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Algebra

# On the genus of a division algebra

## *Sur le genre d'un corps gauche*

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ABSTRACT

We define the genus **gen**$(D)$ of a finite-dimensional central division algebra $D$ over a field $K$ as the set of all classes $[D']$ in the Brauer group $\mathrm{Br}(K)$ that are represented by central division $K$-algebras $D'$ having the same maximal subfields as $D$. We give examples where **gen**$(D)$ is reduced to a single element, and other examples where it is finite.



RÉSUMÉ

Nous définissons le genre **gen**$(D)$ d'un corps gauche central $D$ de dimension finie sur un corps $K$ comme l'ensemble des classes $[D']$ dans le groupe de Brauer $\mathrm{Br}(K)$ qui sont représentées par des corps gauches $D'$ de centre $K$ ayant les mêmes sous-corps maximaux que $D$. Nous donnons des exemples où **gen**$(D)$ est réduit à un seul élément, ainsi que d'autres où **gen**$(D)$ est fini.



### Version française abrégée

Soit $K$ un corps et $\mathrm{Br}(K)$ son groupe de Brauer. Pour une algèbre centrale simple $A$ sur $K$, on note $[A]$ sa classe dans $\mathrm{Br}(K)$. On définit le genre **gen**$(D)$ d'un corps gauche central $D$ sur $K$ comme l'ensemble des classes $[D'] \in \mathrm{Br}(K)$, où $D'$ est un corps gauche central sur $K$ ayant les mêmes sous-corps maximaux que $D$. Dans cette note, on étudie les deux questions suivantes :

**Question 1.** Quand est-ce que le genre est réduit à un seul élément ?

**Question 2.** Quand est-ce que **gen**$(D)$ est fini ?

On observe que **gen**$(D)$ peut être réduit à un seul élément seulement si $[D]$ est d'exposant deux dans $\mathrm{Br}(K)$ ; en effet, dans cette situation, **gen**$(D)$ consiste d'un seul élément si $K$ est un corps global. On prouve, en particulier, que si $K$ est un corps de car. $\neq 2$ qui a la propriété que $|\mathbf{gen}(D)| = 1$ pour tout corps gauche $D$ sur $K$ d'exposant deux, alors le corps de fractions rationelles $K(x)$ a la même propriété. Par conséquent, $|\mathbf{gen}(D)| = 1$ pour tout corps gauche $D$ d'exposant deux sur $K = k(x_1, \ldots, x_r)$, où $k$ est soit un corps de nombres soit un corps fini de car. $\neq 2$.

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On établit aussi le résultat de finitude suivant : soit $K$ un corps de type fini sur son sous-corps premier, soit $D$ un corps gauche central sur $K$ de degré $n$, $(n, \mathrm{car}.\,K) = 1$. Alors $|\mathbf{gen}(D)|$ est fini. La preuve est réduite à la démonstration de la finitude du sous-groupe de $n$-torsion ${}_n\mathrm{Br}(K)_V$ du groupe de Brauer de $K$ non-ramifié par rapport à un ensemble convenable de valuations discrètes de $K$. On donne un exemple d'une borne explicite pour $|\mathbf{gen}(D)|$ dans le cas où $D$ est une algèbre de quaternions sur le corps de fractions $k(E)$ d'une courbe elliptique définie sur un corps de nombres $k$.

## 1. Introduction

Let $K$ be a field, $\mathrm{Br}(K)$ be its Brauer group, and for any integer $n > 1$ let ${}_n\mathrm{Br}(K)$ be the subgroup of $\mathrm{Br}(K)$ annihilated by $n$. For a finite-dimensional central simple algebra $A$ over $K$, we let $[A]$ denote the corresponding class in $\mathrm{Br}(K)$, and we then define the *genus* $\mathbf{gen}(D)$ of a central division $K$-algebra $D$ of degree $n$ to be the set of classes $[D'] \in \mathrm{Br}(K)$ where $D'$ is a central division $K$-algebra having the same maximal subfields as $D$ (in more precise terms, this means that $D'$ has the same degree $n$, and a field extension $P/K$ of degree $n$ admits a $K$-embedding $P \hookrightarrow D$ if and only if it admits a $K$-embedding $P \hookrightarrow D'$).[1] One can ask the following two questions about the genus of a central division $K$-algebra $D$ of degree $n$:

**Question 1.** When does $\mathbf{gen}(D)$ consist of a single class?

**Question 2.** When is $\mathbf{gen}(D)$ finite?

We note that since the opposite algebra $D^{\mathrm{op}}$ has the same maximal subfields as $D$, the genus $\mathbf{gen}(D)$ can reduce to a single element only if $[D^{\mathrm{op}}] = [D]$, i.e. if $D$ has exponent 2 in the Brauer group. On the other hand, as follows from the theorem of Artin–Hasse–Brauer–Noether (AHBN), $\mathbf{gen}(D)$ does reduce to a single element for any algebra $D$ of exponent 2 over a global field $K$ (in which case $D$ is necessarily a quaternion algebra).[2] The following theorem (which for quaternion algebras was established earlier in [16]) expands the class of fields with this property:

**Theorem 1** *(Stability Theorem). Let $K$ be a field of characteristic $\neq 2$.*

(1) *If $K$ satisfies the following property*:
   (∗) *if $D$ and $D'$ are central division $K$-algebras of exponent 2 having the same maximal subfields then $D \simeq D'$ (in other words, for any $D$ of exponent 2, $|\mathbf{gen}(D) \cap {}_2\mathrm{Br}(K)| = 1$),*
   *then the field of rational functions $K(x)$ also satisfies (∗).*
(2) *If $|\mathbf{gen}(D)| = 1$ for any central division $K$-algebra $D$ of exponent 2, then the same is true for any central division $K(x)$-algebra of exponent 2.*

**Corollary 2.** *Let $k$ be either a finite field of characteristic $\neq 2$ or a number field, and $K = k(x_1, \ldots, x_r)$ be a finitely generated purely transcendental extension of $k$. Then for any central division $K$-algebra $D$ of exponent 2 we have $|\mathbf{gen}(D)| = 1$.*

While Question 1 makes sense only for division algebras of exponent 2, Question 2 can be asked for arbitrary division algebras. As above, it follows from (ABHN) that $\mathbf{gen}(D)$ is finite for any finite-dimensional central division algebra $D$ over a global field $K$. For fields other than global, the finiteness question was investigated in [10] for the genus $\mathbf{gen}'(D)$ defined in terms of all finite-dimensional splitting fields (note that $\mathbf{gen}'(D) \subset \mathbf{gen}(D)$) for division algebras $D$ of arbitrary prime exponent $p$ over the field $K = k(x)$ of rational functions, with $p \neq \mathrm{char}\,k$. In particular, it was shown in [10] that if $\mathbf{gen}'(\Delta)$ is finite for any central division algebra $\Delta$ of exponent $p$ over a field $k$, then $\mathbf{gen}'(D)$ is finite for any central division algebra $D$ of exponent $p$ over $K = k(x)$. At the same time, a direct generalization of the construction described in [8, §2] enables one to provide an example of a quaternion division algebra $D$ over an infinitely generated field $K$ with infinite genus $\mathbf{gen}(D)$. So, the following finiteness result seems to cover the most general situation:

**Theorem 3.** *Let $K$ be a finitely generated field (i.e., a finitely generated extension of its prime field). If $D$ is a central division $K$-algebra of exponent prime to* $\mathrm{char}\,K$*, then* $\mathbf{gen}(D)$ *is finite.*

## 2. The genus and the unramified Brauer group

We will now describe a general set-up that allows one to estimate the size of $\mathbf{gen}(D)$, and will then apply it to proving Theorems 1 and 3. Given a discrete valuation $v$ of $K$, we let $\mathcal{O}_{K,v}$ and $\overline{K}_v$ denote its valuation ring and residue field,

[1] At the end of this note, we will discuss a generalization of this notion to absolutely almost simple algebraic $K$-groups in which maximal subfields are replaced with maximal $K$-tori. We observe in this respect that only *separable* maximal subfields of $D$ give rise to maximal $K$-tori of $G = \mathrm{SL}_{1,D}$. So, in order to make our definitions fully compatible, one should define $\mathbf{gen}(D)$ in terms of maximal separable subfields. In the current note, however, the degree $n$ of $D$ will always be assumed to be coprime to the characteristic of $K$, so the issue of separability will not arise.

[2] Indeed, (ABHN) implies that a quaternion algebra over a global field is uniquely determined by its set of ramified places; on the other hand, if two quaternion division algebras have the same maximal subfields, they necessarily have the same ramified places.

respectively. Fix an integer $n > 1$ (which will later be either the degree or the exponent of $D$) and suppose that $V$ is a set of discrete valuations of $K$ that satisfies the following three conditions:

(A) *For any $a \in K^\times$, the set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite;*
(B) *There exists a finite subset $V' \subset V$ such that the field of fractions of*

$$\mathcal{O} := \bigcap_{v \in V \setminus V'} \mathcal{O}_{K,v}$$

*coincides with $K$;*
(C) *For any $v \in V$, the characteristic of $\overline{K}_v$ is prime to $n$.*

(We note that if $K$ is finitely generated, then (B) is an automatic consequence of (A).) Due to (C), for each $v \in V$ one can define the residue map

$$\rho_v : {}_n\mathrm{Br}(K) \to \mathrm{Hom}\big(\mathcal{G}^{(v)}, \mathbb{Z}/n\mathbb{Z}\big),$$

where $\mathcal{G}^{(v)}$ is the absolute Galois group of $\overline{K}_v$ (cf., for example, [17, §10] or [18, Ch. II, Appendix]). As usual, a class $[A] \in {}_n\mathrm{Br}(K)$ (or a central simple $K$-algebra $A$ representing this class) is said to be *unramified* at $v$ if $\rho_v([A]) = 1$, and *ramified* otherwise. We let $\mathrm{Ram}_V(A)$ (or $\mathrm{Ram}_V([A])$) denote the set of all $v \in V$ where $A$ is ramified.

**Proposition 4.** *If $V$ satisfies conditions* (A), (B), *and* (C), *then for any $[A] \in {}_n\mathrm{Br}(K)$, the set $\mathrm{Ram}_V([A])$ is finite.*

**Proposition 5.** *Let $D$ and $D'$ be central division $K$-algebras such that $[D] \in {}_n\mathrm{Br}(K)$ and $[D'] \in \mathbf{gen}(D) \cap {}_n\mathrm{Br}(K)$. Given $v \in V$, we let $\chi_v$ and $\chi'_v \in \mathrm{Hom}(\mathcal{G}^{(v)}, \mathbb{Z}/n\mathbb{Z})$ denote the images under $\rho_v$ of the classes $[D]$ and $[D']$, respectively. Then*

$$\mathrm{Ker}\ \chi_v = \mathrm{Ker}\ \chi'_v$$

*for all $v \in V$. In particular, if $D$ is unramified at $v$ then so is $D'$.*

We define the unramified part of ${}_n\mathrm{Br}(K)$ relative to $V$ as follows:

$${}_n\mathrm{Br}(K)_V := \bigcap_{v \in V} \mathrm{Ker}\ \rho_v.$$

The following statement relates the size of the genus to the size of ${}_n\mathrm{Br}(K)_V$:

**Theorem 6.** *Assume that ${}_n\mathrm{Br}(K)_V$ is finite. Then for any finite-dimensional central division $K$-algebra $D$ of exponent $n$, the intersection $\mathbf{gen}(D) \cap {}_n\mathrm{Br}(K)$ is finite, of size*

$$\big|\mathbf{gen}(D) \cap {}_n\mathrm{Br}(K)\big| \leqslant \big|{}_n\mathrm{Br}(K)_V\big| \cdot \varphi(n)^r, \quad \textit{with } r = \big|\mathrm{Ram}_V(D)\big|,$$

*where $\varphi$ is the Euler function. In particular, if $D$ has degree $n$ then*

$$\big|\mathbf{gen}(D)\big| \leqslant \big|{}_n\mathrm{Br}(K)_V\big| \cdot \varphi(n)^r.$$

We will now specialize to the situation where $K = k(C)$ is the field of rational functions on a smooth absolutely irreducible projective curve $C$ over a field $k$. Set $V$ to be the set of all geometric places of $K$, i.e. those discrete valuations of $K$ that are trivial on $k$. Then the corresponding unramified Brauer group ${}_n\mathrm{Br}(K)_V$ will be denoted by ${}_n\mathrm{Br}(K)_{\mathrm{ur}}$ (this is precisely the $n$-torsion subgroup of the Brauer group of the curve $C$). Applying the techniques outlined above, in conjunction with some considerations involving specialization, we obtain the following:

**Theorem 7.** *Let $n > 1$ be an integer prime to $\mathrm{char}\, k$. Assume that*

- *the set $C(k)$ of rational points is infinite;*
- *$|{}_n\mathrm{Br}(K)_{\mathrm{ur}}/\iota_k({}_n\mathrm{Br}(k))| =: M < \infty$, where $\iota_k : \mathrm{Br}(k) \to \mathrm{Br}(K)$ is the canonical map.*

*Then*

(1) *if there exists $N < \infty$ such that*

$$\big|\mathbf{gen}(\Delta) \cap {}_n\mathrm{Br}(k)\big| \leqslant N$$

*for any central division $k$-algebra $\Delta$ of exponent $n$, then for any central division $K$-algebra $D$ of exponent $n$ we have*

$$\left|\mathbf{gen}(D) \cap {}_n\mathrm{Br}(K)\right| \leqslant M \cdot N \cdot \varphi(n)^r,$$

*where $r = |\mathrm{Ram}_V(D)|$;*

(2) *if $\mathbf{gen}(\Delta) \cap {}_n\mathrm{Br}(k)$ is finite for any central division $k$-algebra $\Delta$ of exponent $n$, then $\mathbf{gen}(D) \cap {}_n\mathrm{Br}(K)$ is finite for any central division $K$-algebra $D$ of exponent $n$.*

One notable case where Theorem 7 applies is $C = \mathbb{P}^1_k$ over an infinite field $k$ (which we can assume without loss of generality). It is well-known that in this case ${}_n\mathrm{Br}(K)_{\mathrm{ur}} = \iota_k({}_n\mathrm{Br}(k))$ (cf. [9, Corollary 6.4.6]), i.e. one can take $M = 1$. Now, let $n = 2$ and assume that $k$ satisfies condition $(*)$ of Theorem 1, i.e. $|\mathbf{gen}(\Delta) \cap {}_2\mathrm{Br}(k)| = 1$ for any central division $k$-algebra $\Delta$ of exponent 2. The latter means that one can take $N = 1$. We then obtain from Theorem 7 that $|\mathbf{gen}(D) \cap {}_2\mathrm{Br}(K)| = 1$ for any central division $K$-algebra $D$ of exponent 2, proving part (1) of Theorem 1. The proof of part (2) is similar.

Furthermore, it follows from Theorem 6 that in order to prove Theorem 3, it is enough to establish the following:

**Theorem 8.** *Let $K$ be a finitely generated field, and let $n > 1$ be an integer coprime to $\mathrm{char}\, K$. Then there exists a set $V$ of discrete valuations of $K$ that satisfies conditions (A), (B) and (C), and for which the unramified Brauer group ${}_n\mathrm{Br}(K)_V$ is finite.*

We originally proved Theorem 8 by a method related to the proof of the Weak Mordell–Weil Theorem (cf. [11, Ch. VI]), which in principle can be used to obtain some estimates on the size of ${}_n\mathrm{Br}(K)_V$, hence of $\mathbf{gen}(D)$ (see below). It was later pointed out to us by J.-L. Colliot-Thélène [4] that a (nonconstructive) proof of the finiteness of ${}_n\mathrm{Br}(K)_V$ can be derived from the following general statement:

**Theorem 9.** *Let $X$ be a scheme of finite type over $U = \mathrm{Spec}\, A$, where $A$ is either a finite field or the ring of $S$-integers in a number field (with $S$ finite). For any integer $n$ invertible in $A$ and any $n$-torsion constructible sheaf $\mathfrak{F}$ on $X$, the étale cohomology groups $H^i_{\text{ét}}(X, \mathfrak{F})$ are finite for all $i \geqslant 0$.*

Given a finitely generated field $K$ and an integer $n > 1$ prime to $\mathrm{char}\, K$, we can pick a smooth affine integral scheme $X$ as in Theorem 9 with the field of rational functions $K$. Applying Theorem 9 to the étale sheaf associated with the group scheme $\boldsymbol{\mu}_n$ of $n$th roots of unity, we obtain the finiteness of $H^2_{\text{ét}}(X, \boldsymbol{\mu}_n)$. Then the Kummer sequence yields the finiteness of ${}_n\mathrm{Br}(X)$. On the other hand, it follows from the absolute purity conjecture proved by O. Gabber (see [7] for an exposition of Gabber's proof, and also [5, p. 153] and [3, discussion after Theorem 4.2] regarding the history of the question) that the latter coincides with ${}_n\mathrm{Br}(K)_V$, where $V$ is the set of discrete valuations of $K$ associated with the divisors of $X$, cf. [7], hence the required fact (obviously, this $V$ satisfies our conditions (A), (B) and (C)).

Since the proof of Theorem 9 is not readily available in the existing literature, we reproduce below an outline of the argument kindly explained to us by J.-L. Colliot-Thélène in [4] (with his permission). Since for our purposes we only need to consider the smooth case, in the situation where $A$ is a finite field the required fact follows from Corollary 4.5 or Corollary 5.5 in [12, Ch. VI] in conjunction with the Hochschild–Serre spectral sequence (cf. [12, Ch. III, Theorem 2.20]).

Let now $A$ be a ring of $S$-integers in some number field $k$, where $S$ is a finite set of places of $k$. Applying to the structure morphism $f : X \to U$ Theorem 1.1 of the chapter "Théorèmes de finitude" in Deligne's book [6, p. 233], we obtain that the direct images $\mathrm{R}^q f_* \mathfrak{F}$ are constructible $n$-torsion sheaves on $U$. Combining Proposition 2.9 in [13, Ch. II] with Theorem 8.3.19 in [14], we obtain that the groups $H^p_{\text{ét}}(U, \mathrm{R}^q f_* \mathfrak{F})$ are finite for all $p \geqslant 0$. Then the Leray spectral sequence $H^p_{\text{ét}}(U, \mathrm{R}^q f_* \mathfrak{F}) \Rightarrow H^{p+q}_{\text{ét}}(X, \mathfrak{F})$ [12, Ch. III, Theorem 1.18] shows that the groups $H^i_{\text{ét}}(X, \mathfrak{F})$ are all finite.

## 3. An example

We will now show how the methods involved in our original proof of Theorem 8 can actually be used to estimate the size of the unramified Brauer group, and hence of the genus of a division algebra, in certain situations. Because of space limitation, we will focus on the following example. Let $k$ be a number field, and let $E$ be an elliptic curve over $k$ given by a Weierstrass equation

$$y^2 = f(x) \quad \text{where } f(x) = x^3 + \alpha x^2 + \beta x + \gamma.$$

Without loss of generality, we may assume that all the coefficients lie in the ring of integers $\mathcal{O}_k$. We will also assume that $E$ splits over $k$, i.e. $f$ has three roots in $k$. Let $\delta \neq 0$ be the discriminant of $f$, and set

$$S = V^k_\infty \cup V^k(2) \cup V^k(\delta)$$

where $V^k$ denotes the set of all valuations of $k$, $V^k_\infty$ the subset of archimedean valuations, and for $a \in k^\times$ we set $V^k(a) = \{v \in V^k \setminus V^k_\infty \mid v(a) \neq 0\}$. Let

$$K := k(E) = k(x, y).$$

For a nonarchimedean $v \in V^k$, let $\tilde{v}$ denote its extension to $F := k(y)$ given by

$$\tilde{v}\big(a_m y^m + \cdots + a_0\big) = \min_{a_i \neq 0} v(a_i)$$

(cf. [1, Ch. VI, §10]). It can be shown that for $v \in V^k \setminus S$, the valuation $\tilde{v}$ has a unique extension to $K$, which we will denote by $w = w(v)$. We now introduce the following set of discrete valuations of $K$:

$$V = V_0 \cup V_1,$$

where $V_0$ is the set of all geometric places of $K$ (i.e., those discrete valuations that are trivial on $k$), and $V_1$ consists of the valuations $w = w(v)$ for all $v \in V^k \setminus S$. It is easy to see that $V$ satisfies conditions (A), (B) and (C).

**Theorem 10.** *The unramified Brauer group ${}_2\mathrm{Br}(K)_V$ is finite of order dividing*

$$2^{|S|-t} \cdot \big|{}_2\mathrm{Cl}_S(k)\big|^2 \cdot \big|U_S(k)/U_S(k)^2\big|^2,$$

*where $t = c + 1$ with $c$ being the number of complex places of $k$, and $\mathrm{Cl}_S(k)$ and $U_S(k)$ are the class group and the group of units of the ring of $S$-integers $\mathcal{O}_k(S)$, respectively.*

**Sketch of proof.** We will use the following description of the 2-torsion ${}_2\mathrm{Br}(K)_{V_0}$ in the geometric Brauer group [2]: *If $E$ splits over $k$, i.e. $f(x) = (x-a)(x-b)(x-c)$ with $a, b, c \in k$, then ${}_2\mathrm{Br}(K)_{V_0} = {}_2\mathrm{Br}(k) \oplus I$, where $I \subset {}_2\mathrm{Br}(K)_{V_0}$ is a subgroup such that every element of $I$ is represented by a bi-quaternion algebra $(r, x-b)_K \otimes_K (s, x-c)_K$ for some $r, s \in k^\times$.* Let $[D] \in {}_2\mathrm{Br}(K)_V$. Then $[D] = [\Delta' \otimes_K \Delta'']$ where $\Delta' = \Delta_0 \otimes_k K$ for some central division $k$-algebra $\Delta_0$ of exponent 2, and $\Delta'' = (r, x-b)_K \otimes_K (s, x-c)_K$ for some $r, s \in k^\times$. Using the corestriction map $\mathrm{cor}_{K/F}$, one shows that $\Delta_0$ is unramified at all $v \in V^k \setminus S$, and hence $\Delta''$ is unramified at all $w \in V_1$. The latter implies that $v(r), v(s) \equiv 0 \pmod 2$ for all $v \in V^k \setminus S$. Let

$$\tilde{\Gamma} = \big\{x \in k^\times \bigm| v(x) \equiv 0 \ (\mathrm{mod}\ 2) \text{ for all } v \in V^k \setminus S\big\},$$

and let $\Gamma$ be the image of $\tilde{\Gamma}$ in $k^\times/k^{\times 2}$. Then there is an exact sequence

$$0 \to U_S(k)/U_S(k)^2 \to \Gamma \to {}_2\mathrm{Cl}_S(k) \to 0$$

(cf. [11, §6.1]), hence $|\Gamma| = |{}_2\mathrm{Cl}_S(k)| \cdot |U_S(k)/U_S(k)^2|$.

Our previous discussion shows that there are at most $|\Gamma|^2$ possibilities for $\Delta''$. On the other hand, it follows from (ABHN) that ${}_2\mathrm{Br}(k)_{V^k \setminus S}$ has order $2^{|S|-t}$, which bounds the number of possibilities for $\Delta'$. Combining this with the above computation of $|\Gamma|$, we obtain our claim. $\square$

**Example.** Consider an elliptic curve $E$ over $\mathbb{Q}$ given by $y^2 = x^3 - x$. We have $\delta = 4$, so $S = \{\infty, 2\}$. Furthermore,

$$|S| - t = 1, \qquad \mathrm{Cl}_S(\mathbb{Q}) = 1 \quad \text{and} \quad U_S(\mathbb{Q}) = \{\pm 1\} \times \mathbb{Z}.$$

So, by Theorem 10, for $K = \mathbb{Q}(E)$ and the set $V$ constructed above, the group ${}_2\mathrm{Br}(K)_V$ has order dividing $2 \cdot 4^2 = 32$. Combining this with Theorem 6, we obtain that for any quaternion algebra $D$ over $K$, we have $|\mathbf{gen}(D)| \leqslant 32$.

## 4. Concluding remarks

The questions considered in this note for division algebras can be analyzed in the broader context of arbitrary absolutely almost simple simply connected (or adjoint) $K$-groups. In this set-up, one can define the genus of such a $K$-group $G$ as the collection of $K$-forms $G'$ of $G$ that have the same isomorphism classes of maximal $K$-tori (as a variation, one can base the notion only on *generic* tori). We note that questions about groups in the same genus arise in the analysis of weak commensurability of Zariski-dense subgroups which in turn is related to some problems in differential geometry, cf. [15]. In view of our Theorem 3, it seems natural to propose the following:

**Conjecture.** *Let $G$ be an absolutely almost simple simply connected algebraic group over a finitely generated field $K$ of characteristic zero (or of "good" characteristic relative to $G$). Then there exists a* finite *collection $G_1, \ldots, G_r$ of $K$-forms of $G$ such that if $H$ is a $K$-form of $G$ having the same isomorphism classes of maximal $K$-tori as $G$, then $H$ is $K$-isomorphic to one of the $G_i$'s.*

Our proof of Theorem 3 yields in fact a proof of this conjecture for inner forms of type $\mathsf{A}_\ell$.

**Theorem 11.** *Let $G$ be an absolutely almost simple simply connected algebraic group of inner type $\mathsf{A}_\ell$ over a finitely generated field $K$ whose characteristic is either zero or does not divide $\ell + 1$. Then the above conjecture is true for $G$.*

(In this regard, we note that if central simple $K$-algebras $A_1 = M_{\ell_1}(D_1)$ and $A_2 = M_{\ell_2}(D_2)$, where $D_1$ and $D_2$ are division algebras, have the same maximal étale $K$-subalgebras, then $\ell_1 = \ell_2$ and $D_1$ and $D_2$ have the same maximal separable subfields, cf. [16, Lemma 2.3].)

We plan to address the general case of the conjecture in our subsequent publications.

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