min\limits_{v \in V^K} \mathrm{rk}_{K_v}\, G$.

Before proceeding to the proof of Theorem 5.7, we observe that since by assumption $L_1 = L_2 =: L$, it follows from condition (1) in the statement of that theorem that

$$\theta_{T_i}(\mathrm{Gal}(L_{T_i}/L)) = W(G_i, T_i) \quad \text{for } i = 1, 2.$$

So, the fact that there is an isogeny $T_1 \to T_2$ defined over L implies that $w_1 = w_2$. Thus, this condition holds in both the Theorems 5.7 and 5.7$'$. As we already mentioned, this implies that either the groups G_1 and G_2 are of the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n for some $n \geq 3$; in particular, the groups have the same absolute rank.

Proof of Theorem 5.7 for types B_n, C_n, E_8, F_4 and G_2. As we mentioned above, for these types we have

$$\mathrm{rk}_K\, G_i = \min_{v \in V^K} \mathrm{rk}_{K_v}\, G_i \quad \text{for} \quad i = 1, 2.$$

Condition (2) of the theorem implies that $\mathrm{rk}_{K_v}\, G_1 = \mathrm{rk}_{K_v}\, G_2$ for all $v \in \mathcal{V}$. On the other hand, for $v \notin \mathcal{V}$, both G_1 and G_2 are split over K_v, which automatically makes the local ranks equal. It follows that $\mathrm{rk}_K\, G_1 = \mathrm{rk}_K\, G_2$. Furthermore, inspecting the tables in [21], one observes that the Tits index of an absolutely almost simple group G of one of the above types over a local or global field K is completely determined by its K-rank, and our assertion about the local and global Tits indices of G_1 and G_2 being isomorphic follows (in case G_1 and G_2 are of the same type). □

Proof of Theorem 5.7′ for types B_n, C_n, E_7, E_8, F_4 and G_2. It is enough to show that G_2 is K_{2v}-isotropic for all $v \in V^{K_2}$, and in fact, since G_2 is assumed to be quasi-split over K_{2v} for all $v \notin \mathcal{V}_2$, it is enough to check this only for $v \in \mathcal{V}_2$. However by our construction, each $v \in \mathcal{V}_2$ is an extension of some $v_0 \in \mathcal{V}_1$. Since G_1 is K_1-isotropic, we have

$$\mathrm{rk}_{K_{2v}}\, T_1 \geq \mathrm{rk}_{K_{1v_0}}\, T_1 = \mathrm{rk}_{K_{1v_0}}\, G_1 > 0,$$

so the existence of a K_2-isogeny $T_1 \to T_2$ implies that $\mathrm{rk}_{K_{2v}} T_2 > 0$, hence G_2 is K_{2v}-isotropic as required. □

Thus, it remains to prove Theorems 5.7 and 5.7′ assuming that G_1 and G_2 are of the same type which is one of the following: A_n, D_n, E_6 and E_7 (recall that Theorem 5.7′ has already been proven for groups of type E_7). Then replacing the isogeny $\pi : T_1 \to T_2$, which is defined over K in Theorem 5.7 and over K_2 in Theorem 5.7′, with a suitable multiple, we may (and we will) assume that $\pi^*(\Phi(G_2, T_2)) = \Phi(G_1, T_1)$. Besides, we may assume through the rest of the appendix that G_1 and G_2 are adjoint, and then π extends to an isomorphism $\bar{\pi} : G_1 \to G_2$ over a separable closure of the field of definition (cf. Lemma 4.3(2) and Remark 4.4 in [14]). This has two consequences that we will need. First, the assumption that $L_1 = L_2$ in Theorem 5.7 implies that the orbits of the $*$-action on a system of simple roots $\Delta_1 \subset \Phi(G_1, T_1)$ correspond under π^* to the orbits of the $*$-action on the system of simple roots $\Delta_2 \subset \Phi(G_2, T_2)$ such that $\pi^*(\Delta_2) = \Delta_1$, and this remains true over any completion K_v. Thus, it is enough to prove for each $v \in V^K$ that $\alpha_1 \in \Delta_1$ corresponds to a distinguished vertex in the Tits index of G_1/K_v if and only if $\alpha_2 := \pi^{*-1}(\alpha_1) \in \Delta_2$ corresponds to a distinguished vertex in the Tits index of G_2/K_v. Similarly, the assumption that $L_2 \subset K_2 L_1$ in Theorem 5.7′ implies (in the above notations) that if $O_1 \subset \Delta_1$ is an orbit of the $*$-action, then $(\pi^*)^{-1}(O_1)$ is a union of orbits of the $*$-action. Consequently, it is enough to prove that if $\alpha_1 \in \Delta$ corresponds to a distinguished vertex in the Tits index of G_1/K_1, then $\alpha_2 := \pi^{*-1}(\alpha_1) \in \Delta_2$ corresponds to a distinguished vertex in the Tits index of G_2/K_{2v} for all $v \in V^{K_2}$.

Second, given two systems of simple roots $\Delta_1', \Delta_1'' \subset \Phi(G_1, T_1)$ and the corresponding systems of simple roots $\Delta_2', \Delta_2'' \subset \Phi(G_2, T_2)$, an identification (induced by an automorphism of the root system) $\Delta_1' \simeq \Delta_1''$ is canonical if and only if the corresponding identification $\Delta_2' \simeq \Delta_2''$ is canonical.

Proof of Theorem 5.7 for the remaining types. As above, fix systems of simple roots $\Delta_i \subset \Phi(G_i, T_i)$ for $i = 1, 2$, so that $\pi^*(\Delta_2) = \Delta_1$. We need to show, for each $v \in V^K$, that a root $\alpha_1 \in \Delta_1$ corresponds to a distinguished vertex in the Tits index of G_1/K_v if and only if $\alpha_2 := \pi^{*-1}(\alpha_1) \in \Delta_2$ corresponds to a distinguished vertex in the Tits index of G_2/K_v. This is obvious if both G_1 and G_2 are quasi-split over K_v as then all the vertices in the Tits indices of G_1/K_v and G_2/K_v are distinguished. So, it remains to consider the case where $v \in \mathcal{V}$. Let S_i^v be the maximal K_v-split subtorus of T_i. Since $\mathrm{rk}_{K_v}\, T_i = \mathrm{rk}_{K_v}\, G_i$, we see that S_i^v is actually a maximal K_v-split torus of G_i for $i = 1, 2$, and besides, π induces an isogeny between S_1^v and S_2^v. Pick compatible orderings on $X(S_1^v) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(T_1) \otimes_{\mathbb{Z}} \mathbb{R}$, and on $X(S_2^v) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(T_2) \otimes_{\mathbb{Z}} \mathbb{R}$ that correspond to each other under π^*, and let $\Delta_i^v \subset \Phi(G_i, T_i)$ for $i = 1, 2$ be the system of simple roots that corresponds to this (new) ordering on $X(T_i) \otimes_{\mathbb{Z}} \mathbb{R}$; clearly, $\pi^*(\Delta_2^v) = \Delta_1^v$. Furthermore, let $\alpha_i^v \in \Delta_i^v$ be the root corresponding to α_i under the canonical identification $\Delta_i \simeq \Delta_i^v$; it follows from the above remarks that $\pi^*(\alpha_2^v) = \alpha_1^v$. On the other hand, α_i corresponds to a distinguished vertex in the Tits index of G_i/K_v if and only if α_i^v restricts to S_i^v nontrivially, and the required fact follows. $\square$

Proof of Theorem 5.7′ for the remaining types. Let T_1^0 be a maximal K_1-torus of G_1 that contains a maximal K_1-split torus S_1^0. As in the definition of the Tits index of G_1/K_1, we fix compatible orderings on $X(S_1^0) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(T_1^0) \otimes_{\mathbb{Z}} \mathbb{R}$, and let Δ_1^0 denote the system of simple roots in $\Phi(G_1, T_1^0)$ corresponding to this ordering on $X(T_1^0) \otimes_{\mathbb{Z}} \mathbb{R}$. Now, pick an element g_1 of G_1, rational over a suitable field extension of K_1, so that $T_1 = i_{g_1}(T_1^0)$, and let

$$\Delta_1 = (i_{g_1}^*)^{-1}(\Delta_1^0) \subset \Phi(G_1, T_1).$$

Furthermore, let $\Delta_2 = \pi^{*-1}(\Delta_1)$; then Δ_2 is a system of simple roots in $\Phi(G_2, T_2)$. It follows from the above discussion that it is enough to prove the following:

($*$) *Let $\alpha_1^0 \in \Delta_1^0$ be distinguished in the Tits index of G_1/K_1, and let $\alpha_1 = (i_{g_1}^*)^{-1}(\alpha_1^0) \in \Delta_1$. Then $\alpha_2 := \pi^{*-1}(\alpha_1) \in \Delta_2$ corresponds to a distinguished vertex of G_2/K_{2v}* for all $v \in V^{K_2}$.

Since G_2 is quasi-split over K_{2v} for $v \notin \mathcal{V}_2$, it is enough to prove ($*$) assuming that $v \in \mathcal{V}_2$. By the description of $\mathcal{V}_2$, v is an extension to K_2 of some $v_0 \in \mathcal{V}_1$. Since $\mathrm{rk}_{K_{1 v_0}}\, T_1 = \mathrm{rk}_{K_{1 v_0}}\, G_1$, the maximal $K_{1 v_0}$-split subtorus $S_1^{v_0}$ of T_1 is a maximal $K_{1 v_0}$-split torus of G_1, so it follows from the conjugacy of maximal split tori (cf. [20], 15.2.6) that we can find an element h_1 of G_1, rational over a finite extension of $K_{1 v_0}$, such that

$$T_1 = i_{h_1}(T_1^0) \quad \text{and} \quad S_1^{v_0} \supset i_{h_1}(S_1^0).$$

We claim that to prove ($*$) it suffices to find a different ordering on $X(T_1^0) \otimes_{\mathbb{Z}} \mathbb{R}$ (depending on v) that induces the same ordering on $X(S_1^0) \otimes_{\mathbb{Z}} \mathbb{R}$ (this ordering on $X(T_1^0) \otimes_{\mathbb{Z}} \mathbb{R}$ will be referred to as the *new* ordering, while the ordering fixed earlier will be called the *old* ordering) such that if $\Delta_1^{0v} \subset \Phi(G_1, T_1^0)$ is the system of simple root corresponding to the new ordering, $i^* \colon \Delta_1^0 \simeq \Delta_1^{0v}$ is the canonical identification, $\alpha_1^{0v} := i^*(\alpha_1^0)$, $\Delta_1^v := (i_{h_1}^*)^{-1}(\Delta_1^{0v})$ and $\alpha_1^v := (i_{h_1}^*)^{-1}(\alpha_1^{0v}) \in \Delta_1^v$, then the root $\alpha_2^v := \pi^{*-1}(\alpha_1^v)$ of the simple system of roots $\Delta_2^v = \pi^{*-1}(\Delta_1^v) \subset \Phi(G_2, T_2)$ corresponds to a distinguished vertex in the Tits index of G_2/K_{2v}. Indeed, the identification $\Delta_1 \simeq \Delta_1^v$ given by $i_{h_1}^* \circ i^* \circ (i_{g_1}^*)^{-1}$ is canonical and takes α_1 to α_1^v. It follows that the canonical identification of Δ_2 with Δ_2^v takes α_2 to α_2^v, so the fact that α_2^v corresponds to a distinguished vertex in the Tits index of G_2/K_{2v} implies that the same is true for α_2, as required. What is crucial for the rest of the argument is that due to the invariance of the Tits index, the root α_1^{0v} is distinguished in the Tits index of G_1/K_1, i.e., its restriction to S_1^0 is nontrivial.

To construct a new ordering on $X(T_1^0) \otimes_{\mathbb{Z}} \mathbb{R}$ with the required properties, we let T_2^{0v} denote a maximal K_{2v}-torus of G_2 that contains a maximal K_{2v}-split torus S_2^{0v} of G_2. Next we find an element h_2 of G_2, rational over a finite extension of K_{2v}, such that

$$T_2 = i_{h_2}(T_2^{0v}) \quad \text{and} \quad S_2^v \subset i_{h_2}(S_2^{0v}), \tag{34}$$

where S_2^v is the maximal K_{2v}-split subtorus of T_2. Since π is defined over K_2, it follows from (34) that for $\varphi := i_{h_2}^{-1} \circ \pi \circ i_{h_1}$

$$S_1^0 \subset \varphi^{-1}(S_2^{0v}) =: \mathcal{S}.$$

Lift the old ordering on $X(S_1^0) \otimes_{\mathbb{Z}} \mathbb{R}$ first to a coherent ordering on $X(\mathcal{S}) \otimes_{\mathbb{Z}} \mathbb{R}$, and then lift the latter to a coherent ordering on $X(T_1^0) \otimes_{\mathbb{Z}} \mathbb{R}$. We claim that this ordering on $X(T_1^0) \otimes_{\mathbb{Z}} \mathbb{R}$ can be taken to be the new ordering. Indeed, as above, let $\Delta_1^{0v} \subset \Phi(G_1, T_1^0)$ be the system of simple roots corresponding to the new ordering, and let $\alpha_1^{0v} \in \Delta_1^{0v}$ be the root corresponding to $\alpha_1^0 \in \Delta_1^0$ under the canonical identification $\Delta_1^0 \simeq \Delta_1^{0v}$; as we already mentioned, α_1^{0v} restricts to S_1^0 nontrivially. By construction, the system of simple roots $\Delta_2^{0v} \subset \Phi(G_2, T_2^{0v})$ such that $\varphi^*(\Delta_2^{0v}) = \Delta_1^{0v}$ corresponds to a choice of compatible orderings on $X(S_2^{0v}) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X(T_2^{0v}) \otimes_{\mathbb{Z}} \mathbb{R}$, and $\alpha_2^{0v} \in \Delta_2^{0v}$ such that $\varphi^*(\alpha_2^{0v}) = \alpha_1^{0v}$ restricts to S_2^{0v} nontrivially, i.e. is a distinguished vertex in the Tits index of G_2/K_{2v}. On the other hand, in the above notations we have

$$i_{h_1}^*(\alpha_1^v) = \alpha_1^{0v}, \quad \pi^*(\alpha_2^v) = \alpha_1^v \quad \text{and} \quad i_{h_2}^*(\alpha_2^v) = \alpha_2^{0v}.$$

Thus, $\alpha_2^v \in \Delta_2^v$ corresponds to a distinguished vertex in the Tits index of G_2/K_{2v}, as required. $\square$

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