

ON THE NOTION OF GENUS FOR  
DIVISION ALGEBRAS AND ALGEBRAIC GROUPS

(joint work with V. Chernousov and I. Rapinchuk)

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AMITSUR SYMPOSIUM – June 24, 2020

- 1 Division algebras with the same maximal subfields
- 2 Genus of a division algebra
- 3 Genus of a simple algebraic group
- 4 “Killing” the genus

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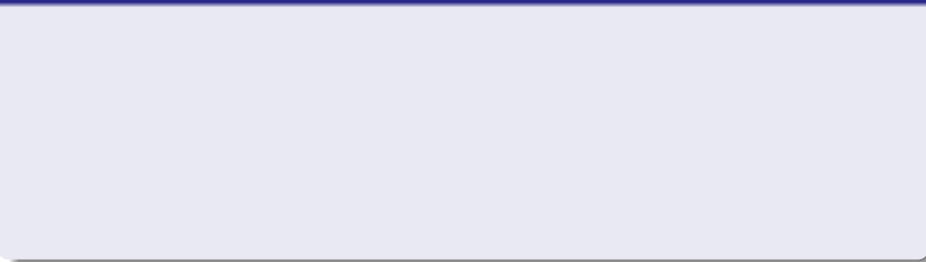
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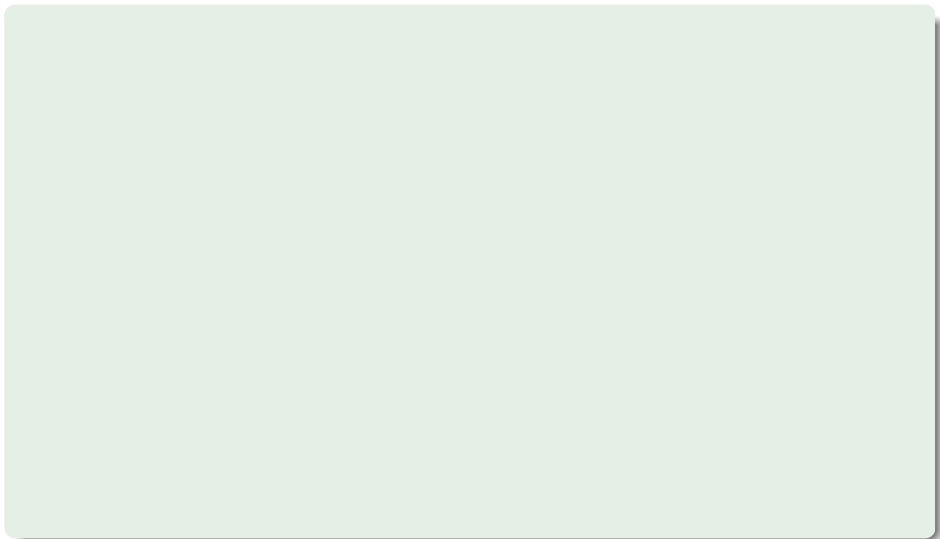
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**Recall:**  $A_\Gamma$  determines commensurability class of  $\Gamma$ .

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$$K_\Gamma = \mathbb{Q}(\text{tr } \gamma \mid \gamma \in \tilde{\Gamma}^{(2)})$$

(trace field).

• If  $\Gamma$  is an *arithmetic* Fuchsian group, **then**

(1)  $K_\Gamma$  is a number field, and

(2)  $A_\Gamma$  is the quaternion algebra involved in the description of  $\Gamma$ .

**Recall:**  $A_\Gamma$  determines commensurability class of  $\Gamma$ .

That is why we want to know  $A_\Gamma$  in this case!





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Assume that  $M_1$  and  $M_2$  are length-commensurable.



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**Theorem (A. Reid, 1992)**

*If two arithmetically defined Riemann surfaces are length-commensurable then they are commensurable.*



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### Theorem

*Let  $M_i = \mathbb{H}/\Gamma_i$  ( $i \in I$ ) be a family of length-commensurable Riemann surfaces where  $\Gamma_i \subset \mathrm{PSL}_2(\mathbb{R})$  is Zariski-dense. Then quaternion algebras  $A_{\Gamma_i}$  ( $i \in I$ ) split into **finitely many** isomorphism classes (over common center).*

- 1 Division algebras with the same maximal subfields
- 2 Genus of a division algebra
- 3 Genus of a simple algebraic group
- 4 “Killing” the genus

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Both facts follow from (AHBN).



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then using weak approximation one finds a quadratic extension  $L/\mathbb{Q}$  which embeds into one algebra but **not** into the other.

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*Let  $\text{char } k \neq 2$ . If  $|\mathbf{gen}(D)| = 1$  for any quaternion algebra  $D$  over  $k$ , then  $|\mathbf{gen}(D')| = 1$  for any quaternion algebra  $D'$  over  $k(x)$ .*

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Then (2) is obvious, and (1) follows from the fact that

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**Note** that  $\mathcal{K}$  is **infinitely generated**.

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Instead of analyzing kernel of global-to-local map (as in (ABHN)),

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In fact, in this case one can give an estimate on size of  $\mathbf{gen}(D)$  that depends on size of  ${}_n\mathrm{Br}(K)_V$  for a fixed  $V$ :

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It is based on finiteness of certain subgroups of  ${}_2\mathrm{Br}(K)_V$ .



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- One can construct examples where  ${}_2\mathrm{Br}(K)_V$  is “large.”

- 1 Division algebras with the same maximal subfields
- 2 Genus of a division algebra
- 3 Genus of a simple algebraic group
- 4 “Killing” the genus

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 $G_1$  &  $G_2$  have *same isomorphism classes of maximal  $K$ -tori* if every maximal  $K$ -torus  $T_1$  of  $G_1$  is  $K$ -isomorphic to a maximal  $K$ -torus  $T_2$  of  $G_2$ , and vice versa.

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$\mathbf{gen}_K(G)$  = set of isomorphism classes of  $K$ -forms  $G'$  of  $G$  having same  $K$ -isomorphism classes of maximal  $K$ -tori.



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(2) If  $G$  is an absolutely almost simple group over a finitely generated field  $K$  of “good” characteristic then  $\mathbf{gen}_K(G)$  is finite.

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Then special fiber (reduction)

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is a connected simple group of same type as  $G$ .



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A. R., I. Rapinchuk, *Linear algebraic groups with good reduction*,  
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**Variations:** one can consider only groups of a specific type, only inner forms, impose restrictions on characteristic etc.





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One expects that divisorial sets are as required.

# Finiteness Conjecture

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This conjecture can be viewed as an analog of Shafarevich’s Conjecture for abelian varieties proved by Faltings, but presents a different set of challenges.



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These and other connections shift focus of current work to  
Finiteness Conjecture.



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Finiteness of  $\mathbf{gen}_K(G)$  is derived from results on Finiteness Conjecture





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Results for  $G = \mathrm{SL}_{1,A}$  are based on (known) finiteness of

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**However**, proving finiteness of unramified cohomology groups  $H^i(K, \mu_n^{\otimes j})_V$  is a very difficult problem for  $i \geq 3$  resolved only in special cases!

At the same time, a precise description of  $K$ -forms in terms of *commutative Galois cohomology* is available only for certain types.



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So, to handle general case in Finiteness Conjecture one will need to develop an *intrinsic approach* to analysis of forms with good reduction.

- 1 Division algebras with the same maximal subfields
- 2 Genus of a division algebra
- 3 Genus of a simple algebraic group
- 4 "Killing" the genus

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So, there appears to be **no** regularity for *finite* extensions of base field.

**Question.** *What happens to genus under base change?*

Using (ABHN), one can construct a *cubic* division algebra  $D$  over  $\mathbb{Q}$  and finite extensions  $F_1$  and  $F_2$  of  $\mathbb{Q}$  such that

$$|\mathbf{gen}(D \otimes_K F_1)| < |\mathbf{gen}(D)| < |\mathbf{gen}(D \otimes_K F_2)|.$$

Similar examples can be constructed for algebraic groups.

So, there appears to be **no** regularity for *finite* extensions of base field.

**However,** for *purely transcendental* extensions we have:

## Theorem 9



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In particular, the latter is finite if  $K$  is a number field.



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consists of  $D' \otimes_K K(x_1, \dots, x_{n-1})$  where  $\langle [D'] \rangle = \langle [D] \rangle$  in  $\mathrm{Br}(K)$ .



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*Can one extend this phenomenon to all simple algebraic groups?*



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General case remains open.