Modeling Noisy Data with Differential Equations Using Observed and Expected Matrices

Article *in* Psychometrika · September 2010

DOI: 10.1007/s11336-010-9168-2

2 authors:

Pascal R Deboeck
University of Utah

Steven Boker
University of Virginia

PSYCHOMETRIKA—VOL. 75, NO. 3, 420–437
SEPTEMBER 2010
DOI: 10.1007/S11336-010-9168-2

MODELING NOISY DATA WITH DIFFERENTIAL EQUATIONS USING OBSERVED AND EXPECTED MATRICES

PASCAL R. DEBOECK

UNIVERSITY OF KANSAS

STEVEN M. BOKER

UNIVERSITY OF VIRGINIA

Complex intraindividual variability observed in psychology may be well described using differential equations. It is difficult, however, to apply differential equation models in psychological contexts, as time series are frequently short, poorly sampled, and have large proportions of measurement and dynamic error. Furthermore, current methods for differential equation modeling usually consider data that are atypical of many psychological applications. Using embedded and observed data matrices, a statistical approach to differential equation modeling is presented. This approach appears robust to many characteristics common to psychological time series.

Key words: intraindividual variability, differential equation model(s)(ing), time series, damped linear oscillator, analytic solution(s).

The analysis of time series is critical to answering questions about how, why, and when individuals change on psychological variables (Nesselroade & Ram, 2004). Classical ergodic theorems suggest that it is unlikely that models based on cross-sectional data and those based on intraindividual data will lead to concurrent conclusions (Molenaar, 2004). Modeling of intraindividual variability requires advancement beyond methods commonly used to examine intraindividual change, as methods designed to analyze trends may average over variability of interest (Boker & Nesselroade, 2002); for example, (latent) growth curve modeling and hierarchical linear modeling.

Differential equation modeling is one promising methodology for modeling of intraindividual variability, as it can be used to model person-specific trajectories. Common characteristics of many psychological time series make unbiased parameter estimation problematic; such characteristics include: short time series (<100 observations), large proportions of measurement error ($>20\%$), low sampling rates (where smoothing may obscure change of interest), significant sources of dynamic error (e.g., daily perturbations to the current state of a system), observations that are unequally spaced in time (e.g., diary data or occasional missing data), and univariate time series. We present a method that can produce unbiased differential equation model estimates over a wide range of such conditions by fitting localized estimates solutions or approximate solutions to a time series. This is done by modifying a common statistical approach where parameters are estimated by minimizing the differences between an observed matrix and an expected matrix. Focus is placed on the damped linear oscillator model, due to its potential uses in psychology as a first-order approximation of many self-regulating systems, although the principles presented should be applicable to any differential equation model.

Requests for reprints should be sent to Pascal R. Deboeck, Department of Psychology, University of Kansas, Lawrence, KS, USA. E-mail: pascal@ku.edu

Differential Equation Modeling

Differential equation models express the relationship(s) between the current state of a system and its derivatives. Modeling these relationships, rather than the observed trajectories, provide several advantages including: expression of complex trajectories using linear equations with few parameters, parameters with meaningful interpretations, parameter estimates that do not depend on the selection of initial time ($t = 0$), and vastly differing trajectories for differing individuals described by a common dynamical system. Parameter estimation of a differential equation model is often approached by either (1) estimating the parameters directly from a time series using a model (e.g., Ramsay & Silverman, 2005), or (2) estimating the derivatives of a time series which are then used to estimate the model parameters (e.g., Boker, Neale & Rausch, 2004). The first approach works well when variables that change over time are not expected to violate the dynamics of the system described by a set of differential equation models; examples are abundant in the physical sciences, where one would not expect a physical object to display a discontinuous change in position. Such random displacements, often called *dynamic error*, seem plausible for many psychological variables due to the occurrence of random, external life events. The second approach is more appropriate for many psychological applications, as it allows for modeling of the *state space*—that is, the instantaneous relationships between observed values and derivatives—without assuming a specific trajectory (i.e., a lack of dynamic error). Unfortunately, estimation error variance rapidly increases as higher order derivatives are estimated (Ramsay & Silverman, 2005). A common approach is to smooth data prior to derivative estimation, but this implicitly assumes sampling rates which are uncharacteristic high for many psychological applications.

As most psychological time series include both dynamic and measurement error, an ideal method would work well with both types of error. We propose that differential equation models can be fit to observed data while (1) maintaining the advantages of derivative-estimation when modeling systems with large proportions of dynamic error, and (2) avoiding problems associated with the estimation of individual derivatives due to measurement error. Paralleling common statistical approaches, one could minimize the difference between an observed time series and the expected values for a time series, based on one's theoretical model. To differentiate this description from the first general approach previously described (which does not work as well with dynamic error) to allow for dynamic error, any parameters of the expected matrix that are dependent on time must be estimated for every observation in time; that is, one would need to estimate a set of observation-specific parameters, in addition to the model parameters. The estimation of both sets of parameters can be accomplished using a time delay embedded matrix (Takens, 1981), thus allowing for consideration of data with both significant amounts of measurement and dynamic error.

Damped Linear Oscillator Model

The damped linear oscillator model can be used to describe changes between extreme values around some equilibrium or "typical" value. As many psychological constructs may be self-regulating, the damped linear oscillator may be a reasonable first approximation of such systems. To model an observed time series $\mathbf{x} = \{x_1, x_2, \ldots, x_t\}$ as a damped linear oscillator, we express the expected value of each observation. One solution for the differential equation model of a damped linear oscillator,

$$\frac{d^2x}{dt^2} = -\omega^2 x + \zeta \frac{dx}{dt}, \tag{1}$$

can be written

$$x_t = A_0 \exp^{\zeta t/2} \cos(\omega' t + \delta),$$
$$\omega' = \sqrt{\omega^2 - \zeta^2/4}, \tag{2}$$

where x_t is the value of some variable at time t, ω describes the frequency of oscillation, ζ is related to the damping or amplification of the observations, A_0 is the amplitude at $t = 0$, and δ is the phase of oscillation at $t = 0$. It should be noted that only the real part of ω' is utilized in calculating x_t; systems where $\omega^2 - \zeta^2/4 < 0$ correspond to systems that return or depart from equilibrium without oscillations occurring. This paper will assume that ω and ζ are constant within a time series (model parameters), but that A_0 and δ are dependent on the time of observation due to dynamic error (observation-specific parameters).

It is assumed that systems being modeled oscillate. This is apparent in Equation (1), as the frequency of oscillation is related to the coefficient $-\omega^2$. Other papers will represent this coefficient using η (e.g., Boker & Nesselroade, 2002). Using $-\omega^2$ will constrain the solutions to lie within the range of systems that may exhibit oscillation; η must be less than zero before oscillation is possible. While not examined here, models that allow for positive η values would allow for systems where individuals could be repelled from, rather than attracted to, their equilibrium.

Observed and Expected Values

For this model, two observation-specific parameters are estimated for each observation in addition to the two model parameters. This is accomplished by using localized vectors (e.g., x_{t-2} to x_{t+2}); these localized vectors are the same conceptually as the rows of a time delay embedded matrix, which is one approach for recreating a state space (Takens, 1981). Embedding matrices have been used previously in psychology (e.g., Boker, Neale & Rausch, 2004), but such studies have initially estimated the derivatives of a time series; this can result in correlations between derivative estimates, which in turn can bias model parameter estimates.

An embedded matrix is created in the same manner as prescribed by other research (Boker et al. 2004; Takens, 1981). Given the time series $\mathbf{x}$, where $\mathbf{x} = x_1, x_2, \ldots, x_T$, a five-dimensional embedding $\mathbf{X}$ is created using five replicates of the time series, offset in time by a selected number of observations. This embedded matrix would have the form:

$$\mathbf{X}_{\text{observed}} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_2 & x_3 & x_4 & x_5 & x_6 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{T-4} & x_{T-3} & x_{T-2} & x_{T-1} & x_T \end{bmatrix}. \tag{3}$$

The number of embeddings will depend on the number of parameters being estimated, but should be close to the minimum required for identification.

The observed matrix will be compared to an expected matrix. Given Equation (2), the first row of $\mathbf{X}_{\text{expected}}$ is equal to

$$\mathbf{X}_{\text{expected}}[1,] = \begin{bmatrix} A_0 \exp^{\zeta(-2t_\Delta)/2} \cos(-2\omega' t_\Delta + \delta) \\ A_0 \exp^{\zeta(-1t_\Delta)/2} \cos(-\omega' t_\Delta + \delta) \\ A_0 \exp^{0} \cos(\delta) \\ A_0 \exp^{\zeta(1t_\Delta)/2} \cos(\omega' t_\Delta + \delta) \\ A_0 \exp^{\zeta(2t_\Delta)/2} \cos(2\omega' t_\Delta + \delta) \end{bmatrix}^T,$$
$$\omega' = \sqrt{\omega^2 - \zeta^2/4} \tag{4}$$

assuming $t = 0$ for the middle observation, and that observations are equally spaced t_{Δ} units apart. The observation-specific parameters A_0 and δ are estimated by constraining these parameters to be constant within a single row of $\mathbf{X}_{\text{expected}}$, but not across rows. The model parameters ω and ζ are constrained to be constant across all rows. Changes of $\mathbf{X}_{\text{expected}}$ can be made to accommodate nonconstant ω and ζ parameters, as well as data that are not equally spaced in time.

Given $\mathbf{X}_{\text{observed}}$ and $\mathbf{X}_{\text{expected}}$, and a function to be minimized or maximized, estimated parameters can be produced that allow simultaneously for both dynamic and measurement error. The minimization of the differences between the observed and expected matrices for differential equations parallels methods for the estimation of complex nonlinear problems and, therefore, inherits similar advantages and disadvantages, for example, the presence of local minima (Ramsay et al., 2007; Esposito & Floudas, 2000). It should also be noted that we have to create $\mathbf{X}_{\text{expected}}$ based on an analytic solution for the damped linear oscillator. It is very probable that approximate analytic solutions and numerical methods such as Runge–Kutta fourth order integration could be used for generating $\mathbf{X}_{\text{expected}}$.

1. Simulated Example

The simulation that follows examines the robustness of the proposed method when estimating the ω and ζ parameters of a damped linear oscillator model with time series that may have characteristics common to much of psychology. The simulated, univariate time series were generated with a variety of less-than-ideal conditions, including: few observations, large proportions of measurement and dynamic error, low sampling rates, and unequally spaced observations.

1.1. Methods

1.1.1. Simulated Data Generation Univariate time series were generated using Mathematica (Mathematica, 2005) and the differential equation for a damped linear oscillator (Equation (1)). Time series of 500 observations were generated for each possible ω, from which initial pairs of x and $\frac{dx}{dt}$ were randomly selected, thus quasi-randomly selecting the initial phase of each time series. Runge–Kutta fourth order numerical integration of the differential equation for a damped linear oscillator was then used to generate each of the time series. Five values were selected for the ω parameter[1]: $\sqrt{8}$, $\sqrt{2}$, $\sqrt{0.5}$, $\sqrt{0.125}$, $\sqrt{0.03125}$. Five values were selected for the ζ parameter: -0.10, -0.05, 0.00, 0.05, 0.10.

Independent, normally-distributed observations (i.e., measurement error) were added to each time series such that the ratio of the variance of the measurement error to the variance of the signal was either 1:1 or 1:4. Two different lengths of time series were examined: 25 and 50 observations. Time series were generated with both equally and unequally spaced observations. For cases with equally spaced observations, the forward integration procedure was advanced one time unit between observations. For cases with unequally spaced observations, the forward integration procedure was advanced an amount of time randomly drawn from a normal distribution with mean of 1 and standard deviation of 0.15. All conditions, including three dynamic error conditions, were crossed for a total of 1000 conditions. Five-hundred time series were generated for each condition.

The first dynamic error condition consisted of true damped linear oscillators, that is, there was no dynamic error. In the second condition, as each observation was integrated, a number

[1] As the time between observations was selected to be 1, in a true damped linear oscillator these ω values correspond approximately to 2.2, 3.1, 8.9, 17.8 and 35.5 observations measured per cycle, assuming equally spacing of observations and $\zeta = 0$. Note that some of the sampling rates are so low that they approach the Nyquist limit.

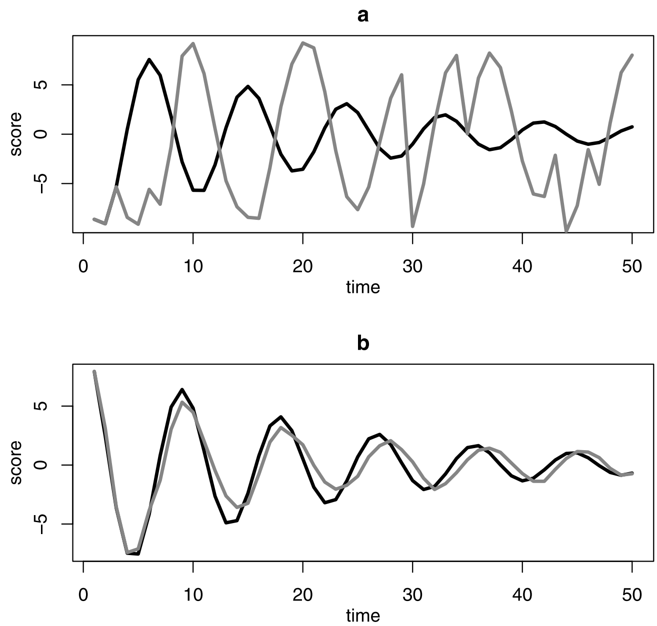

FIGURE 1.
Examples of time series generated with different types of dynamic error (grey line) relative to time series with no dynamic error (black line). In the top graph (a), both phase and amplitude have a 20% chance of reseting at any particular observation. In the bottom figure (b), small perturbations occur at every observation, leading to changes in amplitude and frequency over time.

was randomly drawn from a uniform distribution bounded by 0 and 1. When this value exceeded a criterion of 0.2, the values of x and $\frac{dx}{dt}$ were randomly reselected from the time series originally used to randomize the initial conditions. In the third condition, the observed values used to calculate a subsequent observation was multiplied by a random number from a normal distribution with a mean of 1 and a variance of 0.2, prior to forward integration of the next observation. Figures 1a and 1b show examples of the second and third types of dynamic error.

The first dynamic error condition served as a baseline to ensure that unbiased estimates could be produced with the proposed method. The second type of dynamic error produced sharp, sudden changes. The implementation of this dynamic error will result in the phase being randomly perturbed and the maximum amplitude returned to the initial maximum amplitude occasionally throughout the time series. The third type of dynamic error produced small, constantly occurring changes. By multiplying previous values by a randomly selected value, rather than adding, observations closer to equilibrium (zero) are perturbed less than observations at the extremes.

1.1.2. Analysis The matrices, $\mathbf{X}_{\text{observed}}$ and $\mathbf{X}_{\text{expected}}$, were generated using five embedding dimensions and Equation 2. Minimization of the sum of squared errors was performed using the Broyden, Fletcher, Goldarb, and Shanno method (BFGS; Broyden, 1970; Fletcher, 1970; Goldfarb, 1970; Shanno, 1970) with the function `optim()` in R (2007). The optimization was repeated 30 times for each time series with differing starting values. The amplitude A and phase δ were estimated for each row; ω and ζ were then fixed across all rows. Estimates of ω were transformed into equivalent values bounded by 0 and π; this requires the assumption that the sampling rate is greater than the Nyquist limit (Shannon, 1948). An R script for this analysis is presented in Appendix.

TABLE 1.
Mean parameter estimates, equally spaced intervals and signal to noise ratio of 1:1.

True Damped Linear Oscillator										
$\omega =$	ω Estimates					ζ Estimates				
	2.83	1.41	0.71	0.35	0.18	2.83	1.41	0.71	0.35	0.18
$\zeta = -0.10$	2.85	1.42	0.71	0.34	0.14	−0.115	−0.106	−0.104	−0.126	−0.146
$\zeta = -0.05$	2.85	1.41	0.71	0.35	0.15	−0.066	−0.056	−0.058	−0.058	−0.096
$\zeta = 0.00$	2.84	1.42	0.72	0.35	0.16	0.002	0.000	−0.001	−0.001	−0.011
$\zeta = 0.05$	2.85	1.41	0.72	0.35	0.16	0.056	0.053	0.055	0.056	0.065
$\zeta = 0.10$	2.84	1.42	0.71	0.35	0.16	0.129	0.109	0.114	0.120	0.164
Damped Linear Oscillator with Phase Resets ($p(\text{reset}) = 0.2$)										
$\omega =$	ω Estimates					ζ Estimates				
	2.83	1.41	0.71	0.35	0.18	2.83	1.41	0.71	0.35	0.18
$\zeta = -0.10$	2.73	1.43	0.70	0.42	0.33	−0.006	−0.004	−0.007	0.000	−0.017
$\zeta = -0.05$	2.76	1.44	0.71	0.41	0.30	−0.005	−0.008	−0.008	−0.004	−0.008
$\zeta = 0.00$	2.76	1.41	0.74	0.42	0.31	0.002	0.004	0.003	−0.007	0.005
$\zeta = 0.05$	2.78	1.42	0.69	0.41	0.31	0.002	0.005	0.002	0.009	0.001
$\zeta = 0.10$	2.77	1.40	0.69	0.41	0.31	0.008	0.010	0.014	0.002	0.018
Damped Linear Oscillator with Small Perturbations (perturbations $\sim N(1, .2)$)										
$\omega =$	ω Estimates					ζ Estimates				
	2.83	1.41	0.71	0.35	0.18	2.83	1.41	0.71	0.35	0.18
$\zeta = -0.10$	2.83	1.42	0.72	0.35	0.16	−0.134	−0.120	−0.114	−0.130	−0.192
$\zeta = -0.05$	2.84	1.42	0.72	0.36	0.17	−0.079	−0.073	−0.064	−0.074	−0.087
$\zeta = 0.00$	2.84	1.42	0.72	0.35	0.17	−0.015	−0.009	−0.010	−0.017	−0.020
$\zeta = 0.05$	2.83	1.42	0.72	0.35	0.17	0.060	0.045	0.044	0.059	0.073
$\zeta = 0.10$	2.85	1.42	0.71	0.37	0.17	0.125	0.098	0.099	0.131	0.169

Notes: Mean parameter estimates based on time series of 50 observations. Results for time series consisting of 25 observations were similar.

1.2. Results

All summary statistics presented consist of the calculation of the statistic within the 500 time series created for a specific combination of conditions, unless otherwise noted. Despite the large number of estimated parameters, the optimization method converged on a solution for every single time series. Tables 1 and 2 present the mean estimated values of ω and ζ for time series with equally spaced and unequally spaced observations. Within each combination of conditions, similar results were observed for equally and unequally spaced time series.

With a true damped linear oscillator, the means of the estimated ω and ζ values closely correspond to the values used to generate the data. With phase and amplitude resetting, there are conditions where the expected values of the parameters may deviate from the values used to generate the data. This is apparent with the estimates of ζ which consistently produced estimates near zero, and with the estimates for the two lowest values of ω. The results for the model with small amplitude perturbations suggest that the expected values of the parameters are similar to the values used to generate the damped linear oscillator time series. While small perturbations can cause long-term changes in phase and amplitude, within any small window of time the observed values closely correspond to the model used to generate data.

Tables 3 and 4 present the 95% confidence intervals of the ω and ζ estimates, as well as the mean of the squared errors of the estimates around the true parameter values (MSE). The true

TABLE 2.
Mean parameter estimates, unequally spaced intervals and signal to noise ratio of 1:1.

True Damped Linear Oscillator										
$\omega =$	ω Estimates					ζ Estimates				
	2.83	1.41	0.71	0.35	0.18	2.83	1.41	0.71	0.35	0.18
$\zeta = -0.10$	2.82	1.42	0.74	0.41	0.32	−0.125	−0.111	−0.126	−0.126	−0.115
$\zeta = -0.05$	2.84	1.42	0.74	0.39	0.26	−0.057	−0.057	−0.049	−0.060	−0.061
$\zeta = 0.00$	2.84	1.41	0.73	0.37	0.24	−0.001	0.003	−0.003	0.002	−0.011
$\zeta = 0.05$	2.84	1.42	0.73	0.40	0.24	0.028	0.047	0.054	0.053	0.062
$\zeta = 0.10$	2.79	1.42	0.74	0.38	0.31	0.120	0.108	0.109	0.118	0.159
Damped Linear Oscillator with Phase Resets (p(reset) = 0.2)										
$\omega =$	ω Estimates					ζ Estimates				
	2.83	1.41	0.71	0.35	0.18	2.83	1.41	0.71	0.35	0.18
$\zeta = -0.10$	2.66	1.46	0.82	0.56	0.43	0.001	−0.021	−0.015	−0.015	0.001
$\zeta = -0.05$	2.65	1.46	0.78	0.52	0.42	0.004	−0.012	−0.015	0.002	−0.014
$\zeta = 0.00$	2.71	1.51	0.76	0.49	0.45	0.000	−0.005	0.010	0.005	0.013
$\zeta = 0.05$	2.72	1.47	0.75	0.52	0.46	−0.001	0.006	0.012	0.013	0.006
$\zeta = 0.10$	2.70	1.47	0.78	0.49	0.42	0.027	0.013	0.022	0.024	0.024
Damped Linear Oscillator with Small Perturbations (perturbations $\sim N(1, .2)$)										
$\omega =$	ω Estimates					ζ Estimates				
	2.83	1.41	0.71	0.35	0.18	2.83	1.41	0.71	0.35	0.18
$\zeta = -0.10$	2.78	1.43	0.77	0.44	0.33	−0.134	−0.120	−0.123	−0.128	−0.157
$\zeta = -0.05$	2.82	1.42	0.74	0.41	0.29	−0.073	−0.066	−0.075	−0.089	−0.086
$\zeta = 0.00$	2.82	1.43	0.74	0.40	0.31	−0.021	−0.012	−0.013	−0.017	−0.022
$\zeta = 0.05$	2.81	1.41	0.73	0.39	0.28	0.058	0.046	0.043	0.042	0.047
$\zeta = 0.10$	2.78	1.40	0.73	0.40	0.32	0.125	0.100	0.097	0.113	0.120

Notes: Mean parameter estimates based on time series of 50 observations. Results for time series consisting of 25 observations were similar with one exception—the 0.18 column for the ω estimates, for the small perturbations condition. With 25 observations the mean ω were about 10% higher, suggesting that combining very high sampling rates with measurement of very short time series and very unequally spaced intervals may lead to slightly biased estimates.

damped linear oscillators tended to produce more efficient estimates than time series with dynamic error; however, in many cases the difference in MSE was small. The estimates using time series with equally spaced observations were more efficient than those measured without equally spaced observations; the reduction in MSE with equally spaced observations was pronounced for estimates of frequency. While confidence intervals are large, particularly for the estimates of ζ, the confidence intervals narrow substantially if one considers data with some positive characteristics. For example, Table 5 shows the 95% confidence interval and the MSE of the estimates based on data with a signal to noise ratio of 4:1; this would correspond to a test–retest reliability of approximately 0.80.

1.2.1. Discussion The simulations examined an extreme set of conditions in order to demonstrate the robustness of the proposed method; these extreme conditions were selected as it is difficult at this time to know what realistic error conditions would be, given a lack of empirical research in many areas of psychology. The mean estimated values of the parameters were approximately equal to those used to generate the data, over most conditions. The exceptions

TABLE 3.
Confidence intervals & MSE, equally spaced intervals and signal to noise ratio of 1:1.

True Damped Linear Oscillator							
ω Estimates				ζ Estimates			
True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE	True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE
2.828	3.096	2.712	0.011	−0.100	0.207	−0.629	0.038
1.414	1.538	1.299	0.006	−0.050	0.222	−0.499	0.028
0.707	0.853	0.594	0.006	0.000	0.289	−0.319	0.017
0.354	0.468	0.121	0.009	0.050	0.423	−0.269	0.021
0.177	0.336	0.031	0.013	0.100	0.628	−0.177	0.038
Damped Linear Oscillator with Phase Resets (p(reset) $= 0.2$)							
ω Estimates				ζ Estimates			
True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE	True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE
2.828	3.107	2.436	0.049	−0.100	0.375	−0.403	0.052
1.414	1.893	0.892	0.076	−0.050	0.398	−0.421	0.035
0.707	1.196	0.371	0.056	0.000	0.360	−0.337	0.036
0.354	0.768	0.051	0.041	0.050	0.340	−0.345	0.031
0.177	0.609	0.038	0.056	0.100	0.327	−0.292	0.032
Damped Linear Oscillator with Small Perturbations (perturbations $\sim N(1, .2)$)							
ω Estimates				ζ Estimates			
True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE	True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE
2.828	3.099	2.678	0.013	−0.100	0.152	−0.731	0.050
1.414	1.568	1.291	0.006	−0.050	0.223	−0.537	0.031
0.707	0.878	0.588	0.008	0.000	0.294	−0.369	0.025
0.354	0.499	0.061	0.016	0.050	0.484	−0.253	0.029
0.177	0.368	0.031	0.014	0.100	0.671	−0.204	0.049

Notes: Estimates for ω average over all values of ζ, and estimates for ζ average over all ω values. Estimates are based on time series of 50 observations with equally spaced observations.

occurred when there was a large amount of phase and amplitude resetting. Frequent phase resetting, combined with high sampling rates, tended to produce frequency estimates that reflected higher frequencies than the true frequency; this effect was not apparent with lower sampling rates. Frequent amplitude resetting, on the other hand, produces estimates of the damping parameter approximately equal to zero—reflecting a system where the average amplitude was not changing over time. The biased results for frequent amplitude and phase resetting are what would be expected given the mismatch between the model used to generate the expected values and the true model. The current method's use of embedded matrices to examine the state space has allowed for consideration of systems with significant amounts of measurement and dynamic error, sampling rates close to the Nyquist limit, time series with relatively few observations and both equally and unequally spaced observations. These advantages suggest that the presented method is likely to be useful and robust in a very wide range of potential applications.

The simulations suggest a few considerations for applied researchers. (1) In general, it is not necessary to record every possible perturbation to a construct of interest in order to recover the parameters of a system. (2) It is not necessary to intensively sample constructs, as long as the fundamental dynamics are being appropriately sampled. (3) If there is frequent amplitude resetting, to accurately recover small periods of damping over time a model will be required that allows

TABLE 4.
Confidence intervals & MSE, unequally spaced intervals and signal to noise ratio of 1:1.

True Damped Linear Oscillator							
ω Estimates				ζ Estimates			
True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE	True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE
2.828	3.108	2.529	0.060	−0.100	0.239	−0.589	0.049
1.414	1.621	1.200	0.033	−0.050	0.280	−0.447	0.027
0.707	1.158	0.490	0.068	0.000	0.326	−0.343	0.022
0.354	1.073	0.099	0.093	0.050	0.436	−0.300	0.028
0.177	1.262	0.047	0.130	0.100	0.602	−0.221	0.039
Damped Linear Oscillator with Phase Resets (p(reset) $= 0.2$)							
ω Estimates				ζ Estimates			
True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE	True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE
2.828	3.122	0.907	0.250	−0.100	0.420	−0.485	0.063
1.414	2.886	0.334	0.250	−0.050	0.414	−0.455	0.048
0.707	2.593	0.134	0.243	0.000	0.422	−0.405	0.044
0.354	2.247	0.079	0.250	0.050	0.398	−0.390	0.038
0.177	2.122	0.065	0.268	0.100	0.417	−0.350	0.044
Damped Linear Oscillator with Small Perturbations (perturbations $\sim N(1, .2)$)							
ω Estimates				ζ Estimates			
True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE	True value	97.5%$^{\text{tile}}$	2.5%$^{\text{tile}}$	MSE
2.828	3.107	2.201	0.094	−0.100	0.257	−0.713	0.060
1.414	1.984	0.886	0.066	−0.050	0.263	−0.519	0.042
0.707	1.364	0.389	0.070	0.000	0.336	−0.392	0.028
0.354	1.324	0.086	0.103	0.050	0.399	−0.287	0.028
0.177	1.559	0.049	0.170	0.100	0.594	−0.242	0.045

Notes: Estimates for ω average over all values of ζ, and estimates for ζ average over all ω values. Estimates are based on time series of 50 observations with unequally spaced observations.

for variations in the damping parameter; this model would represent a system that differs significantly from a damped linear oscillator. Expanding the present example to allow model-specific parameters to change with some polynomial scheme is relatively straightforward to implement; such a change would allow relatively simple models such as the damped linear oscillator to be used for relatively complicated first-order approximations of intraindividual change. (4) To produce more efficient estimates low measurement error and longer time series are obviously advantageous, but equally spaced observations can also convey an advantage. (5) Researchers interested in the ζ parameter would be well advised to try to collect data without too many poor characteristics, as otherwise estimates will be very inefficient and subsequently power to test differences in ζ will be dramatically reduced. (6) The examples presented consider the examination of individual time series, but this is not required as the individual rows of $\mathbf{X}_{\text{observed}}$ could correspond to unique individuals.

2. Applied Example

This example considers the modeling of the motions of a dyad as they dance to simple rhythms. As a person dances, the rhythmic back-and-forth movements he/she produces might

TABLE 5.
Confidence intervals & MSE, equally spaced intervals and signal to noise ratio of 4:1.

True Damped Linear Oscillator							
ω Estimates				ζ Estimates			
True value	$97.5\%^{\text{tile}}$	$2.5\%^{\text{tile}}$	MSE	True value	$97.5\%^{\text{tile}}$	$2.5\%^{\text{tile}}$	MSE
2.828	2.874	2.783	0.001	−0.100	−0.006	−0.257	0.004
1.414	1.450	1.381	< 0.0005	−0.050	0.028	−0.15	0.002
0.707	0.739	0.674	< 0.0005	0.000	0.067	−0.076	0.002
0.354	0.393	0.311	< 0.0005	0.050	0.140	−0.023	0.002
0.177	0.241	0.043	0.002	0.100	0.217	0.012	0.003

Damped Linear Oscillator with Phase Resets ($p(\text{reset}) = 0.2$)							
ω Estimates				ζ Estimates			
True value	$97.5\%^{\text{tile}}$	$2.5\%^{\text{tile}}$	MSE	True value	$97.5\%^{\text{tile}}$	$2.5\%^{\text{tile}}$	MSE
2.828	2.917	2.566	0.013	−0.100	0.11	−0.159	0.012
1.414	1.658	1.183	0.016	−0.050	0.119	−0.141	0.006
0.707	0.909	0.465	0.014	0.000	0.121	−0.106	0.004
0.354	0.594	0.253	0.010	0.050	0.112	−0.09	0.004
0.177	0.515	0.135	0.029	0.100	0.105	−0.076	0.010

Damped Linear Oscillator with Small Perturbations (perturbations $\sim N(1, .2)$)							
ω Estimates				ζ Estimates			
True value	$97.5\%^{\text{tile}}$	$2.5\%^{\text{tile}}$	MSE	True value	$97.5\%^{\text{tile}}$	$2.5\%^{\text{tile}}$	MSE
2.828	2.896	2.758	0.001	−0.100	−0.003	−0.362	0.012
1.414	1.463	1.367	0.001	−0.050	0.037	−0.248	0.006
0.707	0.765	0.662	0.001	0.000	0.099	−0.142	0.004
0.354	0.423	0.293	0.001	0.050	0.183	−0.063	0.005
0.177	0.289	0.041	0.004	0.100	0.313	−0.013	0.008

Notes: Confidence intervals (95%) and mean squared error (MSE) of ω and ζ. Estimates for ω average over all values of ζ, and estimates for ζ average over all ω values. Estimates are based on time series of 50 observations with equally spaced observations.

be well described using a model that allows for oscillations. It should be expected, however, that even if an oscillator model is appropriate, individuals will not literally produce oscillatory movements; there will be variations in the frequency of their movements due to factors such as efforts to align their movements with each other, desire to stay synchronized with the rhythm of the music, and the difficulty of producing perfectly rhythmic movement (Boker et al., 2005). Modeling such data could be accomplished using a differential equation model with changing parameters; this can, however, be problematic to implement as such changes can alter a stationary system into one that is nonstationary consequently leading to the breakdown of methods that assume stationarity.

Dancer's movements, however, appear to occur with a relatively constant frequency if examined over a short period of time. Therefore, it should be possible to model dancer movement using a stationary model, such as the damped linear oscillator model, if it is applied to short windows of time. We have chosen an example of motion to capture data from a dyad dancing to a stationary repeating rhythm as a way to demonstrate the statistical approach to differential equations presented above. By applying the damped linear oscillator model to many short windows of time, we track the time-varying frequency changes for each dancer; that is, rather than just focusing on the stable aspects of a dynamical system as was done previously, we now focus on variation in the frequency and damping parameters by fitting the model to many small pieces of the time series. This is an important step toward understanding the time-varying nature of the synchronization between the two dancers, and by extension might inform the understanding of how and why individuals synchronize movements during conversation.

2.1. Methods

2.1.1. Data Participants consisted of a female-male undergraduate dyad from a Midwestern University. Movements of participants were collected using an Ascension Technologies Motionstar magnetic tracking system. Each individual wore eight sensors, one of which was positioned on the back of the head using a baseball cap. Participants were instructed to dance to the rhythms presented while staying within a 1 m by 1 m square that had been marked on the floor and without touching each other. Participants wore headphones during the experiment, which provided the rhythms as well as instructions as to whether to "lead" or "follow." In addition to lead-follow configurations, the experiment also considered configurations where both participants were instructed to lead, or both were instructed to follow. Rhythms were synthesized to consist of either 7 or 8 beats, with intervals of 200 ms per beat. Rhythms were selected such that some consisted of patterns with clear segmentation (nonambiguous) and others where the segmentation was ambiguous as described by Boker & Kubovy (1998). There were a total of 26 trials, each consisting of a different rhythm presented for 40 seconds.

2.1.2. Analysis Movement of the head was recorded as a series of X, Y, Z coordinates at a rate of 80 measurements per second. The root mean square, $X_{\mathrm{RMS}} = \sqrt{X^2 + Y^2 + Z^2}$, of each person's head movement was calculated for each trial, at each measurement observation. A single trial consisted of a times series of 3,200 observations, which were downsampled to 160 observations by retaining every 20th observation. Downsampling should not alter the results substantially, since the motion capture sampling rate was very high relative to the expected oscillation frequency of the dancers (around 0.2 Hz to 0.4 Hz). Furthermore, the downsampled time series better reflect the situations for which the method presented is intended. The 160-observation times series for each individual was divided into overlapping windows 5 seconds (20 observations) in length, with the distance between the initial observations for a window and a subsequent window separated by 1/4 second. The first 5 seconds of each time series were discarded, as it took some time for dyads to become accustomed to rhythms. Consequently, the damped linear oscillator model was fit to each individual's time series using 121 overlapping windows for each trial. Model parameters were not constrained in any manner between different windows.

For each window (121 per trial) for each trial (26 trials) for each dancer the damped linear oscillator model was applied using the method presented in the simulated example. This produced an estimate of frequency and damping for each individual, for every 5-second span of time, within each of the 26 trials. There were three minor changes made to the estimation procedure, so as to better match the analysis to this particular application. First, the random starting values for the ω parameter were drawn from a uniform distribution bounded by ω values equivalent to a half-second and a five-second oscillation. These values seemed reasonable based on the rhythms used. Second, the equilibrium was set to zero by removing the mean within each window of analysis. Third, estimated results where there was large amounts of damping ($|\zeta|/2\omega > 0.75$)[2] or unusually long frequency estimates (greater than 10 seconds) were considered situations where the movement may not have been well described by a damped linear oscillator; these results were marked as "not available."

[2]This is very close to the value at which the damping is so large that oscillations do not occur (i.e., $|\zeta|/2\omega > 1$).

Two figures were produced to examine the results produced by this analysis. These figures highlight the multiple estimates produced for each individual for each trial, how the estimated frequency of movement changes over the course of the trial, and how the individuals in the dyad may be moving relative to each other. In addition, it was anticipated that each of the dancers would move at a frequency equivalent to the frequency of the rhythm, or some harmonic of this frequency. To examine this, the estimates of ω were converted into wavelength, measured in seconds; this metric was selected for its ease of interpretation. For each estimate, the absolute deviation from the nearest harmonic was calculated. The absolute deviations were then predicted, within individual, using who led in each trial (male, female, both, or neither) and whether the rhythm was ambiguous or not (1 or 0, respectively); a random intercept was included for each trial, as changes in performance were expected across trials. These models address what characteristics led each individual in the dyad to remain closer to a harmonic of the rhythm presented. For both models bootstrapped confidence intervals were generated, due to the possible violation of regression assumptions; confidence intervals were calculated using 5,000 bootstrap replicates and were bias corrected and accelerated using the "boot" library in the statistical program R (2007).

2.2. Results

A sample of results are shown in Figures 2 and 3. Figure 2 shows the estimated ω for each of the dancers across 6 different trials. The horizontal and vertical lines correspond to harmonics of the beat frequency, while the diagonal line corresponds to equal ω estimates. In trials 3, 6, and 7 (plots b, e, and f, respectively) the dancers' head movements occur at similar frequencies and at approximately the same frequency of the rhythm. When neither are instructed to lead, as in trial 4 (plot c), there is a breakdown of this coordination. This is also true with trial 5 (plot d), where both are instructed to lead; the male dancer seems to stubbornly stick to the frequency of the rhythm, while the female dancer attempts to lead the male to a variety of other frequencies. Subtle difference in coordination are also evident, such as in trials 6 and 7 (plots e and f respectively). In one case dancers dance at similar frequencies but show a very low correlation (trial 6, $R^2 = 0.033$) and, in the other, the change in one dancer's head movement frequency seems to be more related to that of the other dancer (trial 7, $R^2 = 0.233$).

Figure 3 shows plots of how the estimated frequencies for each of the dancers are changing over the course of a trial. When the estimates are close dancers are moving at similar frequencies, while estimates that are further apart indicate a difference in the frequency of head movements. These plots correspond to trials 2, 5, and 6. In trial 2 (plot a) the dancers seem to move in and out of periods where they are more synchronous. This is in stark contrast to trial 5 (plot b), where each dancer seems to ignore the other, and trial 6 (plot c) where the two dancers are very similar in their movements.

Table 6 shows the results for the models fit to the absolute difference between the female's or male's wavelength at any moment and the closest harmonic wavelength. As the wavelength of the rhythms are 1.4 and 1.6 seconds, the magnitudes of the estimates suggest both dancers stay relatively close to harmonic wavelengths, although the female dancer did a better job of moving at a harmonic of the frequency of the rhythm. Both dancers tend to dance closer to a harmonic wavelength when the female is leading. But when the male led, the dancers both tend to move at rates that differ more from the harmonics of the rhythm length. The differences further increase in the other leading conditions. Ambiguous rhythms tend to also increase the female's mean departure from a harmonic wavelength. The effect for the male was not significant. The most common harmonic was equivalent to a wavelength twice that of the rhythm, which occurred 65% of the time.

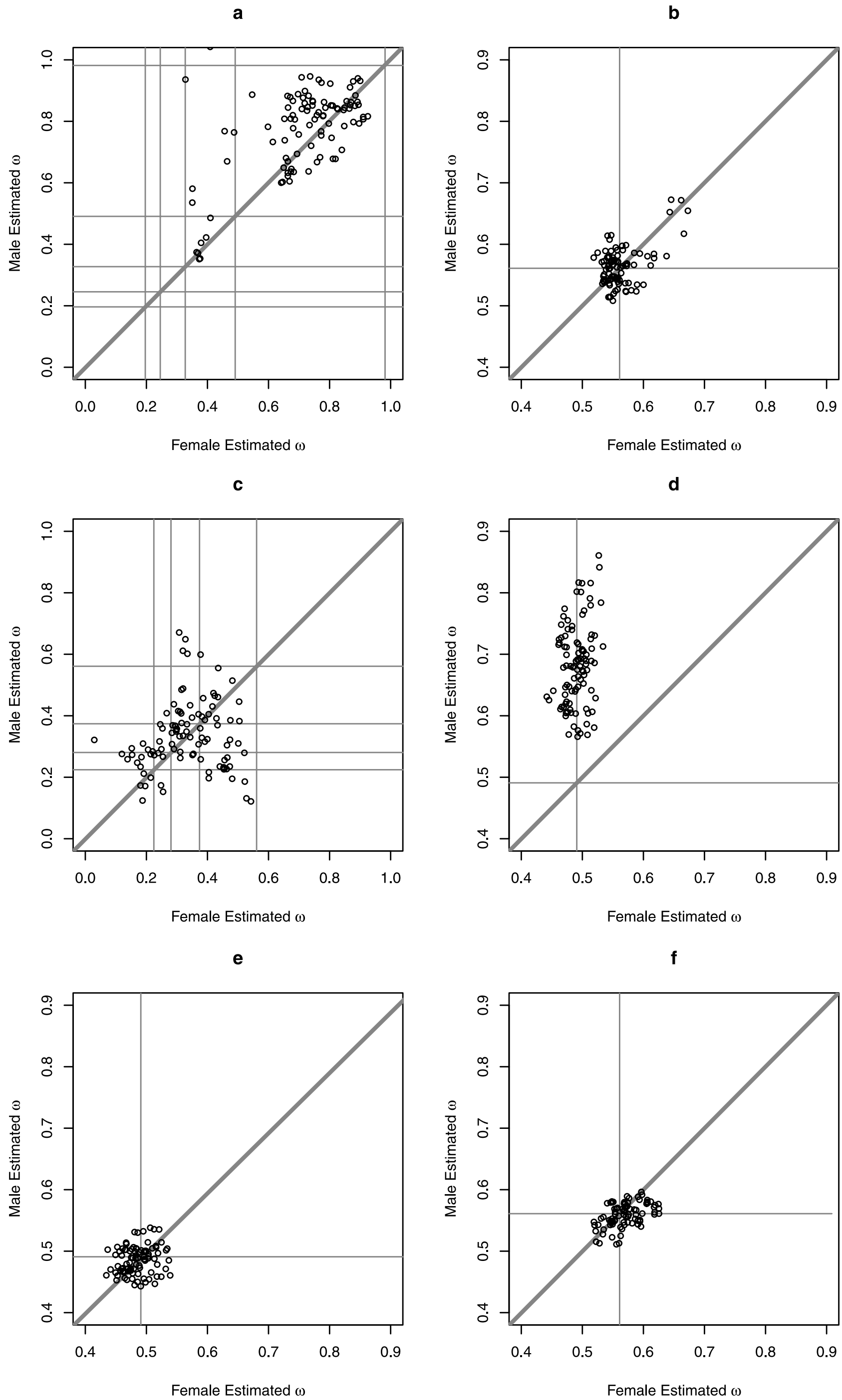

FIGURE 2.
Plot of estimated ω values for each dancer. The points along the diagonal grey line indicate dancers were moving at the same frequency. The horizontal and vertical grey lines indicate different harmonics of the beat frequency. The figures represent the following trial conditions: (a) Trial 2: Ambiguous Rhythm, Female Leads, 8 beats; (b) Trial 3: Ambiguous Rhythm, Female Leads, 7 beats; (c) Trial 4: Non-Ambiguous Rhythm, Neither Leads, 7 beats; (d) Trial 5: Non-Ambiguous Rhythm, Both Lead, 8 beats; (e) Trial 6: Non-Ambiguous Rhythm, Male Leads, 8 beats; (f) Trial 7: Non-Ambiguous Rhythm, Male Leads, 7 beats. Note that Trials 2 and 4 are on a different scale than the other trials.

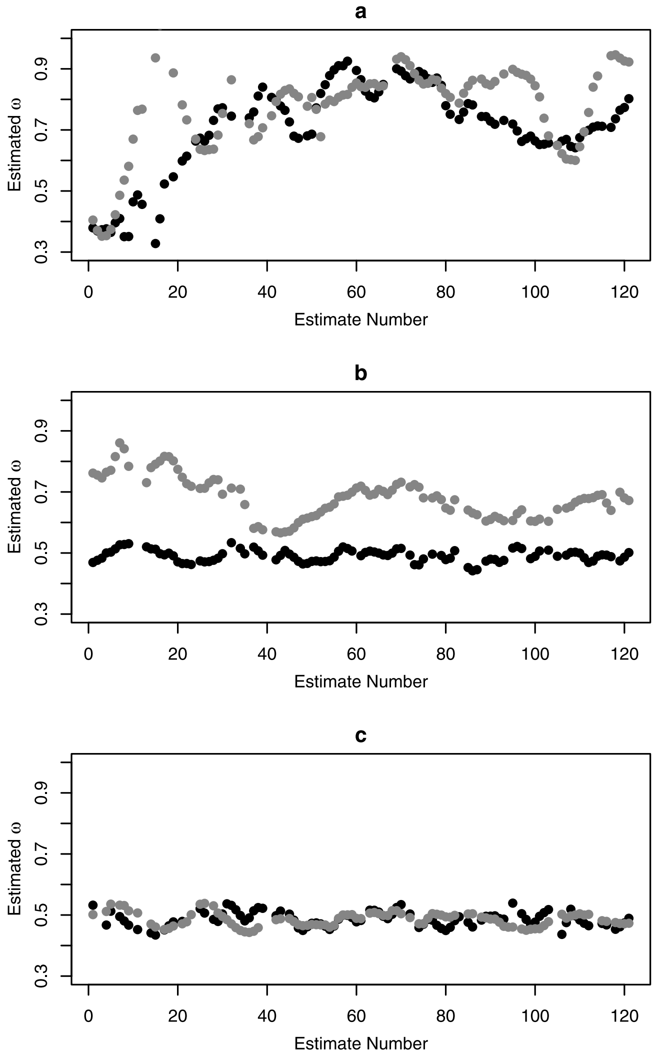

FIGURE 3.
Plot of estimated ω values for Female (black) and Male (grey) dancers. Closer points indicate more similarity in the rate of head movements during the period sampled. The figures represent the following trial conditions: (a) Trial 2: Ambiguous Rhythm, Female Leads, 8 beats; (b) Trial 5: Non-Ambiguous Rhythm, Both Lead, 8 beats; (c) Trial 6: Non-Ambiguous Rhythm, Male Leads, 8 beats.

2.3. Discussion

This example demonstrates one way that the methods presented in this article could be applied to data. In this example, damped linear oscillator parameter estimates were produced for relatively short time series (20 observations). By applying the method to such short time series, we could consider how these parameters were changing over the course of time series, without having to specify a model with intricate changes in parameters over time. The results also helped to confirm that the method produces reasonable values, as estimates were related to the harmonics of the rhythm length.

By examining the estimated parameter on a moment-to-moment basis, it may be possible to understand how individuals synchronize movements with each other over the course of a dance,

TABLE 6.
Models of estimated wavelengths.

(b) Female Absolute Deviations from Harmonic Wavelengths

Predictor	Estimate	95% C.I.		99% C.I.		p-value
		Lower	Upper	Lower	Upper	
Intercept	0.118	0.097	0.141	0.090	0.150	$p < 0.01$
Male Leads = 1	0.034	0.020	0.048	0.015	0.053	$p < 0.01$
Both Lead = 1	0.053	0.037	0.067	0.032	0.072	$p < 0.01$
Neither Leads = 1	0.218	0.198	0.237	0.192	0.243	$p < 0.01$
Ambiguous = 1	0.026	0.014	0.039	0.010	0.042	$p < 0.01$

(c) Male Absolute Deviations from Harmonic Wavelengths

Predictor	Estimate	95% C.I.		99% C.I.		p-value
		Lower	Upper	Lower	Upper	
Intercept	0.411	0.37	0.453	0.357	0.466	$p < 0.01$
Male Leads = 1	0.075	0.061	0.090	0.057	0.095	$p < 0.01$
Both Lead = 1	0.116	0.101	0.130	0.096	0.135	$p < 0.01$
Neither Leads = 1	0.197	0.178	0.217	0.173	0.223	$p < 0.01$
Ambiguous = 1	−0.012	−0.024	0.001	−0.027	0.005	n.s.

or in other situations such as conversation. The results from the models already demonstrate a synchrony between individuals with the rhythms being presented. Unfortunately, building a model of the frequencies as they change over time is not as straightforward as one would hope, as evidenced by comparing trials 2 and 3 in Figure 2a and 2b. These trials occur in succession in the study, and despite the similar circumstances the dancers show remarkable differences in synchrony. It still is left to try to describe these movements using a coherent model, however the results produced are certainly an important first step. Furthermore, if periods of synchrony can be understood in a situation such as dyads dancing, perhaps principles and methods will be extendable to the study of other structures that show periods of synchrony, or lack thereof.

3. Conclusions

The simulation suggested that comparing observed embedded matrices with expected embedded matrices is likely to be a valuable approach for the modeling of differential equations in psychology, as the method works well across a wide range of data conditions. The applied example provided a proof-of-concept and furthermore demonstrated how short windows of analysis could be used to allow for changing parameters over the course of a time series. The primary limitation in both applications was the use of a model with a known analytic solution. It seems likely that numerical methods such as Runge–Kutta integration could be used to produce estimates of x_t, in which case these methods would be universally applicable.[3]

The present paper only discussed the model-specific parameters, ignoring the observation-specific parameters. The observation-specific parameters are based on very few observations, resulting in more variability in these estimates. Through these estimates, however, it is possible to create estimates of x_t. Figure 4 gives an example of the estimated time series that are produced

[3]Due to numerous local minima, estimation of x_t using numerical methods may not be as robust as models with analytic solutions, even when time series are of better quality than examined here (Esposito & Floudas, 2000; Ramsay et al., 2007). Different minimization routines or different minimization/maximization criteria may be required to use numerical integration with the method presented here.

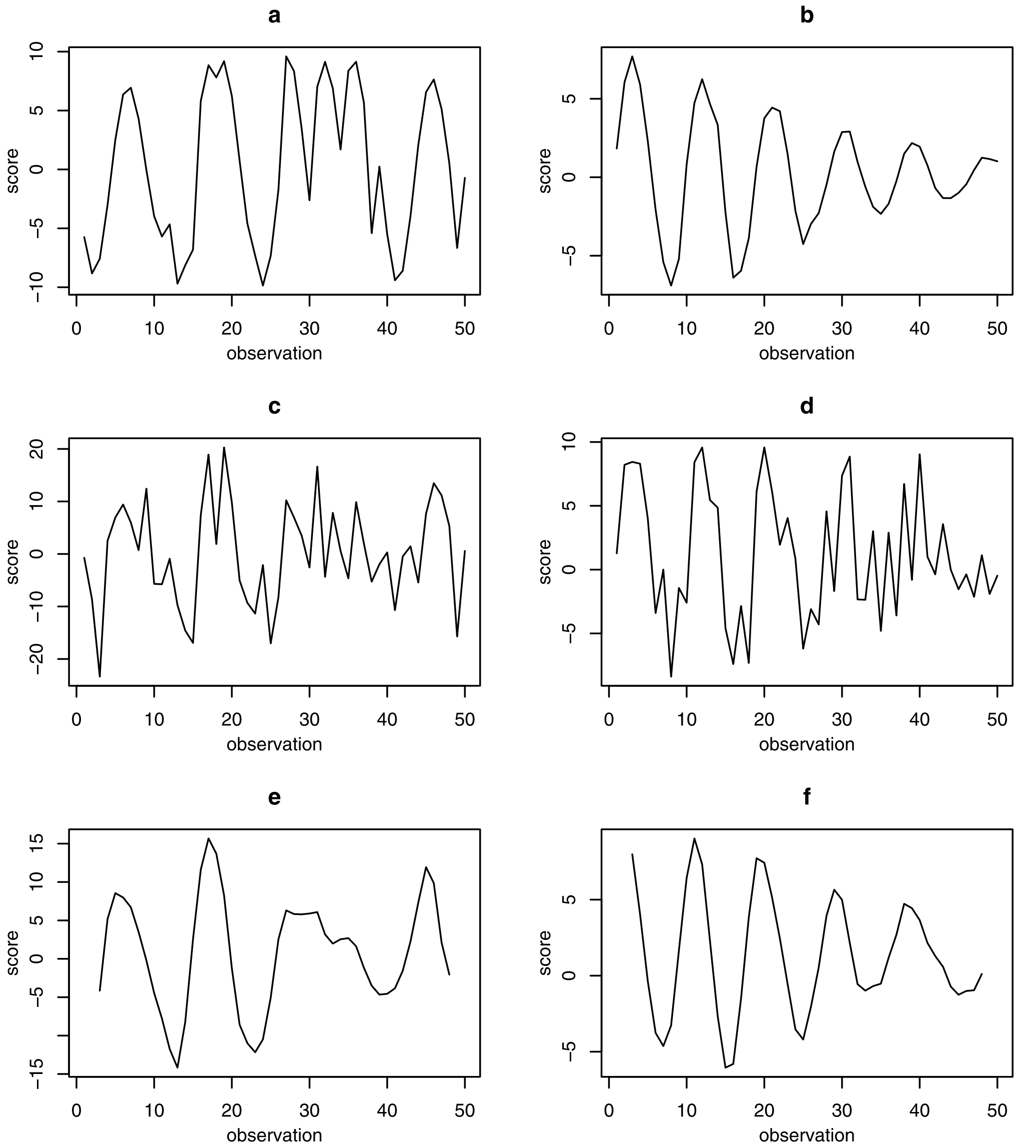

FIGURE 4.
Examples of the time series generated with dynamic error (top row), generated time series plus measurement error (middle row), and recovered time series based on estimated parameters (bottom row). The left and right columns correspond to examples for the two types of dynamic error examined: phase & amplitude resetting and small perturbations at each observation.

using the method presented. These estimates may be potentially useful for several purposes including: (1) creating ways to estimate and compare model fit, and (2) allowing one to identify observations where a significant amount of dynamic error occurred; thereby allowing one to probe further into the understanding as to what may cause disturbances to a particular system.

Psychological time series are likely to possess a complexity and variety of assumption violations unheard of in many of the areas in which differential equation modeling has been developed. Despite these challenges, differential equation modeling is likely to be integral to understanding intraindividual time series. The current paper demonstrates that despite the conditions that may occur in many time series, methods using differential equation modeling are available that will allow for robust, statistical consideration of intraindividual time series. If it were not for the difficulties associated with measurement error and the dynamic changes constantly imposed by both internal and external events of daily life, methods such as those presented here would not be required. But the work presented adapts methods typically reserved for time series measured on physical or biological attributes to time series common to a wide range of psychological studies so that discussion will no longer be limited to the dynamics of variables such as brain waves, but rather branch out to include constructs such as affect, stress and personality.

Appendix

```
EstValues <- function(omega,time,zeta,A,delta) {
    wprimeSQD <- as.complex((omega*omega)-(zeta*zeta/4))
    out <- A*exp(zeta*time/2)*cos((Re(sqrt(wprimeSQD))*time)
           +delta)
    return(out)
    }

FunctionToMinimize <- function(x, xobs, time, len) {
    Estimates <- EstValues(x[1],time,x[2],x[3:(len+2)],
                           x[(len+3):((2*len)+2)])
    SSR <- sum((xobs-Estimates)^2)
    return(SSR)
    }

EstimationFunction <- function(data,time) {
    cat("Reminder: Did you set the equilibrium of x to zero?
        \nProcessing...\n")
    Repetitions <- 30   #Increase this if results don't appear
                        #to be stable
    LowestSSR <- Inf
    BestFit <- NA
    len <- dim(data)[1]
    for(i in 1:Repetitions) {
        startvalues <- c(runif(1,0,pi), rnorm(1,0,.1),
                         runif(len,min(data)/2,max(data))/2,
                         runif(len,0,2*pi))
        fit <- optim(startvalues, FunctionToMinimize, xobs=data,
               time=time, len=dim(data)[1], method="BFGS",
               control=list(maxit=500))
        if(fit$value<LowestSSR) {
            LowestSSR <- fit$value
            BestFit <- fit
         }}
     return(BestFit)
     }

Embed <- function(x,E) {
    out <- x[1:(length(x)-E+1)]
```

```
    for(i in 2:E) { out <- cbind(out,x[i:(length(x)-E+i)]) }
    return(out)
    }

# PROGRAM TO CALL FUNCTIONS
# Requires: 1) observed score time series (x) with equilibrium
#  equal to zero,
# 2)vector of measurement occasions (time)
#
# Returns best fitting results from optim()
#
# Consider running twice. If results are not stable,
# increase repetitions in EstimationFunction

NumberOfEmbeddings <- 5
x.embedded <- Embed(x, NumberOfEmbeddings)
time.embedded <- Embed(time, NumberOfEmbeddings)
EstimationFunction(x.embedded,time.embedded)
```

References

Boker, S.M., Covey, E.S., Tiberio, S.S., & Deboeck, P.R. (2005). Synchronization in dancing is not winner–takes–all: ambiguity persists in spatiotemporal symmetry between dancers. In *2005 proceedings of the North American association for computational, social, and organizational science*.

Boker, S.M., & Kubovy, M. (1998). The perception of segmentation in sequences: local information provides the building blocks for global structure. In Rosenbaum, D.A., & Collyer, C.E. (Eds.) *Timing of behavior: neural, computational, and psychological perspectives* (pp. 109–123). Cambridge: MIT Press.

Boker, S.M., Neale, M.C., & Rausch, J.R. (2004). Latent differential equation modeling with multivariate multi-occasion indicators. In Montfort, K.V., Oud, J., & Satorra, A. (Eds.) *Recent developments on structural equation models: theory and applications* (pp. 151–174). Amsterdam: Kluwer Academic.

Boker, S.M., & Nesselroade, J.R. (2002). A method for modeling the intrinsic dynamics of intraindividual variability: recovering the parameters of simulated oscillators in multi-wave panel data. *Multivariate Behavioral Research*, *37*(1), 127–160.

Broyden, C.G. (1970). The convergence of a class of double-rank minimization algorithms 2. The new algorithm. *Journal of the Institute for Mathematics and Applications*, *6*, 222–231.

Esposito, W.R., & Floudas, C.A. (2000). Deterministic global optimization in nonlinear optimal control problems. *Journal of Global Optimization*, *17*, 97–126.

Fletcher, R. (1970). A new approach to variable metric algorithms. *Computer Journal*, *13*, 317–322.

Goldfarb, D. (1970). A family of variable-metric methods derived by variational means. *Mathematics of Computation*, *24*, 23–26.

Mathematica (2005). Software. Wolfram Research.

Molenaar, P.C. (2004). A manifesto on psychology as idiographic science: bringing the person back into scientific psychology, this time forever. *Measurement*, *2*(4), 201–218.

Nesselroade, J.R., & Ram, N. (2004). Studying intraindividual variability: what we have learned that will help us understand lives in context. *Research in Human Development*, *1*(1–2), 9–29.

R (2007, April). Software. http://www.r-project.org/.

Ramsay, J.O., Hooker, G., Campbell, D. & Cao, J. (2007). Parameter estimation for differential equations: a generalized smoothing approach. *Journal of the Royal Statistical Society B*, *69*, 714–796.

Ramsay, J.O., & Silverman, B.W. (2005). *Functional data analysis*. New York: Springer.

Shanno, D.F. (1970). Conditioning of quasi-newton methods for function minimization. *Mathematics of Computation*, *24*, 647–656.

Shannon, C.E. A mathematical theory of communication, *Bell Systems Technical Journal* 27, 379–423, 623–656 (1948).

Takens, F. (1981). Detecting strange attractors in turbulence. In Rand, D.A., & Young, L.S. (Eds.) *Lecture notes in mathematics: Vol. 898. Dynamical systems and turbulence*. Berlin: Springer.

Manuscript Received: 22 MAY 2008
Final Version Received: 12 SEP 2009
Published Online Date: 27 MAY 2010