On the notion of genus for division algebras and algebraic groups

(joint work with V. Chernousov and I. Rapinchuk)

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Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)$ with $\varepsilon_i = \pm 1$ and $\sum_{i=1}^{r} \varepsilon_i \equiv 0 (\mathrm{mod}\, 3)$.

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One shows that

1. For any $\varepsilon', \varepsilon''$ as above, $D(\varepsilon')$ and $D(\varepsilon'')$ have same finite-dimensional splitting fields,
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Thus, assertion of Amitsur's Theorem cannot be proven for two division algebra sharing only finite-dimensional splitting fields. **So**, Amitsur found the right way of proving his theorem by using infinite-dimensional extensions.

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That is why we want to know A_Γ in this case!

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Here is how A_Γ is linked with geometry of $M = \mathbb{H}/\Gamma$:

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Theorem (A. Reid, 1992)

If two arithmetically defined Riemann surfaces are length-commensurable then they are commensurable.

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Theorem

Let $M_i = \mathbb{H}/\Gamma_i$ $(i \in I)$ *be a family of* length-commensurable *Riemann surfaces where* $\Gamma_i \subset \mathrm{PSL}_2(\mathbb{R})$ *is Zariski-dense. Then quaternion algebras* A_{Γ_i} $(i \in I)$ *split into finitely many isomorphism classes (over common center).*

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Theorem 1 (Stability Theorem)

Let $\operatorname{char} k \neq 2$*. If* $|\mathbf{gen}(D)| = 1$ *for any quaternion algebra* D *over* k*, then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$*.*

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Then (2) is obvious, and (1) follows from the fact that

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remains anisotropic over K_1.

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A central simple algebra K-algebra A is unramified at a discrete valuation v of K if there exists an *Azumaya algebra* $\mathcal{A}$ over valuation ring $\mathcal{O}_v \subset K_v$ such that

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$$\mathrm{Br}(K)_V = \{ x \in \mathrm{Br}(K) \mid x \text{ is unramified at all } v \in V \}.$$

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In fact, in this case one can give an estimate on size of $\mathbf{gen}(D)$ that depends on seize of ${}_n\mathrm{Br}(K)_V$ for a fixed V:

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It is based on finiteness of certain subgroups of ${}_2\mathrm{Br}(K)_V$.

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- The answer is **not** known for *any* finitely generated K.
- One can construct examples where ${}_2\mathrm{Br}(K)_V$ is "large."

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  $\mathbf{gen}_K(G)$ = set of isomorphism classes of K-forms G' of G having same K-isomorphism classes of maximal K-tori.

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Question 1′. *When does* $\mathbf{gen}_K(G)$ *reduce to a single element?*

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Theorem 4 (Prasad-A.R.)

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Then special fiber (reduction)

$$\underline{G}^{(v)} = \mathcal{G} \otimes_{\mathcal{O}_v} K^{(v)}$$

is a connected simple group of same type as G.

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A. R., I. Rapinchuk, *Linear algebraic groups with good reduction*, arXiv: 2005.05484

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Variations: one can consider only groups of a specific type, only inner forms, impose restrictions on characteristic etc.

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One expects that divisorial sets are as required.

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These and other connections shift focus of current work to Finiteness Conjecture.

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Finiteness of $\mathbf{gen}_K(G)$ is derived from results on Finiteness Conjecture

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However, proving finiteness of unramified cohomology groups $H^i(K, \mu_n^{\otimes j})_V$ is a very difficult problem for $i \geqslant 3$ resolved only is special cases!

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So, to handle general case in Finiteness Conjecture one will need to develop an *intrinsic approach* to analysis of forms with good reduction.

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However, for *purely transcendental* extensions we have:

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In particular, the latter is finite if K is a number field.

Both Theorems and Corollary remain valid over $K(x_1, \ldots, x_r)$.

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Can one extend this phenomenon to all simple algebraic groups?

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General case remains open.