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On some finiteness properties of algebraic groups over finitely generated fields

Sur quelques propriétés de finitude des groupes algébriques sur des corps de type fini

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A R T I C L E I N F O

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A B S T R A C T

We present several finiteness results for absolutely almost simple algebraic groups over finitely generated fields that are more general than global fields. We also discuss the relations between the various finiteness properties involved in these results, such as the properness of the global-to-local map in the Galois cohomology of a given K-group G relative to a certain natural set V of discrete valuations of K, and the finiteness of the number of isomorphism classes of K-forms of G having, on the one hand, smooth reduction with respect to all places in V and, on the other hand, the same isomorphism classes of maximal K-tori as G.

R É S U M É

Nous présentons plusieurs résultats de finitude pour les groupes algébriques absolument presque simples définis sur des corps de type fini plus généraux que les corps globaux. Nous discutons aussi des liens entre les propriétés de finitude diverses qui entrent dans le cadre de notre analyse, telles que la propreté de l'application globale–locale dans la cohomologie galoisienne d'un K-groupe G par rapport à un ensemble convenable V de valuations discrètes de K, et la finitude du nombre de K-formes de G ayant, d'une part, bonne réduction en V et possédant, d'autre part, les même classes d'isomorphisme de K-tores maximaux que G.

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Soit K un corps de type fini. Dans cette note, nous discutons la question suivante :

($*$) *(Quand) est-il possible de munir K d'un ensemble naturel V de valuations discrètes tel que, pour un groupe algébrique G simplement connexe absolument presque simple défini sur K, l'ensemble de classes d'isomorphisme sur K de K-formes de G ayant bonne réduction en toutes $v \in V$ (resp. en toutes $v \in V \setminus S$, pour un sous-ensemble $S \subset V$ fini arbitraire) soit fini ?*

Nous relions cette question aux propriétés de l'application globale–locale en cohomologie galoisienne et à l'analyse des K-formes de G possédant les mêmes classes d'isomorphisme de K-tores maximaux que G. D'après des résultats de finitude en cohomologie galoisienne bien connus (cf. [17, Ch. III, §4.6, Theorem 7]), si K est un corps de nombres et V^K désigne l'ensemble de toutes les places de K, alors les ensembles de type $V^K \setminus S$, où $S \subset V^K$ est un sous-ensemble fini contenant toutes les places archimédiennes de K, sont convenables. Sur les corps plus généraux que les corps globaux, la question ($*$) appelle une réponse affirmative pour les formes intérieures de type A_ℓ si la caractéristique de K ne divise pas $(\ell + 1)$ (cf. [4]). Dans la section 2 de cette note, nous construisons un ensemble convenable V pour les groupes spinoriels des formes quadratiques en $\geqslant 5$ variables dans le cas où K est le corps de fonction d'une courbe lisse géométriquement irréductible sur un corps de nombres (Théorème 1) ; le même ensemble marche également pour les groupes de type G_2.

Dans les sections 3 et 4, il s'agit de l'analyse des groupes algébriques absolument presque simples possédant les mêmes classes d'isomorphisme de tores maximaux sur le corps de définition. Plus précisément, si G est un groupe algébrique simplement connexe absolument presque simple défini sur un corps K, nous définissons le genre $\mathbf{gen}_K(G)$ comme l'ensemble des classes d'isomorphisme sur K de K-formes de G possédant les mêmes classes d'isomorphisme de K-tores maximaux que G. Il paraît probable que le genre est fini si K est un corps de type fini de « bonne » caractérisque relativement au type de G. La finitude du genre est connue si K est un corps de nombres ou bien si G est une forme intérieure de type A_ℓ et si K est un corps de type fini dont la caractéristique ne divise pas $(\ell + 1)$ — voir [4, §6]. Dans le cas général, nous relions ce problème à la question ($*$) (voir le Théorème 5), ce qui nous permet d'obtenir de nouveaux résultats de finitude (et même de trivialité) pour le genre des groupes de type G_2.

1. Introduction

The purpose of this note is to discuss several finiteness properties of absolutely almost simple groups over finitely generated fields, such as the properness of the global-to-local map in the Galois cohomology of a K-group G relative to a certain natural set of discrete valuations of K, and the finiteness of the number of isomorphism classes of K-forms of G having, on the one hand, smooth reduction with respect to V and, on the other hand, the same isomorphism classes of maximal K-tori as G. We prove these properties in some cases, and establish relations between them in general. To put these results in context, we recall that if G is an algebraic group over a number field K, and V^K denotes the set of all places, then the main finiteness result (cf. [17, Ch. III, §4.6, Theorem 7]) states that the natural global-to-local map $\rho_G \colon H^1(K, G) \to \prod_{v \in V^K} H^1(K_v, G)$ is *proper*, i.e. the pre-image of any finite set is finite (in particular, the corresponding Tate–Shafarevich set $Ш(G) := \mathrm{Ker}\, \rho_G$ is finite). Then a similar map ρ^S_G constructed using the product over all $v \in V^K \setminus S$ is also proper, for any finite subset $S \subset V^K$. To provide a different perspective on this result, we recall that if a connected K-group G has smooth reduction at a non-Archimedean place $v \in V^K$, then it becomes quasi-split over the completion K_v (cf. [13, Theorem 6.7]). Combining this with the properness of ρ^S_G, one concludes that for an absolutely almost simple simply connected K-group G, the set of K-isomorphism classes of K-forms of G having smooth reduction at all $v \in V^K \setminus S$ is finite, for any finite subset $S \subset V^K$ containing all Archimedean places. (With some additional efforts, this result can be extended to all reductive groups — see [10].) *Conversely*, this property *implies* the properness of ρ^S_G for adjoint semi-simple groups.

The question arises if and to what extent the above finiteness property can be extended to fields other than number fields. More precisely, let K be a finitely generated field. Can one equip K with a "natural" set V of discrete valuations such that, for a given (absolutely almost simple simply connected) K-group G, the set of its K-forms that have smooth reduction at all $v \in V$ (resp., at all $v \in V \setminus S$, where $S \subset V$ is an arbitrary finite subset) consists of finitely many K-isomorphism classes? What makes this question interesting is that the affirmative answer would have important consequences for the global-to-local map as well as other issues. First, the version involving all $v \in V$ would imply that for an adjoint K-group $\overline{G}$ having smooth reduction at all $v \in V$, the kernel $Ш_V(\overline{G})$ of the global-to-local map $\rho_{G,V} \colon H^1(K, \overline{G}) \to \prod_{v \in V} H^1(K_v, \overline{G})$ relative to V, is finite. Second, the version allowing one to remove from V any finite subset would imply the properness not only of $\rho_{G,V}$ but also of $\rho_{G,V \setminus S}$ for any finite $S \subset V$ (cf. [4, §6]). Third, this version (with some additional requirements on V) would imply the finiteness of the genus of absolutely almost simple simply connected groups over finitely generated fields — see §2 below. Yet another application would be a finiteness result for absolutely almost simple algebraic groups containing a finitely generated Zariski-dense subgroup weakly commensurable to a given one (see [15, §7]; note that Conjecture 7.8 there is already a theorem). One can consider some variations of the general problem, for example, by restricting attention only to *inner* forms of G having smooth reduction. But in all cases, a natural candidate for such a V appears to be the set of discrete valuations associated with the prime divisors of a model of K, i.e. a smooth arithmetic scheme with function field K (we call such sets *divisorial*). It is known that divisorial sets V indeed work for inner forms of type A_ℓ provided that char K does not divide $\ell+1$ (cf. [3]); this relies on the finiteness of the unramified Brauer group ${}_{(\ell+1)}\mathrm{Br}(K)_V$ – see [5]. We note that in [5], we considered some specific divisorial sets V and obtained for them explicit bounds on the order of the unramified Brauer group. Until recently, no other types have been considered, and the first result of this note deals with spinor groups when V is as constructed in [5].

2. Spinor groups with smooth reduction

Let C be a smooth geometrically connected projective curve over a number field k with function field $K = k(C)$, and denote by V_0 the set of discrete valuations of K associated with the closed points of C. Furthermore, pick a finite subset $S \subset V^k$ that contains all Archimedean places and all places of bad reduction for a certain model of C. Then every $v \in V^k \setminus S$ has a canonical extension to a discrete valuation $\tilde{v}$ of K. We set $V_1 = \{\tilde{v} \mid v \in V^k \setminus S\}$, and as in [5] consider $V = V_0 \cup V_1$.

Theorem 1. *In the above notations, the number of K-isomorphism classes of spinor groups $G = \mathrm{Spin}_n(q)$ of nondegenerate quadratic forms in $n \geqslant 5$ variables over K that have smooth reduction at all $v \in V$ is finite.*

Theorem 2. *Notations as in Theorem 1, for $G = \mathrm{SO}_n(q)$ the kernel $\text{Ш}_V(G)$ of the map $H^1(K, G) \to \prod_{v\in V} H^1(K_v, G)$ is finite.*

We will derive Theorem 1 from a result that allows for more general sets of discrete valuations of arbitrary fields. To formulate it, we need to introduce some additional notations. Assume that a field K is equipped with a set V of discrete valuations that satisfies the following condition:

(A) For any $a \in K^\times$, the set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite.

We let $\mathrm{Div}(V)$ denote the free Abelian group on the set V, the elements of which will be called "*divisors*." Since V satisfies (A), with any $a \in K^\times$ we can associate the "*principal divisor*"

$$(a) = \sum_{v\in V} v(a)\cdot v,$$

and we let $\mathrm{P}(V)$ denote the subgroup of $\mathrm{Div}(V)$ formed by all principal divisors. We call the quotient $\mathrm{Div}(V)/\mathrm{P}(V)$ the *Picard group* of V and denote it by $\mathrm{Pic}(V)$.

Next, let v be a discrete valuation of K such that the residue field $K^{(v)}$ has characteristic $\neq 2$, and let $\mu_2 = \{\pm 1\}$. Then for any $i \geqslant 2$, there exists a residue map in Galois cohomology

$$\partial_v^i \colon H^i(K, \mu_2) \longrightarrow H^{i-1}(K^{(v)}, \mu_2)$$

(cf. [7, §6.8], [17, Ch. II, Appendix]). We now make the following assumption

(B) char $K^{(v)} \neq 2$ for all $v \in V$.

We then define the ith unramified cohomology group of K with respect to V by

$$H^i(K, \mu_2)_V = \bigcap_{v\in V} \mathrm{Ker}\,\partial_v^i.$$

With these notations, we have the following theorem.

Theorem 3. *Let K be a field equipped with a set V of discrete valuations satisfying conditions* (A) *and* (B)*, and $n \geqslant 5$ be an integer. Assume that*

(1) *the quotient $\mathrm{Pic}(V)/2\mathrm{Pic}(V)$ is finite; and*
(2) *the unramified cohomology groups $H^i(K, \mu_2)_V$ are finite for all $i = 1, \dots, [\log_2 n] + 2$.*

Then the number of K-isomorphism classes of spinor groups $G = \mathrm{Spin}_n(q)$ of nondegenerate quadratic forms q over K in n variables that have smooth reduction at all $v \in V$ is finite.

Sketch of proof of Theorem 1. Let V be the set of places of $K = k(C)$ as in Theorem 1. Without loss of generality, we may assume that V does not contain any dyadic places. It follows from [11] that the group $\mathrm{Pic}(V)$ is finitely generated, implying that condition (1) of Theorem 3 holds. The finiteness of $H^i(K, \mu_2)_V$ for $i = 1$ is well known (cf. [5, §5]), and for $i = 2$ is established in [5]. On the other hand, the finiteness for $i \geqslant 4$ follows from the theorems of Poitou and Tate (see [17, Ch. II, §6.3]). In the remaining (difficult) case $i = 3$ the finiteness (of even $H^3(K, \mu_2)_{V_0}$) follows from results of Kato [12] and Jannsen [9] on cohomological Hasse principles. Now, we obtain Theorem 1 from Theorem 3. (We observe that the groups

$H^i(K, \mu_2)_V$ for $i \neq 3$ are known to remain finite if one deletes from V an arbitrary finite subset, but we do not know if this is also the case for $i = 3$.)

We should mention that Theorem 3 applies to fields that are not necessarily finitely generated. For example, in recent years, there has been a great deal of activity in studying division algebras and algebraic groups over the function fields of p-adic curves (see, for example, [2,6] and [8]). So, we would like to point out a finiteness result for spinor groups over function fields of curves over a class of fields that contains all p-adic fields but is in fact much larger. To formulate it, we need to introduce a generalization of Serre's condition (F) (see [17, Ch. III, §4]). Let K be a field and $m \geqslant 1$ an integer prime to char K. We say that K satisfies condition (F'_m) if

(F'_m) for every finite separable extension L/K, the quotient $L^\times/L^{\times m}$ is finite.

Combining Theorem 3 with some recent results on the finiteness of unramified cohomology with μ_m-coefficients for the function fields of curves over fields satisfying (F'_m) (see [16]), we obtain the following theorem.

Theorem 4. *Let C be a smooth (but not necessarily projective) geometrically connected curve over a field k of characteristic $\neq 2$ that satisfies condition (F'_2), and let $K = k(C)$. Denote by V the set of discrete valuations of K corresponding to the closed points of C. Then the number of K-isomorphism classes of spinor groups $G = \mathrm{Spin}_n(q)$ of nondegenerate quadratic forms q over K in n variables that have smooth reduction at all $v \in V$ is finite.*

One can expect that Theorem 4 extends to any absolutely almost simple simply connected K-group G provided that the base field k satisfies condition (F) (in fact, it is probably enough to require (F'_p) for all primes p dividing the order of the Weyl group of G). In this regard, we observe that Theorems 1 and 4 do remain valid for groups of type G_2 (for the same fields K and the same sets of valuations V).

3. The genus and smooth reduction

As we already mentioned in the introduction, the finiteness question for the number of K-forms of a given absolutely almost simple simply connected algebraic K-group G having smooth reduction at a given set V of discrete valuations of K has applications not only to the properness of the global-to-local map in Galois cohomology, but also to the problem of describing algebraic groups having the same isomorphism classes of maximal tori over the field of definition. More precisely, we say that two absolutely almost simple algebraic K-groups G_1 and G_2 have the *same isomorphism classes of maximal K-tori* if every maximal K-torus T_1 of G_1 is K-isomorphic to some maximal K-torus T_2 of G_2, and vice versa.

Definition. Let G be an absolutely almost simple simply connected algebraic group over a field K. The (K-)*genus* $\mathbf{gen}_K(G)$ of G is the set of K-isomorphism classes of K-forms G' of G that have the same K-isomorphism classes of maximal K-tori as G.

One expects $\mathbf{gen}_K(G)$ to be finite over any finitely generated field of *good* characteristic, and in fact to reduce to just a single element in some special situations (see, for example, [3, Theorem 6.3] as well as Theorem 8 and Corollary 10 below). In previous work, we were able to resolve similar issues for finite-dimensional central division K-algebras, where the genus is defined in terms of the K-isomorphism classes of maximal subfields (see [3–5]). One of the key observations was that if two central division K-algebras D_1 and D_2 of the same degree belong to the same genus, then (under some natural assumptions) they have the same ramification at any discrete valuation v of K (see [3, Lemma 2.5] for the precise statement); in particular, if D_1 is unramified at v then so is D_2. Our next result provides an extension of this statement to arbitrary absolutely almost simple groups.

Theorem 5. *Let G be an absolutely almost simple simply connected algebraic group over a field K, and let v be a discrete valuation of K. Assume that the residue field $K^{(v)}$ is finitely generated and that G has smooth reduction $\underline{G}^{(v)}$ at v. Then any $G' \in \mathbf{gen}_K(G)$ also has smooth reduction at v. Moreover, the reduction $\underline{G'}^{(v)}$ lies in the genus $\mathbf{gen}_{K^{(v)}}(\underline{G}^{(v)})$.*

Corollary 6. *Let G be an absolutely almost simple simply connected algebraic group over a field K. Assume that K is equipped with a set V of discrete valuation that satisfies* (A) *and*

(C) *for any $v \in V$, the residue field $K^{(v)}$ is finitely generated,*

and is such that for any finite subset $V_0 \subset V$, the set of K-forms of G that have smooth reduction at all $v \in V \setminus V_0$ is finite. Then the genus $\mathbf{gen}_K(G')$ is finite for any K-form G' of G.

4. The finiteness and triviality of the genus for type G_2

The finiteness of the genus for an inner form G of type A_ℓ over a finitely generated field K of characteristic not dividing $\ell + 1$ was established in [3, Theorem 5.3], where we also described some situations where the genus is actually trivial (see [3, Theorem 5.3] as well as [4, Theorem 6.3]). We now want to present similar, but more restrictive, results for groups of type G_2.

Theorem 7. *Let G be a simple algebraic K-group of type G_2. Then in each of the following situations:*

(1) *K is a finitely generated transcendence degree one extension of a number field,*
(2) *K is a finitely generated purely transcendental extension of a number field,*

the genus $\mathbf{gen}_K(G)$ is finite.

Theorem 8. *Let k be a number field, and $K = k(x)$ be the field of rational functions over k. Then for any K-group G of type G_2, the genus $\mathbf{gen}_K(G)$ consists of a single element.*

Note that, in general, the genus of a group of type G_2 may be nontrivial — see [1].

Next, we have the following result, which is obtained from Theorem 5 by applying the result of [14].

Theorem 9. *Let G be an absolutely almost simple simply connected algebraic group over a finitely generated field k of characteristic zero, and let $K = k(x)$ be the field of rational functions. Then any $H \in \mathbf{gen}_K(G \otimes_k K)$ is of the form $H = H_0 \otimes_k K$ for some $H_0 \in \mathbf{gen}_k(G)$.*

Note that for $H_0 \in \mathbf{gen}_k(G)$, the group $H = H_0 \otimes_k K$ **may not** lie in $\mathbf{gen}_K(G \otimes_k K)$. In fact, applying Theorem 9 and some results about Pfister forms, we obtain the following for groups of type G_2.

Corollary 10. *Let G be a group of type G_2 over a finitely generated field k of characteristic zero, and let $F = k(x_1, \dots, x_6)$ be the field of rational functions in 6 variables. Then $\mathbf{gen}_F(G \otimes_k F)$ consists of a single element.*

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