RESEARCH STATEMENT

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1. Introduction

My research is at the interface of higher category theory and algebraic topology. In particular, I apply higher categorical techniques to equivariant homotopy theory, the study of spaces with a group action using methods from algebraic topology.

In the equivariant setting, analogues of algebraic invariants of spaces such as homotopy and homology groups gain richness and subtlety from the structure of the acting group. Instead of assigning a single group or ring to a space, the invariants instead assign to a G-space a system of objects indexed by the subgroups of G. These systems often have an intricate structure of maps between the objects assigned to the subgroups, reminiscent of the operations of induction and restriction of group representations. The abstract study and formalization of these systems is the primary focus of my work.

The theories of *Mackey functors* and *Tambara functors* describe some of these underlying structures. Let $R(G)$ denote the representation ring of a finite group G. If $K \leq H \leq G$, there are operations of restriction $R^H_K \colon R(H) \to R(K)$ and induction $T^H_K \colon R(K) \to R(H)$, with $T^H_K(V) = \bigoplus_{hK \in H/K} V$. These operations satisfy certain identities such as the Mackey double coset formula, which abstract to the axioms defining R(-) as a Mackey functor for the group G. Replacing the direct sum with a tensor product defines the multiplicative induction N^H_K and a multiplicative Mackey functor structure on R(-). The maps T^H_K and N^H_K are additive and multiplicative homomorphisms, respectively.

Refining R(-) to a Tambara functor reveals a twisted distributive law between the two forms of induction. Tambara's insight in [15] was to encode inductive distributivity by a construction in the category of finite G-sets. With regard to the running example, let $K_G(X)$ denote the Grothendieck ring of G-equivariant complex vector bundles over a G-set X. Then $K_G(G/H) \cong R(H)$, and if X is the disjoint union of orbits, then $K_G(X)$ is the direct sum of representation rings. Maps of G-sets induce operations of *transfer, norm* and *restriction* between equivariant K-theory rings, extending the operations defined for R(-) and endowing K_G(-) with the structure of a Tambara functor for G.

If $H \leq G$ and Y is a finite H-set, let $G \times_H Y$ be the induced G-set. Then $K_H(Y) \cong K_G(G \times_H Y)$, and the transfer, norm and restriction for K_H agree with the corresponding operations on K_G. This behavior exemplifies the concept of a *global Tambara functor*, a collection of compatible Tambara functors for all finite groups. The theory of global Mackey functors has been studied, notably in Schwede's book [13], but the various formulations lack a categorical framework for expressing global distributivity. In my thesis [7], I solve this problem: I construct global Tambara functors and initiate a development of a rigorous and comprehensive theory of these structures.

A key point in the construction necessitated the use of quasicategories as a model for ∞-categories, as developed in the work of Lurie [9]. Given a suitable ∞-category $\mathcal{C}$, I construct an ∞-category Bispan($\mathcal{C}$) of *bispan diagrams* in $\mathcal{C}$. I then define a global Tambara functor as a product-preserving functor from the homotopy category of Bispan(FinGpd) to the category of sets, where FinGpd is the bicategory of finite groupoids. A global Tambara functor restricts to a Tambara functor for all finite groups, and variations of my construction recover Tambara's definition for a fixed finite group, as well as models for global commutative ring spectra, global analogues of the equivariant spectra admitting all norm maps as in [6]. Further variations may provide new insights into the theory of ∞-categorical polynomial functors as first considered in [5].

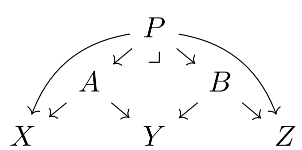
2. Background and context

Before discussing Tambara functors, it is helpful to consider the simpler construction of Mackey functors. Lindner [8] shows that the category of Mackey functors for a finite group G is the ordinary category of product-preserving functors $\mathrm{Span}(\mathrm{Fin}_G) \to \mathrm{Set}$, where $\mathrm{Span}(\mathrm{Fin}_G)$ has objects finite G-sets, and morphisms equivalence classes of *spans* $[X \leftarrow A \to Y]$. The composition is illustrated in diagram (a) below: the resulting span is represented by the curved arrows, defined using an auxiliary pullback (P).

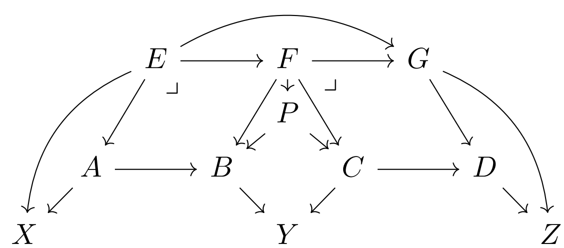
Any morphism $\omega = [X \xleftarrow{f} A \xrightarrow{g} Y]$ in $\mathrm{Span}(\mathrm{Fin}_G)$ factors as a composition $\omega = T_g R_f$ of two spans of a distinct form: $T_g = [X \xleftarrow{=} X \xrightarrow{g} Y]$ and $R_f = [Y \xleftarrow{f} X \xrightarrow{=} X]$. The images of T_g and R_f under a Mackey functor are the *transfer* and *restriction* associated to g and f. While previous constructions of Mackey functors conveniently packaged the axioms governing induction and restriction — most interestingly, the rather complicated Mackey double coset formula is expressed through a simple axiom involving pullback diagrams of G-sets — Lindner's formulation lifts all the axioms to properties of the category $\mathrm{Span}(\mathrm{Fin}_G)$ and presheaves on (the opposite of) this category.

First defined in [15], a *Tambara functor* for a group G is a product-preserving functor from $\mathrm{Bispan}(\mathrm{Fin}_G)$ to sets, where the objects of $\mathrm{Bispan}(\mathrm{Fin}_G)$ are finite G-sets and the morphisms are equivalence classes of *bispan diagrams* of G-sets. A morphism from X to Y is an equivalence class $\omega = [X \xleftarrow{f} A \xrightarrow{g} B \xrightarrow{h} Y]$, which can be written as a composition $\omega = T_h N_g R_f$ of three bispans of special form: $T_h = [B \xleftarrow{=} B \xrightarrow{=} B \xrightarrow{h} Y]$, $N_g = [A \xleftarrow{=} A \xrightarrow{g} B \xrightarrow{=} B]$ and $R_f = [X \xleftarrow{f} A \xrightarrow{=} A \xrightarrow{=} A]$ (*transfer*, *norm*, *restriction*). The composition of two bispans in canonical TNR form, shown in diagram (b), is a three step construction, involving a pullback (P), then an exponential diagram (pentagon $PCDGF$, see §3.1) and another pullback (E).

(a)



(b)



When taking bispans in ordinary categories, showing that the composition is well-defined with respect to the equivalence relation, as well as associative and unital, is onerous, but manageable due to the use of universal properties to construct the composite diagrams. These direct methods do not work when attempting to construct a theory of bispans in higher categories, which my work provides.

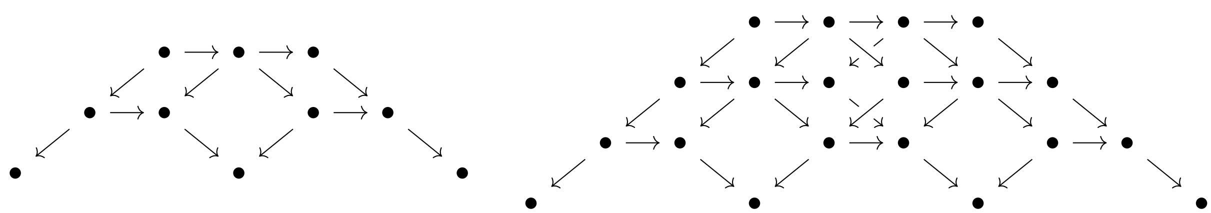
There are a number of constructions of global Mackey functors. One approach is via bisets as in [13], while another is to consider the ∞-category of admissible spans of finite groupoids, as in [2, 10], where the legs of the spans inducing the transfers must be *discrete fibrations*, or coverings, of groupoids.

My approach to global Tambara functors is via bispans of finite groupoids. There are a number of predecessors: [11], drawing influence from biset functors, and [5] which develops a theory of polynomial functors in spaces. In [16], a tricategory of bispans in a *strict* 2-category is constructed, but composition in the bicategory of finite groupoids is only well-defined up to natural isomorphism, and applying the construction of [16] leads into a thicket of homotopy coherence issues. In his thesis [4], supervised by Strickland, author of [14], Cranch constructs an ∞-category of bispans in the category of finite sets, and this is the construction I generalize, obtaining an ∞-category of bispans in any suitable ∞-category. Specializing to the bicategory of finite groupoids attains the desired definition of global Tambara functors.

3. My work

3.1. **Overview.** The bicategory FinGpd of finite groupoids is a convenient setting for global constructions. Every finite groupoid is isomorphic to the *action groupoid* $B_G(X)$ associated to some G-set X, whose objects are the elements of X and morphisms are of the form $(g, x)\colon x \to gx$. A single group takes many guises in FinGpd, for the action groupoid $B_H(H/H)$ is equivalent to $B_G(G/H)$ for any G containing H, and the collapse map $B_G(G/H) \to B_H(H/H)$ is a discrete fibration. Since diagrams in FinGpd only commute up to natural isomorphism, constructing a theory of bispans in FinGpd calls for the use of ∞-categories to efficiently and gracefully handle the data of the large diagrams involved in bispan composition with weakened commutativity assumptions.

The composition of bispans is determined by the existence and properties of certain classes of diagrams. Thus, at this juncture it is reasonable to generalize and consider bispans in a suitable ∞-category $\mathcal{C}$. An ∞-category of bispans in $\mathcal{C}$ consists of a collection of diagrams in $\mathcal{C}$ encoding n-fold composites of bispans as n varies. Let $\mathrm{TNR}(n)$ denote the shape of the diagram encoding an n-fold composite of bispans, so that $\mathrm{TNR}(0)$ is a point and $\mathrm{TNR}(1)$ is a single bispan. The following figure depicts $\mathrm{TNR}(2)$ and $\mathrm{TNR}(3)$:



Observe that $\mathrm{TNR}(2)$ lacks the explicit choice of auxiliary pullback (P) as seen in Figure (b) in §2. This amounts to replacing the exponential diagrams with a similar universal construction tailored to the composition of bispans, the *cromulent diagrams* of [4], discussed and generalized to the setting of ∞-categories in §3.2. These appear in the construction of Theorem 3, from which Theorem 1 follows.

The main theorem of my work is the following, establishing an ∞-categorical bispan construction.

Theorem 1. [7] *Let $\mathcal{C}$ be a suitable ∞-category. There exists an ∞-category* $\mathrm{Bispan}(\mathcal{C})$ *whose n-simplices are diagrams of shape* $\mathrm{TNR}(n)$ *in $\mathcal{C}$.*

Setting $\mathcal{C} = \mathrm{FinGpd}$ and requiring that the legs of the bispan diagrams corresponding to the norms and transfers are discrete fibrations of groupoids makes FinGpd suitable for the theorem. Applying the bispan construction to FinGpd and taking the homotopy category of the ∞-category Bispan(FinGpd) produces an ordinary category ho(Bispan(FinGpd)) and a *global Tambara functor* is a product-preserving functor ho(Bispan(FinGpd)) $\to$ Set. The ordinary category $\mathrm{Bispan}_1(\mathrm{Fin}_G)$ is recovered as $\mathrm{ho}(\mathrm{Bispan}(\mathrm{Fin}_G))$.

Corollary 1. [7] *The action groupoid construction defines a restriction functor from the category of global Tambara functors to the category of Tambara functors for any fixed finite group.*

Corollary 2. [7] *There are two inclusions of the twisted arrow category of the ordinal* $[n]$ *into* $\mathrm{TNR}(n)$*, each inducing a restriction* $\mathrm{Bispan}(\mathcal{C}) \to \mathrm{Span}(\mathcal{C})$*, where* $\mathrm{Span}(\mathcal{C})$ *is the effective Burnside category of* [2]*. As a consequence, global Tambara functors restrict to additive and multiplicative global Mackey functors.*

A more sophisticated direction is to consider *homotopical Tambara functors*, where product-preserving functors of ∞-categories $\mathrm{Bispan}(\mathrm{FinGpd}) \to \mathcal{D}$ are considered for various choices of $\mathcal{D}$, taking full advantage of the homotopy coherent properties of my bispan construction. Natural generalizations beyond Tambara functors in sets may take $\mathcal{D}$ to be the ∞-categories of spaces or spectra.

3.2. **Exponential and cromulent diagrams.** The composition of bispan diagrams involves a universal construction known as an *exponential diagram*, depicted in (c) below, while the ∞-categorical bispan construction uses the related *cromulent diagrams*, depicted in (d).

$$\text{(c)}\quad \begin{array}{ccccc} & \overset{\varepsilon}{\swarrow} & M & \overset{h}{\dashrightarrow} & N \\ & & \downarrow & \lrcorner & \downarrow v \\ A & \xrightarrow[p]{} & X & \xrightarrow[f]{} & Y \end{array} \qquad \text{(d)}\quad \begin{array}{ccccc} A & \leftarrow & M & \dashrightarrow & N \\ p\downarrow & & \downarrow & \lrcorner & \downarrow \\ X & \xleftarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

Exponential diagrams encode the distributivity of multiplicative induction over additive induction. An exponential diagram like (c) above induces an equality $N_f T_p = T_v N_h R_\varepsilon$ of bispans. These diagrams satisfy a universal property: they are terminal amongst all diagrams of the same shape whose bottom row is (p, f) and where the square above f is a pullback square. Cromulent diagrams about (p, f, g) are terminal with respect to the solid arrows and the square above g being a pullback, and arise by pasting pullbacks of p along f to exponential diagrams whose bottom row is the resulting pullback and g. In the following, the *base* of an exponential or cromulent diagram will refer to the subdiagrams on the solid arrows.

The suitability of an ∞-category $\mathcal{C}$ for the bispan construction is conditional on the existence of a right adjoint Π_f to the pullback functor $\Delta_f\colon \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ for certain classes of morphisms in $\mathcal{C}$. The following theorem is crucial, combining the general theory of Lurie [9] and work of Riehl and Verity [12] on adjunctions of ∞-categories, as well as my construction of a slice ∞-category for functors of ∞-categories. The first part establishes the existence of exponential and cromulent diagrams in a rigorous homotopy-coherent context while the second characterizes them in terms of an adjunction.

Theorem 2. [7] *Let $\mathcal{C}$ be a suitable ∞-category, and let π be the functor restricting diagrams of exponential or cromulent shape in $\mathcal{C}$ to their base.*

(a) The fibers of π vary functorially with respect to the base, and each fiber has a terminal object. The terminal objects of each fiber are the exponential or cromulent diagrams in $\mathcal{C}$, and the sub-∞-category spanned by these diagrams is equivalent to the the codomain of π.

(b) A diagram in the fiber $\pi^{-1}(p, f)$ or $\pi^{-1}(p, f, g)$ is terminal if and only if it is equivalent in the fiber to the diagram

$$\begin{array}{ccccc} & \swarrow & \Delta_f \Pi_f(A) & \longrightarrow & \Pi_f(A) \\ & & \downarrow & & \downarrow \\ A & \xrightarrow[p]{} & X & \xrightarrow[f]{} & Y \end{array} \quad \text{or} \quad \begin{array}{ccccc} A & \longleftarrow & \Delta_g \Pi_g \Delta_f A & \longrightarrow & \Pi_g \Delta_f A \\ p\downarrow & & \downarrow & \lrcorner & \downarrow \\ X & \xleftarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

3.3. **A distributive law for bispans.** To prove Theorem 1 it is necessary to first construct an alternate bispan construction generalizing Theorem 6.9 of [4]. This alternate construction is subtly different from that of Theorem 1, as the n-simplices of the alternate bispan ∞-category are no longer diagrams of shape $\mathrm{TNR}(n)$, but instead compatible collections of subdiagrams of simpler shapes. The following theorem provides the foundation for establishing Theorem 1.

Theorem 3. [7] *Let $\mathcal{C}$ be a suitable ∞-category. There is a bisimplicial set $\mathcal{D}^+_\times(\mathcal{C})$ and a sequence of bisimplicial sets $\{\Xi^n\}_{n\geq 0}$ called* distributahedra *such that the simplicial set whose n-simplices are suitable maps of bisimplicial sets $\Xi^n \to \mathcal{D}^+_\times(\mathcal{C})$ is an ∞-category equivalent to* $\mathrm{Bispan}(\mathcal{C})$.

Distributahedra are polytopes modeling the coherent distributivity of morphisms between two sub-∞-categories. The bisimplicial set $\mathcal{D}^+_\times(\mathcal{C})$ has (m, n)-simplices given by diagrams of shape $\Delta^m \times \mathrm{TwAr}(\Delta^n)$ in $\mathcal{C}$ with certain subdiagrams being pullback squares and cromulent diagrams.

Theorem 3 is proved by finding arrows filling in the dashed parts of the following diagrams:

$$\begin{array}{ccc} \Lambda^{k-i,n-(k-i)}_{k-i,i} & \longrightarrow & \mathcal{D}^+_\times(\mathcal{C}) \\ \downarrow & \nearrow & \\ \Delta^{k-i,n-(k-i)} & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \Lambda^{k+j,n-(k+j)}_{k,0} & \longrightarrow & \mathcal{D}^+_\times(\mathcal{C}) \\ \downarrow & \nearrow & \\ \Delta^{k+j,n-(k+j)} & & \end{array}$$

for all $n \geq 2$, $0 < k < n$, $0 \leq i \leq k$ and $0 \leq j \leq n-k$, where $\Lambda^{a,c}_{b,d}$ and $\Delta^{a,c}$ are defined as in Chapter 6 of [4]. These are bisimplicial analogues of the horns Λ^n_k and n-simplices Δ^n in simplicial sets. I show $\mathcal{D}^+_\times(\mathcal{C})$ is a (horizontally) Reedy fibrant simplicial set, ensuring the existence of all fillings. Cromulence is automatic for all cases except for when all superscripts are at most 2, due to dimensional reasons. This leaves 7 cases which can be individually checked, applying Theorem 2 and the Beck-Chevalley conditions for ∞-categories in [5].

4. Future work

I plan to further develop the theory of global Tambara functors and bispan categories, as well as to explore new projects using my background in higher category theory and homotopy theory.

(1) Establish a comparison theorem between global Tambara functors and global power functors, defined by Schwede in [13] as global Mackey functors with *power operations*. Then, provide new constructions for global power functors, e.g. those arising from the generalized character rings for Morava E-theories as in [1]. This is my main priority and the primary motivation for the current work.

(2) Investigate the relationship between global Tambara functors and Tambara functors for finite groups: are there examples of Tambara functors for a group G which cannot be realized as restrictions of global ones? Such examples exist for Mackey functors.

(3) If $\underline{S}$ is a Mackey functor, then transfer along the fold map $X \sqcup X \to X$ gives $\underline{S}(X)$ the structure of a commutative semigroup, while the transfer and norm of Tambara functor make $\underline{S}(X)$ a commutative semiring. Bispan constructions can thus encode ring-like objects in higher categories. Work of Berman on higher Lawvere theories in [3] suggests that spectrally-valued functors from various bispan categories provide models for highly structured ring spectra with less theoretical overhead than in other models.

(4) Develop the "commutative algebra" of global Tambara functors, analogous to the equivariant theory developed in [14] and work of Nakaoka as well as Blumberg, Hill and Hopkins.

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