Algebraic and Abstract Simple Groups: Old and New

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The classification of finite simple groups (cf., for example, [16]) tells us that any nonabelian finite simple group which is not one of the alternating groups A_n, $n \geqslant 5$, or one of the 26 sporadic groups, can be obtained via the following construction or its variations: take an absolutely simple simply connected algebraic group $\mathbf{G}$ over a finite field F; then the group of F-rational points $\mathbf{G}(F)$ "typically" (i.e. with finitely many exceptions occurring over "small" F) does not have proper noncentral normal subgroups (in which case we say that it is projectively simple), yielding thereby a desired finite simple group $\mathbf{G}(F)/Z(\mathbf{G}(F))$. Thus, (absolutely simple simply connected) algebraic groups appear to be a universal source of abstract simple groups, at least in the case of algebraic groups over finite fields. From this perspective, it is only natural to ask about the relationship between *infinite* simple groups and the groups of rational points of algebraic groups over *infinite* fields. One needs to bear in mind, however, that the class of infinite simple groups that in any case should be regarded as "sporadic" will be incomparably larger as there are numerous constructions of finitely generated infinite simple groups, which cannot be linear over any field. On the other hand, we are not aware of projectively simple *linear* groups that are essentially different from the groups of rational points, so some "classification" of such groups is not totally out of the question, even though it appears to be a very challenging problem.[1] In this article we will discuss the other side of the story, which is the question *if* (or *when*), *given an absolutely simple simply connected algebraic group* $\mathbf{G}$ *over an infinite field* K, *the group* $\mathbf{G}(K)$ *is projectively simple?* This question has received a lot of attention at various periods in the past two centuries, with important results obtained in the last couple of years. The purpose of this article is to survey the landmarks in this direction and to call attention to remaining old and emerging new problems. The author would like to

[1] This problem may be connected with another difficult problem whether finitely generated "semi-simple" linear groups with the congruence subgroup property are in fact arithmetic, but no direct links have been found so far.

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1 The Classics and Classical Groups

It is interesting that the idea to employ matrix groups in order to produce simple groups is basically as old as group theory itself. Namely, among the first concepts of the latter was the notion of nonsolvable groups introduced and used by Galois to prove his celebrated theorem that the general polynomial equation of degree $\geqslant 5$ is not solvable by radicals. One of the key ingredients of Galois's argument was the fact that the symmetric group S_n for $n \geqslant 5$ is nonsolvable which, of course, is closely related to the simplicity of the corresponding alternating group A_n. Apparently, Galois tried to find other examples of nonabelian simple groups of substitutions within the theory of substitution groups proper, but did not succeed (the classification of finite simple groups enables us to understand the reason for his "failure": in addition to the alternating groups A_n, $n \geqslant 5$, only five sporadic groups M_{11}, M_{12}, M_{22}, M_{23}, and M_{24}, discovered by Mathieu, are "naturally" represented as groups of substitutions). So, Galois began to explore a different construction involving linear fractional substitutions

$$z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0,$$

with coefficients in the field F_p of p elements, where p is a prime. All such substitutions form a group, currently denoted $PGL_2(F_p)$, and Galois mentions explicitly that its subgroup $PSL_2(F_p)$, made up of substitutions satisfying $\alpha\delta - \beta\gamma = 1$, is a nonsolvable group for $p \geqslant 5$ (cf. his *Oeuvres Mathématiques,* Gauthier-Villars, Paris, 1897). Today we can only speculate whether Galois had a proof of the simplicity of $PSL_2(F_p)$, or how close he was to finding one, but we know for a fact that he obtained a lot of information about this group including a delicate arithmetic result describing when $PSL_2(F_p)$ has a subgroup of index $< p + 1$ (this result, described in the last letter of Galois, cf. [8], has to do with the question of obvious interest to Galois, when $PSL_2(F_p)$, along with its realization as a group of substitutions (of degree $p + 1$) of the projective line $\mathbb{P}^1(F_p)$, has a realization as a group of substitutions of a smaller degree – the answer turns out to be iff $p = 2, 3, 5, 7, 11$;

cf. [5], §§262-263 and the footnote on p. 286). Subsequently, the group $PSL_2(F_p)$ was actively studied by other mathematicians, including, for example, Serret (co-author of the famous Frenet-Serret formulas), but the first documented proof of the simplicity of $PSL_2(F_p)$ appeared in Jordan's *Traité des substitutions et des équations algébriques* (1870). Regarding credit for this and other results in the book, we would like to quote the footnote on p. 100 of [4] which in turn quotes R. Brauer's remark that Jordan in the introduction to his *Traité* refers to the book as a "commentary" on the work of Galois, making one of the most modest statements in the history of mathematics. However, Jordan's fundamental work is remembered not only for extraordinary modesty. In particular, it was in this book that Jordan introduced the general linear group $GL_m(F_{p^n})$ of arbitrary dimension $m \geqslant 2$ over an arbitrary finite (Galois) field F_{p^n} as the collection of matrices with entries in this field having nonzero determinant, as well as other important groups (orthogonal, symplectic), and analyzed their normal subgroups over F_p. Surprisingly enough, it took almost 25 years to extend Jordan's result about the simplicity of $PSL_2(F_p)$ to $PSL_2(F_{p^n})$ for arbitrary n : this was accomplished independently by and E.H. Moore (1893) and W. Burnside (1894). Burnside included results on linear groups over finite fields into his fundamental book *Theory of Groups of Finite Order* (1897). In fact, the books by Jordan and Burnside contain the analysis of "factors of composition" of $GL_m(F_{p^n})$ for m arbitrary (with $n = 1$ in Jordan's case), resulting in the statement that unless $m = 2$ and $p^n = 2$ or 3, the group $SL_m(F_{p^n})$ does not contain proper noncentral subgroups normalized by $GL_m(F_{p^n})$, however according to Dickson [5], p. 84, the argument in both sources was incomplete. So, the first flawless proof of the simplicity of $PSL_m(F_{p^n})$ should probably be credited to Dickson (cf. his dissertation (1896) and his book [5], §§103-108).

Already at an early stage, it became apparent that the general linear group contains very many interesting (and potentially projectively simple) subgroups: as we mentioned above, symplectic and orthogonal groups (over finite fields) were considered by Jordan in *Traité*, while unitary groups of hermitian and skew-hermitian forms appeared in Dickson's book [5] under the names "hyperorthogonal" and "hyperabelian" groups (nowadays, all these groups are called "classical," following H. Weyl who used the term in the title of his famous book [20]). In [5], Dickson carefully examined the composition series for each type of classical group, having discovered that apart from finitely many exceptions arising only in "small" dimensions and over "small" fields, each group has a composition factor which is a noncommutative finite simple group (it should be noted that some of these results were already known to Jordan). In spite of many similarities in argument for different types (in particular, the crucial role of certain specific unipotent elements, the role of which we will discuss a bit later), there were technical as well as more essential differences. For example, while the symplectic group is projectively simple, the composition series for the special orthogonal group $SO_n(f)$, $n \geqslant 3$, $n \neq 4$, over a field of characteristic $\neq 2$ turns out to be longer:

$$SO_n(f) \supset O'_n(f) \supset Z,$$

where $O'_n(f)$ is a subgroup of index two in $SO_n(f)$, which Dickson describes in terms of generators, and Z is the center of $O'_n(f)$ (having order one or two). Of course, this phenomenon is easily explained by the theory of algebraic groups: the symplectic group is simply connected, while the special orthogonal group has a 2-sheeted covering $\pi\colon \mathbf{Spin}_n(f) \to \mathbf{SO}_n(f)$[2]; however whenever we have have a central isogeny $\pi\colon \tilde{\mathbf{G}} \to \mathbf{G}$ of connected algebraic groups (of which the previous covering is an example) defined over a field K, the image $\pi(\tilde{G}(K))$ is a normal subgroup of $G(K)$ with the quotient $G(K)/\pi(\tilde{G}(K))$ being an abelian group of exponent dividing that of the finite group $F(\overline{K}) := \operatorname{Ker}\pi$, where $\overline{K}$ is the algebraic closure of K. Moreover, if K is a finite field, then $[G(K) : \pi(\tilde{G}(K))] = \mid F(K) \mid$ (by a theorem due to S. Lang), hence the existence of a normal subgroup $O'_n(f) \subset SO_n(f)$ of index two over finite fields of characteristic $\neq 2$ (in the current terminology, this subgroup is described as the kernel of the spinor norm). To avoid the presence of "additional" normal subgroups of $G(K)$ of the form $\pi(\tilde{G}(K))$, we from the outset impose the assumption that G be simply connected (i.e. without nontrivial connected coverings).

Of course, Dickson did not have the necessary techniques from the theory of algebraic groups to present his simplicity results from a general perspective. Moreover, it appears that at least when writing his book [5], he was not well aware of the landmark developments in the theory of Lie groups achieved around the same time by Lie, Cartan, Killing and others. Later, however, he became familiar with these results and started looking for some parallels. In particular, there was clear analogy between the series of finite simple groups he had constructed and compact simple Lie groups of types A_n, B_n, C_n and

[2]We will use bold face to denote algebraic groups.

D_n, so one of the problems that occupied him was how to construct finite analogs of compact Lie groups of types E_6, E_7, E_8, F_4, G_2 (in Dickson's own words: "After determining four systems of simple groups ..., the author was led to consider five isolated continuous groups of 78, 133, 248, 52 and 14 parameters"). Dickson was able to construct a finite analog of the compact Lie group of type G_2 and prove its projective simplicity (in the introduction to the 1958 edition of [5], W. Magnus writes that after that "no new simple groups of finite order were discovered for half a century"). Dickson also managed to construct a finite analog of the compact Lie group of type E_6 and to compute its order. However, for the other three exceptional types, finite analogs were not found until 1955.

After a long period of hibernation, the subject of simplicity of classical groups re-emerged in the early 1940s in the works of J. Dieudonné. Although Dickson stated results in his book for groups over finite fields, he indicated in the preface to [5] that his "method of investigation is applicable to groups in an infinite field," and subsequently published several papers to validate this claim (in particular, he, in fact, had constructed groups of types G_2 and E_6 over arbitrary fields). Dieudonné, however, from the very beginning considered classical groups not only over general (commutative) fields, but, in fact, over arbitrary (noncommutative) skew fields (we observe that the latter do not occur in the context of finite fields as according to Wedderburn's theorem, any finite skew field is commutative). As a technical tool, Dieudonné began to make extensive use of geometric notions which in many instances made tedious computations with matrices unnecessary. One of the most celebrated results of Dieudonné is his construction of a determinant map over skew fields. More precisely, he showed that for an arbitrary skew field D and any $m \geqslant 2$, there exists a unique group homomorphism

$$\delta\colon GL_m(D) \longrightarrow D^*/[D^*, D^*]$$

(where $[D^*, D^*]$ is the commutator subgroup of the multiplicative group D^*) that enjoys two most important properties of the usual determinant: it is invariant under elementary row and column operations and satisfies $\delta(\mathrm{diag}(1, \ldots, 1, d)) = d[D^*, D^*]$ (we note, however, that in the noncommutative situation, δ is not invariant under taking the transpose of a matix!). The kernel of δ, denoted $SL_m^+(D)$, can be described in geometrical terms as the subgroup of $GL_m(D)$ generated by *transvections,* which are defined to be transformations of the (left) vector space $V = D^m$ of the form $\tau(v) =$

$v + \varphi(v)a$, where $a \in V$ is a fixed vector, and φ is a fixed linear functional on V that vanishes on a. It turns out, furthermore, that except when $m = 2$ and D is a field of two or three elements, the group $SL_m^+(D)$ is projectively simple. The use of transvections allows one to give a very short proof of this fact for $m \geqslant 3$: in this case, all transvections form a single conjugacy class in $SL_m^+(D)$, so if a normal subgroup $N \subset SL_m^+(D)$ contains a transvection, it must coincide with $SL_m^+(D)$; on the other hand, using an arbitrary noncentral element $g \in N$, it is easy to create a transvection inside N. This argument highlights the role of (certain) unipotent elements (in this case, transvections) in proofs of simplicity, and leads to the general "philosophy" that the subgroup generated by such elements should be projectively simple. So, developing this idea for other classical groups (orthogonal, unitary), Dieudonné had to limit his considerations to the case when these groups contain a nontrivial unipotent element (in the form of a unitary transvection, etc.) which amounts to the requirement that the corresponding sesqui-linear form is *isotropic,* i.e. it admits a nonzero vector of "length" zero. (We observe that Dickson did not need to impose this assumption as it holds automatically over finite fields due to the fact that any quadratic form in $n \geqslant 3$ variables is isotropic.) The results on classical groups were presented in Dieudonné's monograph [6] which went through three editions and soon became a mathematical classic (among other features, we would like to mention that it contains a very complete bibliography on the subject). In fact, (especially) the first edition (1955) played a notable role in stimulating research in the area, and some important results such as G.E. Wall's construction of a unitary analog of the spinor norm for isotropic skew-hermitian forms over a skew field with an involution of the second kind, were obtained after the first edition. Another publication that helped to promote the structural aspect of classical groups was Artin's book [1], containing an exposition of simplicity results for the special linear, symplectic and orthogonal groups. The most recent systematic account of the theory of classical groups over general rings, with connections to algebraic K-theory, is given in the fundamental book [7] by A. Hahn and O.T. O'Meara.

2 Simplicity for isotropic algebraic groups

It is a kind of historical law that the general properties of the simple groups have been verified first in the various groups, and afterwards one has sought and found general explanations that do not require the examination of special cases.

E. Cartan (1936)[3]

A general construction over arbitrary fields of groups corresponding to all types of simple compact Lie groups was given by C. Chevalley [3] in 1955. (It is worth mentioning that he developed this construction after having worked out separately the cases E_6, E_7 and F_4– apparently he was aware of Dickson's work on G_2, but not with his construction of E_6, cf. [3]). It was the first time in history when simple groups in the context of general fields were constructed not in terms of an explicit geometric realization, but using general properties of Lie algebras (so, [3] was really the first publication directly related to the title of our article). More precisely, Chevalley starts out with an arbitrary simple complex Lie algebra $\mathfrak{g}$, and fixes a Cartan subalgebra $\mathfrak{h}$ and the associated Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in R} \mathfrak{g}_\alpha),$$

where R is the root system and $\mathfrak{g}_\alpha$ are 1-dimensional root subspaces (eigenspaces for the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ with nonzero weights). Chevalley shows that it is possible to pick an element $X_\alpha \in \mathfrak{g}_\alpha$ for each $\alpha \in R$ and a basis $H_1, \dots, H_r$ of $\mathfrak{h}$ so that the basis $\{H_i\}_{i=1}^r \cup \{X_\alpha\}_{\alpha \in R}$ of $\mathfrak{g}$, called the Chevalley basis, has "nice" (in particular, integral) structure constants; then the $\mathbb{Z}$-span $\mathcal{L}$ of this basis is a Lie algebra over $\mathbb{Z}$ (called a Chevalley lattice). For $\alpha \in R$, one considers the following formal power series in an indeterminate t :

$$x_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n (\operatorname{ad} X_\alpha)^n}{n!}. \tag{1}$$

[3]Quoted after L. Solomon's review in MR of [2]

Since ad X_α is a nilpotent endomorphism of $\mathfrak{g}$, the sum (1) (interpreted as an element of End $\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[t]]$) is actually a polynomial in t. Moreover, properties of Chevalley bases guarantee that the matrix entries of $x_\alpha(t)$ with respect to the Chevalley basis are polynomials with integral coefficients. Thus, given any field K and any $t \in K$, one can substitute this t in (1) to get an endomorphism of the Lie algebra $\mathcal{L} \otimes_{\mathbb{Z}} K$. One easily verifies that $x_\alpha(t_1)x_\alpha(t_2) = x_\alpha(t_1 + t_2)$, so $\{x_\alpha(t) \mid t \in K\}$ is a 1-parameter group of automorphisms of $\mathcal{L} \otimes_{\mathbb{Z}} K$. Chevalley defines the group G associated with $\mathfrak{g}$ and K as the subgroup of $\mathrm{Aut}(\mathfrak{g} \otimes_{\mathbb{Z}} K)$ generated by $x_\alpha(t)$ for all $t \in K$ and $\alpha \in R$. Clearly, the transformations $x_\alpha(t)$ are unipotent, so the Chevalley groups are *by definition* generated by unipotent elements. Moreover, it turns out that these elements satisfy relatively simple relations which depend on the root system and generalize the usual commutator relations $[e_{ij}(a), e_{jk}(b)] = e_{ij}(ab)$ for elementary matrices provided that $i \neq k$. By manipulating with these elements using these relations, Chevalley was able to prove that G is a simple group provided that K is not "too small."

From our perspective, it is important to understand whether Chevalley groups are in fact groups of rational points of some algebraic groups. The answer is "almost." The thing is that the above construction presents Chevalley groups as subgroups of (the identity component of) the automorphism group of the Lie algebra which is an adjoint algebraic group (an antipode of a simply connected group). But as we pointed out in §1, for a nonsimply connected group, the group of rational points typically contains proper normal subgroups, indicating that the corresponding Chevalley group is strictly smaller than the group of rational points. However, Chevalley later adapted his construction by employing a representation with maximal possible lattice of weights in place of the adjoint representation, and the Chevalley groups obtained by this procedure are precisely the groups of K-points of simply connected algebraic groups which can, in fact, be obtained via the Chevalley construction applied to the algebraic closure of K. It should be added that such an algebraic group $\mathbf{G}$ contains a connected subgroup $\mathbf{T}$ whose Lie algebra is obtained from the original Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This subgroup turns out to be a maximal K-torus which moreover is diagonalizable over K. Such a torus is said to be *split* over K, and then the group $\mathbf{G}$ itself is also said to be split (i.e. possessing such a maximal torus). Conversely, one proves that the group of rational points of any simple simply connected split algebraic group can be obtained via the Chevalley construction. For this reason, we often do not make any distinction between Chevalley groups and (the groups of rational points of) simple simply connected split groups (for example, $SL_n(K)$ is the simply connected Chevalley group of type A_{n-1} over K).

The class of Chevalley groups, however, does not contain all of the simple groups constructed by Dickson, even over finite fields (among those left out are, for example, special unitary groups). To include the rest, Steinberg in 1959 proposed the following addition to Chevalley's construction. Suppose the Dynkin diagram of $\mathfrak{g}$ has an automorphism (symmetry) $\alpha \mapsto \bar{\alpha}$, the order of which is 2 for the types A_n ($n \geqslant 2$), D_n ($n \geqslant 5$), or E_6, and 2 or 3 for the type D_4, and let L/K be a cyclic extension of degree equal to the order of the symmetry and σ be a generator of the Galois group $\mathrm{Gal}(L/K)$. Then the corresponding (simply connected) Chevalley group G over L has the automorphism $\bar{\sigma}$ such that $x_\alpha(t) \mapsto x_{\bar{\alpha}}(\sigma(t))$ for all $t \in L$ and all simple roots α. Steinberg proposed to consider the group $G_{\bar{\sigma}}$ of fixed points of $\bar{\sigma}$, which is said to be obtained from G by twisting using the symmetry $^-$ of the Dynkin diagram and the field automorphism σ (when identifying the type of $G_{\bar{\sigma}}$, one appends the order of $^-$ to the Lie type of G, e.g. ${}^2A_n$). Steinberg proved for $G_{\bar{\sigma}}$ analogs of most of the fundamental results discovered by Chevalley for Chevalley groups. In particular, $G_{\bar{\sigma}}$ is generated by unipotent subgroups, which, however, need not be 1-dimensional, and is projectively simple provided that K has enough elements.

It is important to emphasize that the family of Chevalley groups and their twisted analogs over finite fields already contain *all* projectively simple finite groups found by Dickson (cf. [17], §11). For example, if σ is a nontrivial automorphism of the quadratic extension $F_{p^{2n}}/F_{p^n}$, then the corresponding automorphism $\bar{\sigma}$ of $G = SL_m(F_{p^{2n}})$ is given by

$$\bar{\sigma}((x_{ij})) = f^{-1}(\sigma(x_{ji}))^{-1}f, \quad \text{where} \quad f = \begin{pmatrix} 0 & \dots & 0 & 1 \\ & & \cdot & \\ & \cdot & & \\ 1 & 0 & \dots & 0 \end{pmatrix},$$

so $G_{\bar{\sigma}} = SU_m(f)$ (thus, the latter has type ${}^2A_{m-1}$). In terms of the theory of algebaic groups, $G_{\bar{\sigma}}$ can be realized as the group of K-points of a simple simply connected algebraic K-group which is quasi-split over K, i.e. possesses a Borel subgroup defined over K. In fact, just as Chevalley groups are linked to split groups, their twisted analogs are linked to nonsplit quasi-split groups, with the sole exception of quasi-split groups of type ${}^6D_4$ that require twisting not by a single automorphism, but by the whole group of symmetries of the Dynkin diagram of type D_4, which is isomorphic to S_3. The latter case, however, cannot occur over finite fields as there are no Galois extensions with Galois group S_3. On the other hand, according to a theorem due to Lang, every algebraic group over a finite field is quasi-split, so the Chevalley and Steinberg constructions account, in fact, for the groups of rational points of all simple simply connected groups, providing thereby a uniform proof of simplicity in this case.

For an infinite field K, however, there are typically various simple K-groups that are not split or quasi-split. For example, given a central division algebra D over K of (finite) degree d, for any $m \geqslant 1$, the reduced norm $\mathrm{Nrd}_{A/K}\colon A \to K$ on the simple algebra $A = M_m(D)$ is represented by a homogeneous polynomial of degree d, with coefficients in K, in terms of the coefficients of an element (e.g. if D is the algebra of Hamiltonian quaternions with the standard basis 1, i, j, k, then $\mathrm{Nrd}_{D/K}(a + bi + cj + dk) = a^2 + b^2 + c^2 + d^2$); then the equation $\mathrm{Nrd}_{A/K}(x) = 1$ defines an algebraic K-group, denoted $\mathbf{SL}_{1,A}$ or $\mathbf{SL}_{m,D}$, whose group of K-rational points coincides with the group $SL_1(A)$ (also denoted $SL_m(D)$) of elements of A with reduced norm 1. Over the algebraic closure of K, this group becomes isomorphic to $\mathbf{SL}_{md}$ (in other word, it is a K-form of $\mathbf{SL}_{md}$), hence simple and simply connected. If $d > 1$, it is not split or quasi-split; moreover, if $m = 1$, it is even anisotropic. (We recall that a simple algebraic K-group G is said to be *anisotropic* over K if $G(K)$ does not contain any nontrivial unipotent element, otherwise G is said to be *isotropic* over K; the latter is equivalent to the existence of a K-split torus in G of positive dimension, and the dimension of such a maximal torus is called the K-rank of G.) Other examples of nonsplit and nonquasi-split (resp., anisotropic) groups can be obtained by considering special orthogonal groups $\mathbf{SO}_n(f)$ (or the corresponding spinor groups) such that the Witt index of the quadratic form f is $< \left[\dfrac{n-1}{2}\right]$ (resp., is equal to zero[4]). Combination of these two examples leads one to consider finite dimensional division algebras with an involution and the (special) unitary

[4]This is equivalent to the fact that f does not represent zero, i.e. the equation $f(x) = 0$ has only zero solution, in which case f is said to be anisotropic

groups of Hermitian or skew-Hermitian forms on finite dimensional vector spaces over these algebras – it turns out that these groups (together with groups of the form $\mathbf{SL}_{m,D}$) cover all possible K-forms of simple algebraic groups of classical types A_n, B_n, C_n and D_n (excluding triality forms ${}^3D_4$ and ${}^6D_4$). We observe that nonsplit or even anisotropic K-forms typically exist also for groups of exceptional types. Thus, over a general field K there may be a whole variety of simple K-groups for which the groups of K-rational points are not covered by the results of either Chevalley or Steinberg. On the other hand, as we discussed in §1, simplicity theorems for large families of isotropic classical groups over not necessarily finite dimensional division algebras (more precisely, for their subgroups generated by certain unipotent elements) were already obtained by Dieudonné and other authors by special examination of each type (cf. [6]); some exceptional groups were considered by Freudenthal and Jacobson. So, the question was whether these results (at least, in the finite dimensional case) can be obtained by general methods; in other words, whether the breakthrough achieved by Chevalley and continued by Steinberg could be completed to include all simple isotropic algebraic groups? (It was clear from the beginning that anisotropic groups require special consideration, cf. next section). This task was accomplished by J. Tits in his celebrated paper [18], the title of which we borrowed for the present article as it reflects most accurately the nature of the problems being discussed here.

Given a simple isotropic algebraic group $\mathbf{G}$ over a field K, we let $\mathbf{G}(K)^+$ denote the (normal) subgroup of $\mathbf{G}(K)$ generated by those unipotent elements that are contained in the unipotent radical of a parabolic K-subgroup of $\mathbf{G}$. It is known that if the field K is perfect (in particular, if $\mathrm{char}\, K = 0$), then every unipotent element of $\mathbf{G}(K)$ is contained in the unipotent radical of some parabolic K-subgroup, so $\mathbf{G}(K)^+$ is simply the subgroup generated by *all* unipotents in $\mathbf{G}(K)$. For a nonperfect field K, the group $\mathbf{G}(K)$ may possess unipotents other than those contained in the unipotent radical of some parabolic K-subgroup, but it is still an open question whether $\mathbf{G}(K)^+$ coincides with the subgroup generated by all unipotents.

Theorem 1. (Tits) *Suppose that K contains at least four elements. Then any subgroup of $\mathbf{G}(K)$ which is normalized by $\mathbf{G}(K)^+$ is either central in $\mathbf{G}$, or it contains $\mathbf{G}(K)^+$. In particular, the group $\mathbf{G}(K)^+$ is projectively simple.*

Tits's proof of this theorem was based on the fundamental idea of recasting Chevalley's simplicity argument in a purely group-theoretic way using the new notion of a BN-pair. Tits defined a BN-pair in an abstract group G as a pair of subgroups B and N generating G and such that $B \cap N$ is normal in N, and the quotient $N/(B \cap N)$ is generated by a set of involutions satisfying two simple conditions. First, Tits proves a general simplicity criterion for a group with a BN-pair which to some extent was inspired by the proof of the simplicity of $PSL_n(K)$ given by K. Iwasawa (1941), cf. [7], §§2.2B-C. Then, using results of his joint paper with A. Borel *Groupes réductifs,* Tits constructed a BN-pair in the group of rational points $\mathbf{G}(K)$ of an arbitrary simple isotropic K-group and showed that if $\mid K \mid \geqslant 4$, this BN-pair satisfies the assumptions of his simplicity criterion, completing the proof of the above theorem. (Without getting into technical details, we just indicate that one takes $B = \mathbf{P}(K)$ and $N = \mathbf{N}(K)$, where $\mathbf{P}$ is a minimal parabolic K-subgroup containing a fixed maximal K-split torus $\mathbf{S}$, and $\mathbf{N}$ is the normalizer of $\mathbf{S}$ in $\mathbf{G}$. Then $\mathbf{P} \cap \mathbf{N}$ is the centralizer of $\mathbf{S}$, so $N/(B \cap N) = \mathbf{N}/(\mathbf{P} \cap \mathbf{N})$ is the corresponding Weyl group, and the involutions appearing in the definition of a BN-pair are precisely the reflections in simple roots of the root system of $\mathbf{G}$ with respect to $\mathbf{S}$ corresponding to the ordering associated with $\mathbf{P}$. For example, if $\mathbf{G} = \mathbf{SL}_n$, one can take for B and N the subgroups of upper triangular and monomial matrices, respectively.) It should be mentioned that Tits's simplicity criterion applies to certain groups other than the groups of rational points of algebraic groups: in his paper, Tits proves with its help the simplicity of the Ree group of type F_4 over the field of two elements (Ree groups are obtained from Chevalley groups of type F_4 and G_2 over a finite field of characteristic 2 and 3 respectively as fixed subgroups of automorphisms similar to those considered by Steinberg except that instead of a symmetry of the Dynkin diagram one uses the correspondence that switches short and long roots; such automorphisms are "nonalgebraic," so the fixed subgroup does not correspond to any algebraic group; for the sake of completeness, we mention also Suzuki groups which are obtained in a similar way from Chevalley groups of type $B_2 = C_2$ over a finite field of characteristic 2).

In view of Tits's theorem, the question about projective simplicity of $\mathbf{G}(K)$, where $\mathbf{G}$ is K-isotropic, reduces to the question whether $\mathbf{G}(K)$ coincides with $\mathbf{G}(K)^+$. In [18], Tits stated a conjecture that $\mathbf{G}(K) = \mathbf{G}(K)^+$ for any simply connected K-simple isotropic group $\mathbf{G}$ over an arbitrary field K. He mentions that this conjecture was suggested to him by M. Kneser, so it became known as *the Kneser-Tits conjecture.* Of course, the results of Chevalley and Steinberg imply the truth of the Kneser-Tits conjecture for split and quasi-split groups (the case of quasi-split groups of type ${}^6D_4$, not considered by Steinberg, was worked out by Tits). Thus, the Kneser-Tits conjecture is always true over an algebraically closed field K (in particular, over the field $\mathbb{C}$ of complex numbers), and hence $\mathbf{G}(K)$ is projectively simple as an abstract group. It was shown by E. Cartan that $\mathbf{G}(K)$ is projectively simple for any (not necessarily isotropic) simple simply connected algebraic group over $K = \mathbf{R}$, implying, in particular, the truth of the Kneser-Tits conjecture over the reals (cf. [12], §7.2). In the general case, however, the Kneser-Tits conjecture turned out to be a very complicated problem. The difficulties are caused by a subgroup, called the anisotropic kernel, which is typically related to some noncommutative division algebras, and the properties of these algebras required to prove the Kneser-Tits conjecture turned out to be equivalent to some long-standing algebraic problems. As an example, let us consider the group $\mathbf{G} = \mathbf{SL}_{m,D}$, where $m > 1$ and D is a finite dimensional central division algebra over K. Then $\mathbf{G}(K) = SL_m(D)$ and $\mathbf{G}(K)^+ = SL_m^+(D)$ in the above notations. So, the Kneser-Tits conjecture in this case is equivalent to the question whether $SL_m(D) = SL_m^+(D)$; in other words, whether the reduced norm on $GL_m(D)$ coincides with the Dieudonné determinant. Since the image of $SL_m(D)$ under the Dieudonné determinant is precisely $SL_1(D)/[D^*, D^*]$, the latter amounts to the question if $SL_1(D) = [D^*, D^*]$, posed in 1943 by Artin and Tannaka. From the perspective of algebraic K-theory, the quotient $SL_1(D)/[D^*, D^*]$ is isomorphic to the reduced Whitehead group $SK_1(D)$, so yet another reformulation of these problems, given in Bass's book *Algebraic K-theory,* is whether the group $SK_1(D)$ is trivial for all finite dimensional division algebras. For a long time since its formulation, the only cases for which the Tannaka-Artin problem was settled were for division algebras over p-adic fields, i.e. finite extensions of the field of p-adic numbers $\mathbb{Q}_p$ (Nakayama-Matsushima, 1943) and number fields (S. Wang, 1950). The answer in these cases was in the affirmative and the proofs relied heavily on the arithmetic properties of these fields, primarily on the fact that the reduced norm is surjective over p-adic and "almost" surjective over number fields, and could not be extended to more general fields. On the other hand, these and other results for groups of classical types have created necessary prerequisites to attack the Kneser-Tits conjecture for general algebraic groups over these fields.

In 1969, V.P. Platonov [10] proved the Kneser-Tits conjecture for all groups over p-adic fields. So, for a simple simply connected isotropic algebraic group $\mathbf{G}$ over a p-adic field K, the group $\mathbf{G}(K)$ is projectively simple. Platonov used this fact to prove the strong approximation property for simply connected groups over number fields. It also has other important applications, for example, in the analysis of the congruence subgroup problem. Platonov's proof of the Kneser-Tits conjecture for groups over p-adic fields employed the classification of simple groups over these fields and an ingenious procedure for reducing the problem to a subgroup of smaller rank by deleting certain vertices in the Dynkin diagram. Prasad and Raghunathan (1985) reduced the proof of the Kneser-Tits conjecture to groups of K-rank one, for an arbitrary field K. This reduction allows one to give a short proof of the Kneser-Tits conjecture over p-adic fields, since over such fields all groups of K-rank one belong to classical types, and for groups of classical types over p-adic fields the truth of the Kneser-Tits conjecture has long been known. The case of number fields is more complicated as here there exist K-rank one groups belonging to the exceptional types ${}^{3,6}D_4$, ${}^2E_6$ and F_4. For isotropic groups of type F_4 over an arbitrary field the Kneser-Tits conjecture is known to hold. It was observed by Prasad and Raghunathan (unpublished) that the Kneser-Tits conjecture for rank one groups of these types can be reduced to the Margulis-Platonov conjecture (abbreviated (MP), see next section) for anisotropic groups of types A_1 and ${}^2A_3$, respectively. As (MP) for groups of type A_1 has been proven, the Kneser-Tits conjecture for type ${}^{3,6}D_4$ over number fields follows, however (MP) for type ${}^2A_3$ and consequently the Kneser-Tits conjecture for the only rank one form of type ${}^2E_6$ remain open.

In addition to the results for special fields described above, the Kneser-Tits conjecture has been confirmed for groups of types B_n, C_n, F_4 and some other exceptional groups over arbitrary fields (see [19] for details), which lead to the expectation that the answer should be affirmative in the general case. So, it came as a surprise when in 1975 Platonov found examples of division algebras D with nontrivial $SK_1(D)$. The simplest example of such an algebra can be constructed as follows. Let $K = \mathbb{Q}(x, y)$, where x and y are indeterminates, and let D_1 and D_2 be the algebras of generalized quaternions over K corresponding to the pairs $(2, x)$ and $(3, y)$; then $D = D_1 \otimes_K D_2$ is a division algebra, and $SK_1(D) \neq 1$. By exploring this construction, Platonov was able to show that any countable abelian group of finite exponent can be realized as $SK_1(D)$ for an appropriate division algebra D (cf. his ICM-78 talk [11]). Later, similar counterexamples to the Kneser-Tits conjecture were found for types ${}^2A_n$ and D_n. To the best of our knowledge, exceptional types have not been investigated systematically.

In all known cases, the quotient $\mathbf{G}(K)/\mathbf{G}(K)^+$, which Tits [19] called the Whitehead group of the algebraic group $\mathbf{G}$ over K, is abelian and has finite exponent (for example, the group $SK_1(D)$ is such), so it would be interesting to find out if this is true in general. Another interesting question is whether this quotient is always finite if K is finitely generated.

3 The anisotropic case

As we saw in the previous section, for a simple simply connected isotropic algebraic group $\mathbf{G}$ over a field K containing at least four elements, the group $\mathbf{G}(K)$ always contains a "big" (in particular, Zariski dense) normal projectively simple subgroup $\mathbf{G}(K)^+$, with the quotient $\mathbf{G}(K)/\mathbf{G}(K)^+$ typically being an abelian group of finite exponent. Anisotropic groups, however, may exhibit a different behavior. For example, let f be an *anisotropic* quadratic form in three variables over the field of p-adic numbers $\mathbb{Q}_p$, and $\mathbf{G} = \mathbf{SO}_3(f)$. It is known (due to Eichler) that in an appropriate basis of $\mathbb{Q}_p^3$, the group $G(\mathbb{Q}_p)$ is represented by integral p-adic matrices, i.e. $G(\mathbb{Q}_p) \subset GL_3(\mathbb{Z}_p)$. We recall that for any $l \geqslant 1$, the congruence subgroup

$$\Gamma_l = \{X \in GL_m(\mathbb{Z}_p) \mid X \equiv E_3 (\operatorname{mod} p^l)\}$$

is a normal subgroup of $GL_3(\mathbb{Z}_p)$ of finite index. So, letting $N_l = \mathbf{G}(\mathbb{Q}_p) \cap \Gamma_l$, we obtain a family of normal subgroups of $\mathbf{G}(\mathbb{Q}_p)$ such that $\cap_{l=1}^{\infty} N_l = \{E_3\}$; in particular, $\mathbf{G}(\mathbb{Q}_p)$ does not have any infinite projectively simple subgroups.[5] We observe that this particular example, due to Dieudonné (cf. [6], Ch. II, §12) does not rely in any way on the fact that the group $\mathbf{SO}_3(f)$ is not simply connected and has a general nature: if $\mathbf{G}$ is anisotropic over a p-adic field K (a finite extension of $\mathbb{Q}_p$), then the group $\mathbf{G}(K)$ is compact and totally disconnected with respect to the topology induced by that of K, hence possesses a fundamental system of neighborhoods of the identity consisting

[5] In fact, a bit more thorough analysis shows that the finite subgroups of $\mathbf{G}(\mathbb{Q}_p)$ are all solvable.

of open normal subgroups, which prevents $\mathbf{G}(K)$ from containing any infinite projectively simple subgroups. On the other hand, the supply of anisotropic groups over a p-adic field K is rather limited: these are precisely the groups of the form $\mathbf{SL}_{1,D}$, where D is a finite dimensional division algebra over K, in particular, they are all of type A_n (we notice that the group $\mathbf{SO}_3(f)$ in the above example is of type $B_1 = A_1$ and that anisotropic quadratic forms do not exist in dimensions $\geqslant 5$). It is helpful to compare the p-adic situation with the situation over $\mathbb{R}$: in the latter case, for any simple $\mathbb{R}$-anisotropic group $\mathbf{G}$, the group $\mathbf{G}(\mathbb{R})$ is projectively simple (E. Cartan), and anisotropic forms exist for all Lie types. We see that the supply and properties of anisotropic groups significantly vary for different fields. For a long time there was not even a conjectural description of normal subgroups of the groups of rational points of anisotropic groups over arbitrary fields (we will propose such a conjecture at the end of this section), and as a consequence, the entire effort was concentrated on the study of anisotropic groups over well-understood fields.

As a partial replacement for a simplicity theorem for anisotropic groups over p-adic fields, one proves using methods of Lie theory that in this situation every noncentral normal subgroup N of $\mathbf{G}(K)$ is *open* with respect to the topology given by the valuation on K that extends the usual p-adic valuation on $\mathbb{Q}_p$, hence is of finite index. Furthermore, using the description of $\mathbf{G}$ as $\mathbf{SL}_{1,D}$, where D is a finite dimensional central division algebra over K, one finds that $\mathbf{G}(K)$ is an extension of a pro-p group by a finite cyclic group, and all noncentral normal subgroups of $\mathbf{G}(K)$ can be precisely determined (C. Riehm). Combining this with the results of the previous section, we see that in the cases where a simple simply connected algebraic group $\mathbf{G}$ over a p-adic field K is respectively isotropic and anisotropic, we have the following two mutually exclusive possibilities: $\mathbf{G}(K)$ is either projectively simple, or residually finite with all noncentral normal subgroups being open and of finite index.

In terms of complexity of the ground field, the next case to consider should be the case of a number field K.[6] The first thing one notices here is local obstructions to simplicity: if there is a nonarchimedean place v of K such

[6]Most of the material below extends to global fields of positive characteristic without any substantial changes, however to avoid technical details we will limit our exposition to the case of number fields.

that the group $\mathbf{G}(K_v)$ has a proper noncentral normal subgroup, where K_v is the completion of K with respect to v,[7] then as we have seen above, $\mathbf{G}(K_v)$ is residually finite, so $\mathbf{G}(K)$ is also residually finite, and therefore it can not be projectively simple. In his ICM-74 talk, V.P. Platonov conjectured that there are no other obstructions to the projective simplicity; in other words, the following local-global principle holds: $\mathbf{G}(K)$ *is projectively simple if and only if the local groups* $\mathbf{G}(K_v)$ *are projectively simple for all places* v *of* K. However, before this principle was formulated in the context of general simple simply connected groups, M. Kneser (1956) proved that for a quadratic form f over K in $m \geqslant 5$ variables and $\mathbf{G} = \mathbf{Spin}(f)$, the group $\mathbf{G}(K)$ is projectively simple (it was the first result about simplicity that allowed the group under consideration to be anisotropic; we remark that Kneser in fact worked not with the spinor group itself, but with the kernel of the spinor norm in the corresponding orthogonal group, but his result is equivalent to the one stated above). We observe that the local obstructions to simplicity noted above were not explicitly present in Kneser's theorem simply because any quadratic form in $m \geqslant 5$ variables over a p-adic field is isotropic! At the end of his paper, Kneser conjectured that the projective simpliciy of $\mathbf{G}(K)$ should hold also if $m = 3$ and the quadratic form f is isotropic at all nonarchimedean places. Platonov generalized this conjecture of Kneser to arbitrary simple simply connected groups and formulated it in the elegant form of a local-global principle for projective simplicity.

As we already mentioned, a simple group $\mathbf{G}$ of type other than A_n is automatically isotropic at all nonarchimedean places, so Platonov's conjecture for $\mathbf{G}$ of type different from A_n is equivalent to the projective simplicity of $\mathbf{G}(K)$. However, a group $\mathbf{G}$ of type A_n may well be anisotropic at some nonarchimedean places in which case $\mathbf{G}(K)$ is not projectively simple, so one should rather ask for a description of its normal subgroups. To be able to deal with both situations uniformly, Platonov's conjecture was adapted by Margulis, and the resulting conjecture became known as the Margulis-Platonov conjecture (MP).

Conjecture (MP). *Let* $\mathbf{G}$ *be a simple simply connected algebraic group over a global field* K. *Denote by* $\mathcal{A}$ *the set of all nonarchimedean places* v *of* K *such that* $\mathbf{G}$ *is* K_v*-anisotropic. Then for any noncentral normal subgoup*

[7] Of course, v extends the p-adic valuation on $\mathbb{Q}$ for some prime p, and then K_v is a finite extension of $\mathbb{Q}_p$.

$N \subset \mathbf{G}(K)$ there is an open normal subgroup $W \subset \mathbf{G}_{\mathcal{A}} := \prod_{v \in \mathcal{A}} \mathbf{G}(K_v)$ such that $N = \delta^{-1}(W)$, where $\delta \colon \mathbf{G}(K) \to \mathbf{G}_{\mathcal{A}}$ is the diagonal map. In particular, if $\mathcal{A} = \emptyset$ (which is always the case if $\mathbf{G}$ is not of type A_n), then $\mathbf{G}(K)$ is projectively simple.

We notice that the set $\mathcal{A}$ in (MP) is always finite, and the group $\mathbf{G}_{\mathcal{A}}$ is endowed with the product topology. The topology induced on $G(K)$ from that on $\mathbf{G}_{\mathcal{A}}$ in terms of the map δ is sometimes referred to as the $\mathcal{A}$-adic topology. Thus, (MP) is equivalent to the claim that all noncentral normal subgroups of $\mathbf{G}(K)$ are open in the $\mathcal{A}$-adic topology. Since the local group $\mathbf{G}(K_v)$ has proper noncentral normal subgroups only if $v \in \mathcal{A}$, in which case all such subgroups are open, (MP) has the nature of a local-global principle. Although (MP) does not differentiate between isotropic and anisotropic groups, for isotropic $\mathbf{G}$ it just asserts the projective simplicity of $\mathbf{G}(K)$, and hence is equivalent to the Kneser-Tits conjecture. Since the latter has already been established for most groups over global fields, the real focus of (MP) is on anisotropic groups. It should also be noted that while (MP) deals with a precise description of normal subgroups of $\mathbf{G}(K)$, there is a general qualitative result which follows from a theorem of Margulis on lattices and the strong approximation property, according to which for a simple simply connected algebraic group $\mathbf{G}$ defined over a global field K, any noncentral normal subgroup of $\mathbf{G}(K)$ has finite index (cf. [9]).

For more than twenty years, Kneser's theorem remained the only result about projective simplicity of groups of rational points that applied to a class of anisotropic groups. A breakthrough occurred in the late 1970s and 1980s. First, Platonov and Rapinchuk (1978) showed that for $\mathbf{G} = \mathbf{SL}_{1,D}$, where D is a quaternion algebra, the group $\mathbf{G}(K)$ is prefect, i.e. $\mathbf{G}(K) = [\mathbf{G}(K), \mathbf{G}(K)]$, provided that $\mathcal{A} = \emptyset$. (This result is directly related to Kneser's original conjecture as $\mathbf{Spin}(f) \simeq \mathbf{SL}_{1,D}$ for some quaternion algebra D, which in fact is the Clifford algebra of f.) A year later, G.A. Margulis proved (MP) for the groups $\mathbf{SL}_{1,D}$, where D is a quaternion algebra, in full. Subsequently, Platonov and Rapinchuk extended their result from quaternion algebras to division algebras of arbitrary degree and showed that for $\mathbf{G} = \mathbf{SL}_{1,D}$, the commutator subgroup $[\mathbf{G}(K), \mathbf{G}(K)]$ is $\mathcal{A}$-adically open in $\mathbf{G}(K)$, i.e. satisfies (MP). Raghunathan elaborated on this result and showed that

$$\text{if a normal subgroup } N \text{ of } \mathbf{G}(K) \text{ is } \mathcal{A}-\text{adically open, then so is } [N, N]. \tag{1}$$

Tomanov (1991) extended Margulis's result from quaternion algebras to arbitrary algebras of degree 2^d and in fact reduced the problem to algebras of odd index. However, the proof of (MP) for groups of the form $\mathbf{SL}_{1,D}$ required essentially new techniques, which we will discuss a bit later on, and was found only recently.

In the meantime, (MP) was established for most anisotropic groups of type other than A_n. What makes a difference relative to proving projective simplicity is the fact that groups of type different from A_n, at least over number fields, abound in simple simply connected K-subgroups, while groups of type A_n may have very few of these (e.g. $\mathbf{G} = \mathbf{SL}_{1,D}$, where D is a division algebra of prime degree, does not have any connected K-subgroups except for tori). The crucial observation is that a large supply of "nice" K-subgroups creates a possibility for an inductive proof of projective simplicity of $\mathbf{G}(K)$. More precisely, suppose $\{\mathbf{G}_\alpha\}$ is a family of connected K-subgroups of $\mathbf{G}$ such that $\mathbf{G}_\alpha(K)$ is projectively simple for each α, and the $\mathbf{G}_\alpha(K)$'s together generate $\mathbf{G}(K)$. Now, if N is a noncentral normal subgroup of $\mathbf{G}(K)$, then it has finite index by Margulis's theorem, implying that $N \cap \mathbf{G}_\alpha(K)$ is a noncentral normal subgroup of $\mathbf{G}_\alpha(K)$, for each α. By the projective simplicity of $\mathbf{G}_\alpha(K)$, one concludes that $\mathbf{G}_\alpha(K) \subset N$, and therefore $N = \mathbf{G}(K)$, proving the projective simplicity of $\mathbf{G}(K)$. As we will see below, the required K-subgroups can be constructed using geometric considerations, and structural information for groups of exceptional types.

For classical groups this idea, which is different from the approach employed by Kneser, was put forward M. Borovoi. He used it to prove the projective simplicity of anisotropic groups of type C_n ($n \geqslant 2$), quaternionic groups of type D_n ($n \geqslant 4$) and the special unitary group $\mathbf{SU}(f)$, where f is a hermitian form in $m \geqslant 3$ variables over a quadratic extension L/K, belonging to type ${}^2A_{m-1}$. However, to keep the exposition as simple as possible, we will demonstrate the method using the group $\mathbf{G} = \mathbf{Spin}(f)$ where f is a nondegenerate quadratic form in $n \geqslant 6$ variables over K. We consider the natural n-dimensional representation of $\mathbf{G}$, and for $x \in K^n$, we let $\mathbf{G}_x$ denote the stabilizer of x in $\mathbf{G}$. Then one can pick anisotropic vectors $x, y \in K^n$ so that

$$\mathbf{G}(K) = \mathbf{G}_x(K)\mathbf{G}_y(K)\mathbf{G}_x(K). \tag{2}$$

Without getting into details, we mention that this decomposition is obtained by using Witt's theorem in conjunction with the fact that f restricted to the orthogonal complement of the 2-dimensional space spanned by x and y represents "almost all" elements of K, this is a consequence of the Hasse-Minkowski theorem. However, $\mathbf{G}_x$ and $\mathbf{G}_y$ are the spinor groups of quadratic forms in $n-1 \geqslant 5$ variables, and we may assume by induction that the projective simplicity of $\mathbf{G}_x(K)$ and $\mathbf{G}_y(K)$ has already been established. Then (2) shows that the subgroups $\mathbf{G}_x$ and $\mathbf{G}_y$ are as required. Obviously, this argument allows one to reduce the general case to the case $n = 5$. Borovoi used it to reduce the case of B_n to B_2. But projective simplicity of $\mathbf{G}(K)$ for $\mathbf{G}$ of type B_2 follows from the result of Kneser since $\mathbf{G}$ in this case is the spinor group of a quadratic form in 5 variables. Similarly, the case of $\mathbf{SU}(f)$, f a hermitian form in m variables, can be reduced to the case $m = 3$, but the descent from $\mathbf{SU}_3$ to $\mathbf{SU}_2$ (which is of type A_1, hence isomorphic to $\mathbf{SL}_{1,D}$ for some quaternion algebra D and therefore satisfying (MP) by Margulis's theorem), required more advanced arithmetic techniques than the Hasse-Minkowski theorem. For type D_n, one again easily reduces the general case to the case of groups of type D_4, which then needs to be reduced to those groups of type $D_3 = A_3$ for which we already know projective simplicity (this requires results about skew-hermitian forms over quaternion algebras). Adopting the idea of Borovoi, Platonov and Rapinchuk found a general argument which allows one to consider all groups of types B_n ($n \geqslant 2$), C_n ($n \geqslant 2$), D_n ($n \geqslant 4$, except ${}^{3,6}D_4$), and G_2 simultaneously by way of eventual reduction to groups of type A_1 and requires very few arithmetic techniques beyond the Hasse-Minkowski theorem. Independently, Tomanov (and B. Sury) showed that the original argument of Kneser can be extended to groups of type D_n, and also considered *all* groups of type $D_3 = A_3$.

For exceptional types E_7, E_8 and F_4, the required K-subgroups were constructed by V. Chernousov, proving thereby the projective simplicity of $\mathbf{G}(K)$ for these types (type F_4 was independently considered by Tomanov using a geometric argument). Chernousov's argument was based on the fact that a simple group $\mathbf{G}$ of one of these types, defined over a global field K, splits over a suitable quadratic extension L/K, and therefore contains a maximal K-torus $\mathbf{T}$ that splits over L. We can assume in addition that $\mathbf{G}$ is K-anisotropic, and then the 3-dimensional subgroups of $\mathbf{G}$ corresponding to the roots of $\mathbf{G}$ with respect to $\mathbf{T}$ are defined over K (this observation was originally made and used by B. Weisfeiler). One can obtain a sufficiently large number of K-subgroups by taking several roots at a time and considering the subgroup generated by the 3-dimensional subgroups corresponding to each root, and also by varying $\mathbf{T}$. Chernousov showed that $\mathbf{G}(K)$ is generated by the group of K-rational points of simple simply connected K-subgroups of this type of (absolute) rank two. Thus, these subgroups split over L and are of type A_2 or B_2. However, for these types we already know that (MP) holds. Hence, the rank two subgroups constructed above are adequate to complete the proof of projective simplicity of $\mathbf{G}(K)$ for groups of type E_7, E_8 and F_4.

Thus, by the late 80s, (MP) had been proven in all cases except anisotropic groups of type A_n, triality forms ${}^{3,6}D_4$, and groups of type E_6 (a detailed exposition of most of these results is contained in Chapter IX of [12]). Among the remaining groups those of type A_n presented the most formidable challenge: it suffices to mention that even for $\mathbf{G} = \mathbf{SL}_{1,D}$, where D is a cubic division algebra, (MP) remained open for quite some time. A proof of (MP) for the groups $\mathbf{SL}_{1,D}$, which are precisely the anisotropic inner forms of type A_n, was obtained only recently. The completely novel techniques used in this proof have generated a feeling that an understanding of normal subgroups of $\mathbf{G}(K)$, for an arbitrary field K, may not be far off. In fact, it was only due to a formulation of the problem in the context of arbitrary fields that (MP) could be settled for anisotropic groups of inner type A_n. First, Potapchik and Rapinchuk showed that (MP) for $\mathbf{G} = \mathbf{SL}_{1,D}$, where D is a finite dimensional central division over a global field K, is equivalent to the fact that the multiplicative group D^* does not have a nonabelian finite simple group as a quotient. Unlike the original statement of (MP), this reformulation does not explicitly involve any arithmetic attributes of the field K (such as valuations) and therefore makes perfect sense over arbitrary fields, and Potapchik and Rapinchuk conjectured its truth in this generality. They managed to verify this conjecture for division algebras of degree two (quaternion algebras) and three (cubic algebras). While the argument for quaternion algebras was elementary and short, the case of cubic algebras relied on the classification of finite simple groups, viz. it was shown that all quotients of D^*, where D is a cubic division algebra, have a certain abstract property, which fails for every single finite nonabelian simple group. The proof of the conjecture (of Potapchik and Rapinchuk) for division algebras of arbitrary degrees was obtained by Y. Segev and G. Seitz, which completed the proof of (MP) for the groups of the form $\mathbf{SL}_{1,D}$. Among the new techniques introduced and efficiently used by Segev [15] was the notion of the commuting graph $\Delta(\mathcal{G})$ of a (finite) group $\mathcal{G}$: the vertices of $\Delta(\mathcal{G})$ are in one-to-one correspondence with the nonidentity elements of $\mathcal{G}$, and two vertices are connected if the corresponding elements commute. Segev proved that if $\mathcal{G}$ is a finite simple group such that either diam $\Delta(\mathcal{G}) \geqslant 5$ or diam $\Delta(\mathcal{G}) = 4$ and an additional technical condition is satisfied (in which case Segev says that $\Delta(\mathcal{G})$ is "balanced"), then $\mathcal{G}$ cannot be a quotient of the multiplicative group of a finite dimensional division algebra. Then Segev and Seitz, using the classification of finite simple groups, showed that the commuting graph of any nonabelian finite simple group either has diameter $\geqslant 5$, or is balanced, which completed the argument. Although valuations did not appear in [15] explicitly, some constructions therein were equivalent to constructing ones. In the joint work of Rapinchuk and Segev [13] these ideas were developed to prove the following theorem which is the first congruence subgroup theorem over arbitrary fields.

Theorem 2. *Let D be a finite dimensional division algebra over a finitely generated field, and $N \subset D^*$ be a normal subgroup of finite index. If the commuting graph of the quotient D^*/N has diameter $\geqslant 4$, then N is open in D^* in the topology defined by a nontrivial height one valuation v of D.*

We recall that a height one valuation is a group homomorphism $v \colon D^* \to \mathbb{R}$ satisfying the "triangle inequality" $v(a+b) \geqslant \min\{v(a), v(b)\}$ for all $a, b \in D^*$, $b \neq -a$. Then $\mathcal{O}_v := \{a \in D^* \mid v(a) \geqslant 0\} \cup \{0\}$ is a subring of D, called the valuation of ring v. Furthermore, for any $\varepsilon > 0$, $\mathfrak{m}_v(\varepsilon) := \{a \in D^* \mid v(a) > \varepsilon\} \cup \{0\}$ is a 2-sided ideal of D, with $\mathfrak{m}_v = \mathfrak{m}_v(0)$ being a maximal ideal, so that $D_v = \mathcal{O}_v/\mathfrak{m}_v$ is a division algebra, called the residue algebra of D with respect to v. The ideals $\{\mathfrak{m}_v\}$ form a fundamental system of neighborhoods of zero for the topology on D associated with v. So, the openness of N in Theorem 2 means that N contains the congruence subgroup $1 + \mathfrak{m}_v(\varepsilon)$ for some $\varepsilon > 0$. Thus, Theorem 2 is a version of the congruence theorem for D^*, and in fact the first result of this type over general fields.

We now briefly indicate how this theorem can be used to re-prove Segev's result [15] that nonabelian finite simple groups cannot occur as quotients of the multiplicative group of a finite dimensional division algebra. The only information about finite simple groups needed for this argument is that their commuting graphs have diameter $\geqslant 4$, which is easier to verify than the balance condition. So, let D be a finite dimensional central division algebra D over a field K such that the multiplicative group D^* has a nonabelian finite simple group F as a quotient. Without any loss of generality, one can assume that K is finitely generated. Then using Theorem 2 one concludes that F will also appear as a quotient of the multiplicative group $\bar{D}_v^*$ of the residue algebra. This process of replacing a given division algebra with the residue algebra can be continued, and F will still appear as a quotient of the multiplicative group of each of the resulting algebras. Eventually, however, we obtain a finite dimensional division algebra over a finite field. This algebra is commutative by Wedderburn's theorem, and therefore its multiplicative group cannot have F as a quotient – a contradiction.

Inspired by his theorem, Segev conjectured that finite quotients of the multiplicative group of a finite dimensional division algebra should in fact be solvable, and this conjecture was one of the motivations for proving Theorem 2 in [13]. It turned out however that Theorem 2, which, as we showed above, allows one to eliminate all nonabelian finite simple groups as potential quotients of D^*, fell short of eliminating all finite *nonsolvable* groups. More precisely, there are finite minimal nonsolvable groups for which the diameter of the commuting graph is 3. However, the conclusion of Theorem 2 is no longer true if one weakens the assumption from diam $\geqslant 4$ to diam $\geqslant 3$. Nevertheless, further refinement of these methods, carried out by Rapinchuk, Segev and Seitz [14], resulted in a proof of Segev's conjecture.

Theorem 3. *Let D be a finite dimensional division algebra. Then any finite quotient of D^* is solvable.*

The proof uses a new condition on the commuting graph which is stronger than the "diam $\geqslant 3$" condition, but weaker than the "diam $\geqslant 4$" condition and for this reason is called "condition $(3\frac{1}{2})$." The argument can be divided into two parts: first, it is shown that Theorem 2 remains true under the assumption that D^*/N satisfies condition $(3\frac{1}{2})$, and then, it is verified (using the classification of finite simple groups) that every minimal nonsolvable group satisfies condition $(3\frac{1}{2})$. Finally, using the elimination method described above, one argues that none of the minimal nonsolvable groups can be a quotient of D^*, implying that all finite quotients are solvable. The methods developed to prove Theorem 3 can undoubtedly be used to obtain more precise information about finite quotients of D^* (we recall that finite *subgroups* of D^* were described by S. Amitsur and that Amitsur's list of possible subgroups contains a single nonsolvable group, viz. $SL_2(F_5)$). For this purpose, it would be helpful to extend the congruence subgroup theorem to all quotients D^*/N for which the commuting graph has diameter $\geqslant 3$: as we explained above, in this case N may not be open with respect to a *single* valuation of D, but it seems plausible that N will always be open with respect to an appropriate finite set of valuations of D. If true, this statement would provide an analog of (MP) for the multiplicative group D^* over arbitrary fields as in the number-theoretic situation, the set $\mathcal{A}$ of anisotropic nonarchimedean places for the group $\mathbf{SL}_{1,D}$ over a global field K can be identified with the set of all valuations of D. Since quotients by congruence subgroups have rather specific structure, this would provide information about those nonnilpotent finite groups that can appear as quotients of D^*. It would also allow one to simplify the proof of Theorem 3 as the fact that the commuting graph of any minimal nonsolvable group has diameter $\geqslant 3$ (Segev) is much easier to verify than condition $(3\frac{1}{2})$.

To remain in line with the title of this article, we would like to put these results in the context of general algebraic groups and propose, with a certain amount of trepidation, the following conjecture which came up in our discussions with G. Prasad.

Conjecture. *Let* $\mathbf{G}$ *be a reductive algebraic group over an infinite field* K*. Then any finite quotient of* $\mathbf{G}(K)$ *is solvable.*

We observe that if $\mathbf{G}$ is absolutely simple and K-isotropic, then it follows from Theorem 1 that every finite quotient of $\mathbf{G}(K)$ is in fact a quotient of $\mathbf{G}(K)/\mathbf{G}(K)^+$, and the latter group is known to be abelian for most types. This supports our conjecture. Thus, the most difficult and very little explored case in this conjecture is that of K-anisotropic groups. The only available result here is Theorem 3 for $\mathbf{GL}_{1,D}$, and even the transition to the group $\mathbf{G} = \mathbf{SL}_{1,D}$ presents a problem: of course, it follows from Theorem 3 that for any subgroup of finite index N of $\mathbf{G}(K)$ which is *normalized by* D^*, the quotient $\mathbf{G}(K)/N$ is solvable, but it is by no means obvious that a subgroup M of $\mathbf{G}(K)$ of finite index will always contain a subgroup of finite index normalized by D^*! The latter is equivalent to the fact that among the conjugates $g^{-1}Mg$, $g \in D^*$, there are only finitely many distinct ones, which can be proved if one knew that $\mathbf{G}(K)$ possesses a finitely generated subgroup Γ dense in the profinite topology. One case where this is indeed true is number fields where for Γ one can take an appropriate S-arithmetic subgroup. Over general fields the situation is quite complicated: first of all, the profinite topology needs to be replaced by a weaker topology, but the most problematic part is the absence of any results on finite generation of "arithmetic" subgroups of anisotropic groups over fields more general than global fields. Among a number of natural questions arising in this context, we point out the following: let $\mathbf{G}$ be a reductive algebraic group over a finitely generated field K, and let $\mathcal{O} \subset K$ be a finitely generated subring whose field of fractions is K; is it true that $\mathbf{G}(\mathcal{O})$ is contained in a finitely generated subgroup of $\mathbf{G}(K)$? It may very well be the case that these difficult problems can be bypassed as far as the above question is concerned; at least there is the following easier argument for global fields. Suppose $M \subset \mathbf{G}(K)$ has index d, then the subgroup $N \subset M$ generated by g^d, $g \in \mathbf{G}(K)$, is obviously normalized by D^* and has finite index in $\mathbf{G}(K)$ by Margulis's theorem. In any case, the above conjecture does hold for $\mathbf{G} = \mathbf{SL}_{1,D}$ where D is a finite dimensional central division algebra over a global field K, and as an application of this result we point out, following [14], that this fact allows one to give a quick proof of (MP) for $\mathbf{G}$. Indeed, any noncentral normal subgroup N of $\mathbf{G}(K)$ has finite index, and therefore the quotient $\mathbf{G}(K)/N$ is solvable. This means that N contains some term of the derived series of $\mathbf{G}(K)$. But repeated application of (1) shows that all these terms are $\mathcal{A}$-adically open, and the truth of (MP) for N follows. This example shows that the above conjecture is likely to provide a uniform approach to many results on projective simplicity, including those on the conjecture (MP), and we hope that it will stimulate research in the area in the years to come.

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