Utility of atomic kicked-rotor interferometers for precision measurements

R. A. Horne, R. H. Leonard, and C. A. Sackett
Physics Department, University of Virginia, Charlottesville, Virginia 22904, USA
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We theoretically investigate a proposed scheme to use an atomic δ-kicked rotor resonance for high-precision measurements of accelerations and the photon recoil frequency. Although the technique offers rapid scaling of the measurement sensitivity with pulse number, it also features a high sensitivity to initial atomic momentum. We find that for realistic atom sources, the momentum sensitivity significantly limits the achievable precision. We consider several different variations on the technique, but find similar limitations in all cases.

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I. INTRODUCTION

The atomic δ-kicked rotor (ADKR) consists of a collection of cold atoms subjected to a periodic sequence of impulses from a potential which varies sinusoidally in space. The resulting dynamics are equivalent to those of a rotating body subjected to a sequence of impulsive torques, a system which is classically chaotic. The ADKR has proven very useful for exploring the notion of quantum chaos, and exhibits a wide variety of interesting dynamical features [1–5].

Recently, the ADKR has also been considered for applications in precision measurements, because it exhibits dynamical resonances that are highly sensitive to the photon recoil frequency and to accelerations of the system, two important quantities for practical and fundamental applications [6–9]. The resonance widths can exhibit sub-Fourier scaling, where the width decreases faster than the inverse of the number of applied pulses [10,11]. A particularly attractive proposal by McDowall *et al.* [12] suggested a type of interferometric measurement that exhibits a width scaling as the inverse cube of the pulse number. This rapid scaling was experimentally verified by Talukdar *et al.* [13], and it suggests that high sensitivity could be obtained with a relatively short measurement duration.

We show in this paper, however, that the utility of McDowall's approach is constrained by the sensitivity of the measurement to the initial velocity of the atoms. Even if the atom source is taken to be a condensate with momentum width limited by the uncertainty principle, the achievable measurement resolution for a practical condensate size is not competitive with other techniques. Some indication of this was already observable in the numerical calculations reported in [12].

In an attempt to improve on this result, we consider several variant schemes featuring reduced velocity sensitivity. However, the sensitivity to the recoil frequency and to accelerations is similarly reduced, resulting in no net benefit. We emphasize that these conclusions do not detract from the value of the ADKR for studies of quantum dynamics and chaos. Furthermore, other proposed measurement schemes that use the ADKR may not have the same limitations [14–17].

II. ANALYSIS OF FIDELITY TECHNIQUE

The Hamiltonian for the proposed scheme has the form [12]

$$H = \frac{p^2}{2m} + max + V_{\text{pulse}} \tag{1}$$

with

$$V_{\text{pulse}} = \hbar\phi\cos(2kx)\left[\sum_{n=0}^{N-1}\delta(t-nT) - N\delta(t-NT)\right]. \tag{2}$$

Here x and p are the atomic position and momentum, m is the mass, k sets the spatial period of the pulse potential, ϕ is the potential amplitude, T is the time between pulses, and N specifies the number of pulses. The max term represents a constant acceleration a, which could be either inertial or gravitational [13].

In practice, the sinusoidal potential is implemented using pulses from an off-resonant standing-wave laser with wave number k [2]. The amplitude ϕ is determined by the intensity and frequency of the laser. With careful selection of the laser parameters, the possibility for spontaneous emission during a pulse can typically be made negligible. The laser pulses can be approximated by delta functions so long as the distance moved by the atoms during a pulse is small compared to the spatial period of the potential, or $kvdt \ll 1$ for atom velocity v and pulse duration dt.

Each pulse of the laser causes the atomic wave function to diffract, acquiring momentum kicks of $2n\hbar k$ for integer n. The distribution of momenta p produced by a single pulse is given by Bessel functions [18],

$$|p\rangle \to \sum_{n=-\infty}^{\infty}(-i)^n J_n(\phi)|p + 2n\hbar k\rangle. \tag{3}$$

If the the initial momentum of an atom p_i is zero and the acceleration $a = 0$, then the time evolution between pulses is given by

$$|2n\hbar k\rangle \to e^{-4in^2\omega_r T}|2n\hbar k\rangle, \tag{4}$$

where $\omega_r = \hbar k^2/(2m)$ is the photon recoil frequency. The measurement scheme is based on the fact that for $\omega_r T = q\pi/2$ with integer q, the phases in (4) are all multiples of 2π and thus the final state is identical to the initial state. The net effect of the pulse sequence is therefore the same as if the pulses were all delivered at once. Since the amplitudes of simultaneous pulses are simply summed, the net effect will be that of a pulse of amplitude $\sum\phi - N\phi = 0$, and the atoms will all return to rest at the end of the sequence. The actual fraction of atoms brought back to rest gives the fidelity of the operation, F.

When $\omega_r T \neq q\pi/2$, $p_i \neq 0$, or $a \neq 0$, then the time evolution will be more complicated and F will be less than one.

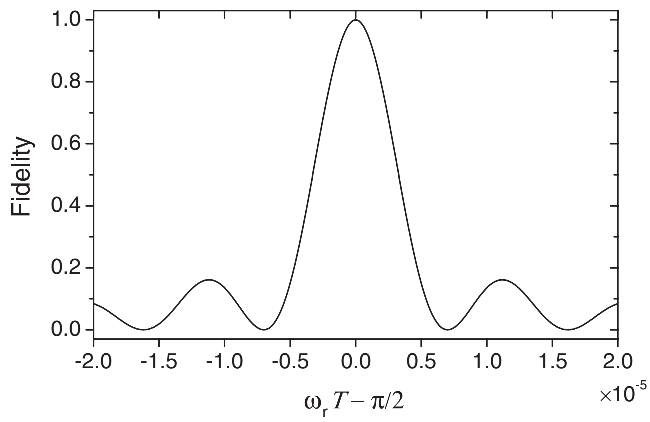

FIG. 1. Sensitivity of fidelity to pulse period. The fidelity is the probability that an atom initially at rest is brought back to rest after application of the Hamiltonian (2). Here T is the pulse period and ω_r is the photon recoil frequency. The calculation uses $N = 50$ pulses with amplitude $\phi = 2$.

Figure 1 shows a representative calculation of the dependence of F on T. The full width at half maximum, ΔT, characterizes the sensitivity to the pulse period and thus to the atomic recoil frequency. Analytical calculations in Refs. [12,13] show that the fidelity widths scale for $N\phi \gg 1$ as

$$\Delta T \to \frac{T_0}{N^3\phi^2}, \tag{5}$$

$$\Delta p_i \to \frac{p_0}{N^2\phi q}, \tag{6}$$

$$\Delta a \to \frac{a_0}{N^3\phi q^2}, \tag{7}$$

with constants $T_0 \approx 3.3\,\omega_r^{-1}$, $p_0 \approx 0.7\hbar k$, and $a_0 \approx 1.4\,\omega_r^2/k$.

The cubic dependence on the pulse number is attractive for precision measurements. However, the usable number of pulses is limited by the sensitivity to the initial momentum. If the atom source has a momentum range δp_i that is large compared to the sensitivity Δp_i, the fidelity will be suppressed and sensitivity to T or a will be lost. In order to avoid this, N, ϕ, and q must be chosen so that $\delta p_i \lesssim \Delta p_i$. For instance, given a pulse number N and period multiple q, the pulse amplitude must satisfy

$$\phi < \frac{1}{N^2 q}\frac{p_0}{\delta p_i}, \tag{8}$$

which in turn implies

$$\Delta T > T_0 N q^2 \left(\frac{\delta p_i}{p_0}\right)^2, \tag{9}$$

$$\Delta a > \frac{a_0}{Nq}\frac{\delta p_i}{p_0}. \tag{10}$$

In the case of time measurements, the best resolution is achieved for $N = q = 1$, where the fractional resolution becomes

$$\frac{\Delta T}{T} \approx 4.3\left(\frac{\delta p_i}{\hbar k}\right)^2. \tag{11}$$

For a cigar-shaped condensate, the largest dimension is typically $\sim$100 μm, which gives $\delta p_i \approx 10^{-3}\hbar k$ for visible light. The optimum $\Delta T/T$ is then of order 10^{-6}, which does not compete with other techniques for determining ω_r that reach 10^{-9} relative precision [8,9].

In the case of acceleration measurements, the sensitivity improves linearly with N and q. However, according to the condition (8), large Nq would require a small ϕ. Over the course of the pulse sequence, the maximum momentum transferred to the atoms is of order $N\phi \times \hbar k$. If $N\phi \ll 1$, then most of the population will remain in the $|p_i\rangle$ state and the fidelity will tend to one, regardless of a. Sensitive operation therefore requires $N\phi$ greater than some $n_{\rm min}$. Numerical calculation indicates that $n_{\rm min} \approx 1.0$ in order to maintain a variation in fidelity of at least 0.5. Together with (8), this gives

$$n_{\rm min} < N\phi < \frac{p_0}{Nq\delta p_i} \tag{12}$$

and thus $Nq < p_0/(n_{\rm min}\delta p_i)$. Applying this constraint to (10) yields

$$\Delta a > n_{\rm min} a_0 \left(\frac{\delta p_i}{p_0}\right)^2. \tag{13}$$

For rubidium atoms with $\delta p_i \approx 10^{-3}\hbar k$, this yields $\Delta a \approx 10^{-5} g$ for earth gravity g. This is again uncompetitive with other techniques [6,7,19].

The physical basis for these constraints can be understood. Typically, the initial state can be characterized by a spatial coherence length $\ell \approx \hbar/\delta p_i$. The first N pulses of the sequence produce a superposition of wave packets with momentum values up to $\sim N\phi\hbar k$. These packets will spread out in space, and once they are separated by more than the coherence length, they will no longer interfere when subjected to further pulses. The measurement technique relies on this interference, and will therefore fail. The spatial spread of the packets is thus limited to a size $\sim\ell$. If each laser pulse provides a momentum kick of order $\phi\hbar k$, then the atoms undergo an average acceleration $\alpha \approx (\phi\hbar k)/(mT)$. The separation of the packets thus grows as $L = \frac{1}{2}\alpha(NT)^2 \approx \frac{1}{2}\phi N^2 T(\hbar k/m)$. Setting $L < \ell$ yields

$$\delta p_i < \frac{\hbar k}{N^2\phi\omega_r T} \tag{14}$$

in agreement with (6).

III. VARIANT PULSE SEQUENCES

We attempted to circumvent this limitation by considering variations on the sequence (2). Since the momentum dependence arises from the inability of the sequence to recombine momentum packets once they spread out, we investigated a scheme that more completely reverses the atom dynamics, using the sequence

$$V_{\rm pulse} = \hbar\phi\cos(2kx)\left[\frac{N}{2}\delta(t) - \sum_{n=1}^{N}\delta(t - nT) + \frac{N}{2}\delta[t - (N+1)T]\right]. \tag{15}$$

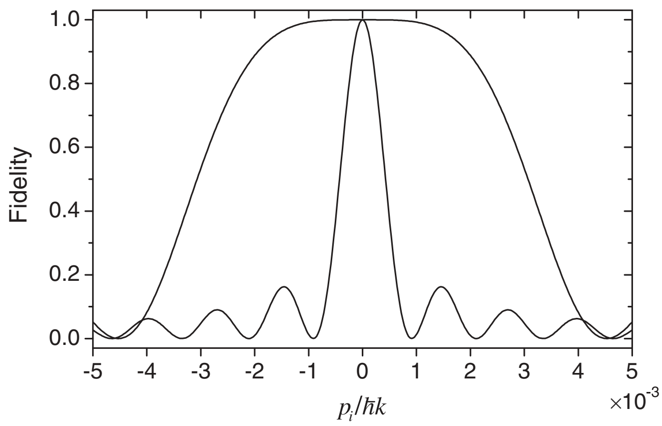

FIG. 2. Fidelity as a function of initial momentum p_i for the original (2) and variant (15) pulse sequences. The broader peak is the variant sequence. Both sequences use $N = 20$, $\phi = 1$, and $q = 2$.

This consists of a sequence as in (2) preceded by its time reversal. We hoped that the second half of the sequence would recombine packets that spread out during the first half.

The additional complexity of the variant sequence makes analytical calculations difficult, so we instead investigated the behavior numerically. Matrix multiplication was used to implement the pulse operator (3), and the time evolution between pulses was evaluated using [20]

$$|2n + \delta\rangle \rightarrow e^{i[(2n+\delta)^2\tau - (2n+\delta)\kappa\tau^2]}|2n + \delta - \kappa\tau\rangle \qquad (16)$$

with dimensionless variables $\delta = p_i/(\hbar k)$, $\tau = \omega_r T$, and $\kappa = ka/(2\omega_r^2)$.

The new sequence does successfully reduce the dependence on the initial momentum, as illustrated in Fig. 2. Unfortunately, this does not result in improved sensitivity to T and a. We find the large-N scaling relations

$$\Delta T \rightarrow \frac{T_0'}{N^3\phi^2}, \qquad (17)$$

$$\Delta p_i \rightarrow \frac{p_0'}{N^{3/2}\phi^{1/2}q}, \qquad (18)$$

$$\Delta a \rightarrow \frac{a_0'}{N^3\phi q^2}, \qquad (19)$$

with $T_0' \approx 12\,\omega_r^{-1}$, $p_0' \approx 1.1\,\hbar k$, and $a_0' \approx 4.8\,\omega_r^2/k$. When the momentum width is taken as a limiting constraint, this yields optimum time and acceleration sensitivities of

$$\Delta T > T_0' N^3 q^4 \left(\frac{\delta p_i}{p_0'}\right)^4, \qquad (20)$$

$$\Delta a > a_0' \left(\frac{\delta p_i}{p_0'}\right)^2. \qquad (21)$$

The T dependence of (20) is interesting. The measurement works best for $N = 2$ and $q = 1$, where we find $\Delta T \approx 150\,\omega_r^{-1}(\delta p_i/\hbar k)^4$. Using $\delta p_i \sim 10^{-3}\hbar k$ suggests that a recoil frequency measurement with a relative precision of order 10^{-10} might be possible. However, achieving this precision requires the use of a very large $\phi \approx (1/8)(\hbar k/\delta p_i)^2$, which excites high momentum states with $mv \sim \phi\hbar k$. Various constraints on the pulse amplitude can be considered, such as spontaneous emission or photoionization. The most significant, however, is that such fast-moving atoms require a short duration dt for the standing wave pulse in order to maintain $kvdt \ll 1$, a condition which can also be expressed as $dt \ll (\phi\omega_r)^{-1}$. Obtaining short pulses is technologically feasible, but they will unavoidably comprise a range of light frequencies ω_L, with $\Delta\omega_L \gtrsim 1/dt$. This in turn implies a range of recoil frequencies, since $\omega_r \propto \omega_L^2$. Thus, as the pulse duration is decreased to permit a high-precision measurement, the recoil frequency itself becomes uncertain. Optimum performance is achieved when the two uncertainties are equal, for which we obtain $\phi \lesssim (\omega_L/10\omega_r)^{1/3}$. For typical atoms, this limits ϕ to about 1000, for which a relative recoil precision of about 10^{-6} is obtained. This is unfortunately again uncompetitive with other techniques.

The acceleration dependence is similar to that of (2), but here fairly good sensitivity can be obtained even for low N. At $N = 2$, $q = 1$, we find an acceleration sensitivity $\Delta a \approx 6\,(\omega_r^2/k)(\delta p_i/\hbar k)^2$. This is not exceptional, but it is obtained using a quite short measurement time of $1.5\pi\omega_r^{-1}$ ($= 200\ \mu$s for Rb atoms). Such a technique could be useful when δp_i is small and rapid measurements are advantageous, either due to a need for fast response times or the possibility of averaging many measurements.

In addition to the variant sequence (15), we also investigated sequences in which all pulses had the same amplitude, specifically

$$V_{\text{pulse}} = \hbar\phi\cos(2kx)\left[\sum_{n=0}^{N-1} - \sum_{n=N}^{2N-1}\right]\delta(t - nT) \qquad (22)$$

and

$$V_{\text{pulse}} = \hbar\phi\cos(2kx)\left[\sum_{n=0}^{N/2-1} - \sum_{n=N/2}^{3N/2-1} + \sum_{n=3N/2}^{2N-1}\right]\delta(t - nT). \qquad (23)$$

These sequences behaved similarly to those of (2) and (15), respectively. They might, however, prove simpler to implement: For large N, the original sequences require intensity modulation with both high dynamic range and high peak amplitude. The more uniform sequences reduce these requirements by a factor of N. In addition, the original sequences are sensitive to fluctuations in the pulse amplitude, since the amplitudes of the large pulses must accurately cancel the sum of the small pulses. By using many small pulses at each phase, amplitude fluctuations tend to average out.

Finally, we considered a sequence of equal-amplitude pulses with alternating phase,

$$V_{\text{pulse}} = \hbar\phi\cos(2kx)\sum_{n=0}^{N-1}(-1)^n\delta(t - nT). \qquad (24)$$

We hoped that the pairs of canceling pulses would allow errors to build up with N, effectively averaging over many two-pulse sequences. Numerical analysis indicates sensitivity scalings of

$$\Delta T \rightarrow \frac{T_0''}{N\phi^2}, \qquad (25)$$

$$\Delta p_i \rightarrow \frac{p_0''}{N\phi q}, \qquad (26)$$

$$\Delta a \to \frac{a_0''}{N^2\phi q^2}, \tag{27}$$

with $T_0'' \approx 2.2\,\omega_r^{-1}$, $p_0'' \approx 0.7\hbar k$, and $a_0'' \approx 2.0\,\omega_r^2/k$. These scalings would be favorable if the measurement functioned for $N\phi > n_{\min}$, as was the case for the previous sequences considered. However, here the alternating phases inhibit the momentum transfer from building up over multiple pulses, so in order to drive the atoms out of the $|p_i\rangle$ state, ϕ itself must be large. We find that $\phi > n_{\min} \approx 1.1$ is necessary to achieve a fidelity variation larger than 0.5. Combining these constraints yields

$$\Delta T > T_0'' N q^2 \left(\frac{\delta p_i}{p_0''}\right)^2, \tag{28}$$

$$\Delta a > a_0'' n_{\min} \left(\frac{\delta p_i}{p_0''}\right)^2, \tag{29}$$

essentially the same as for the original sequence.

IV. CONCLUSIONS

On the basis of these investigations, we conclude that fidelity measurements in an ADKR system are unlikely to be of practical use in precision measurements of acceleration or the photon recoil frequency. This does not detract from their utility in advancing our understanding of quantum dynamical systems. Indeed, further study of them may yield insights on how to circumvent the limitations noted here. Such studies might be facilitated by the reduced momentum sensitivity exhibited in our variant sequences.

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