Докл. Акад. Наук СССР
Том 247 (1979), № 2

Soviet Math. Dokl.
Vol. 20 (1979), No. 4

ON THE GROUP OF RATIONAL POINTS OF THREE-DIMENSIONAL GROUPS

UDC 513.6

V. P. PLATONOV AND A. S. RAPINČUK

Let G be a simple algebraic group defined over a field K, and let G_K be the group of K-rational points of G. The study of the structure of G_K is an important and, in general, very difficult problem. In the last decade much progress has been achieved in investigating G_K when G is K-isotropic (see [1] and [2]). But little is known so far concerning the structure of G_K for K-anisotropic groups G. One of the key questions that arises here is: When is the group G_K projectively simple? Even in the minimal case, when G is three dimensional, there are as yet insurmountable difficulties. The situation for algebraic number fields is more encouraging, since in that case we have the following plausible conjecture [1]: the group G_K is projectively simple, i.e. the quotient of G_K by its center is simple, if and only if for all non-Archimedean valuations v of the field K the local groups G_{K_v} are projectively simple. It is not hard to show that this is actually equivalent to the following conjecture: if G is a simple and simply connected algebraic group, and if G is K_v-isotropic for all non-Archimedean valuations v, then G_K is projectively simple. For $\dim G = 3$ this conjecture was stated by Kneser in 1956 in a somewhat different form (see [3] or [4], Chapter II, §12), and there still has not been any significant progress toward proving it. In this case G_K is the group $SL(1, D)$, where D is the algebra of quaternions over K and $SL(1, D) = (a \in D| \mathrm{Nrd}_{D/K}(a) = 1)$, where for v non-Archimedean $D \otimes_K K_v$ is the full matrix algebra $M_2(K_v)$.

The basic purpose of this paper is to prove an essential part of Kneser's conjecture. In addition, we compute the commutator length of the group $SL(1, D)$, which is of independent interest.

THEOREM 1. *Let D be the algebra of quaternions over an algebraic number field K. The group $SL(1, D)$ coincides with its commutator if and only if, for all non-Archimedean valuations v of the field K, the algebra $D_v = D \otimes_K K_v$ is the full matrix algebra $M_2(K_v)$. Under this condition every element of $SL(1, D)$ is a product of no more than three commutators.*

Let H be any noncentral normal subgroup of $G_K = SL(1, D)$. It follows from the Prasad-Margulis theorem (see [5]) that the index of H in G_K is finite. Hence H always has a subgroup of finite index $H_0 \subseteq H$ which is normal in the group $D^* = GL(1, D)$. This means that, when proving the above conjecture, we may suppose that H is a normal subgroup of $GL(1, D)$. In fact, under the conditions of this conjecture we have the following stronger assertion.

1980 *Mathematics Subject Classification*. Primary 20G30; Secondary 20G25, 15A33.

THEOREM 2. *Every normal subgroup H of $SL(1, D)$ is a normal subgroup of $GL(1, D)$*

For any $g \in GL(1, D)$ we let $\varphi_H(g)$ denote the automorphism induced by $\operatorname{Int}(g)$ in $SL(1, D)/H$.

THEOREM 3. *Suppose that for all non-Archimedean v the algebra $D_v = D \otimes_K K_v \simeq M_2(K_v)$. Then for every $g \in GL(1, D)$ the automorphism $\varphi_H(g)$ of $SL(1, D)/H$ leaves the conjugacy classes invariant, and all of the characters of $SL(1, D)/H$ are real. On the other hand, if $\varphi_H(g)$ is an inner automorphism of $SL(1, D)/H$, then $SL(1, D) = H$.*

We shall give the proof of the most important Theorem 1 below. The proofs of Theorems 2 and 3 use similar arguments.

In what follows we shall let V denote the set of all nonequivalent valuations of the field K; V_∞ and V_f are the sets of Archimedean and non-Archimedean valuations of K, respectively; $V_0 = \{v \in V_\infty | D_v$ is a division ring$\}$. As usual, if $v \in V_f$, we let U_v denote the group of units of K_v, and $V(a) = \{v \in V_f | \, a \notin U_v\}$.

We shall say that a subset $F \subset SL(1, D)$ has the *weak approximation property* if for any finite set $v_1, \ldots, v_r \in V$ the diagonal imbedding $F \longrightarrow \Pi_{i=1}^r \, SL(1, D_{v_i})$ has dense image.

LEMMA 1. *Under the conditions of Theorem 1 the set $F = (s^{-1}\, t^{-1}\, st | \, s, t \in SL(1, D))$ has the weak approximation property.*

PROOF. Suppose that $v_1, \ldots, v_r \in V$ and $a = (a_1, \ldots, a_r) \in \Pi_{i=1}^r \, SL(1, D_{v_i})$. We note that every element a_i is a commutator in $SL(1, D_{v_1})$. In fact, for $SL(1, D_{v_i}) \sim SL(2, K_{v_i})$ this is proved in [6]; but if D_{v_i} is a division ring, then it must be the usual ring of quaternions over the field $K_{v_i} = R$, in which case the claim is verified immediately. Since $SL(1, D)$ is dense in $\Pi_{i=1}^r \, SL(1, D_{v_i})$ by the classical weak approximation theorem, the conclusion of Lemma 1 then follows in the obvious manner.

LEMMA 2. *Under the conditions of Theorem 1 any element $z \in SL(1, D)$ can be represented in the form $z = x^{-1}y^{-1}\, xy$, $x \in SL(1, D)$, $y \in GL(1, D)$.*

PROOF. We look for a representation in the form $xz = y^{-1}\, xy$, which is equivalent to the required form. By the Skolem-Noether theorem (see [7], §10, no. 1), this equality follows when the characteristic polynomials of x and xz over K coincide. Since $\operatorname{Nrd}_{D/K}(z) = 1$, we have

$$xz = y^{-1}xy \Leftrightarrow \operatorname{Trd}_{D/K}(x) = \operatorname{Trd}_{D/K}(xz),$$

where $\operatorname{Trd}_{D/K}$ denotes the reduced trace from D to K. Then the desired element x must satisfy the following system of equations:

(1) $\operatorname{Trd}_{D/K}(x) - \operatorname{Trd}_{D/K}(xz) = 0;$

(2) $\operatorname{Nrd}_{D/K}(x) = 1.$

Let x_1, x_2, x_3, x_4 be the coordinates of x in some basis of D/K. Then (1) is a linear equation in the x_i, and the system (1), (2) is equivalent to the equation $f(x_1, \ldots, x_4) = 1,$

where f is a quadratic form. As already noted, over the completion of K_v in D_v there exist elements x_v and y_v such that $x_v z = y_v^{-1} x_v y_v$; that is, f represents 1 over all of the K_v. Then, by the Hasse-Minkowski theorem, f represents 1 over K. Lemma 2 is proved.

LEMMA 3. *Suppose that for all $v \in V_f$ the algebra $D_v \simeq M_2(K_v)$, and suppose that there exists a subset $F \subset GL(1, D)$ such that, for suitable $g \in GL(1, D)$, $gF \subset SL(1, D)$ and has the weak approximation property.*

Then, for any maximal subfield $T = K(\sqrt{\alpha})$ of D and any $t \in GL(1, D)$, there exist $\tau \in T^*$ and $f \in F$ such that

$$\mathrm{Nrd}_{D/K}(\tau t) \in \mathrm{Nrd}_{D/K}(K(f)).$$

PROOF. By assumption, if $v \notin V_0$, then $SL(1, D_v) \simeq SL(2, K_v)$ and hence

$$\mathrm{Trd}_{D_v/K_v}(SL(1, D_v)) = K_v,$$

and, moreover,

$$\mathrm{Trd}_{D_v/K_v}(g^{-1} SL(1, D_v)) = K_v.$$

From the continuity of the reduced trace and the weak approximation property of gF it then follows immediately that the set $\theta = (\mathrm{Trd}_{D/K}(f) | f \in F)$ is dense in $\Pi_{i=1}^r K_{v_i}$ for any finite set of valuations $v_1, \ldots, v_r \in V \backslash V_0$, and also $\theta^2 = (\mathrm{Trd}_{D/K}(f)^2 | f \in F)$ is dense in $\Pi_{i=1}^r K_{v_i}^2$. Since the set $K_v^2 - 4\,\mathrm{Nrd}_{D/K}(g^{-1}) = (a^2 - \mathrm{Nrd}_{D/K}(g^{-1}) | a \in K_v)$ is open in the v-adic topology of K_v, it follows that the set $K_v^2 \cap (K_v^2 - 4\,\mathrm{Nrd}_{D/K}(g^{-1}))$ is also open, and the corresponding intersection with θ^2 is dense in this set. This implies that for a finite set $W \subset V \backslash V_0$ there exists an $f_W \in F$ such that

$$[\mathrm{Trd}_{D/K}(f_W)]^2 - 4\mathrm{Nrd}_{D/K}(g^{-1}) \in K_v^{*2}, \qquad v \in W.$$

In particular, suppose we have chosen f_{V_1} for the set $V_1 = V(2) \cup V(\mathrm{Nrd}_{D/K}(t)) \cup (V_\infty \backslash V_0) \cup V(\alpha)$. Then for $\delta = [\mathrm{Trd}_{D/K}(f_{V_1})]^2 - 4\mathrm{Nrd}_{D/K}(g^{-1})$ we have

$$K(\sqrt{\delta}) \simeq K(f_{V_1}).$$

We shall show that the element $f = f_{V_1}$ satisfies the requirements of the lemma. Let $(a, b)_v$ denote the v-adic Hilbert symbol. Then it suffices to show that there exists an element $d \in K^*$ for which

(3) $\quad (\delta, d^{-1}\mathrm{Nrd}_{D/K}(t))_v = 1 \quad \forall v \in V;$

(4) $\quad (\alpha, d)_v = 1 \quad \forall v \in V.$

In fact, by Hasse's theorem (see [8]) it follows from (4) that $d \in \mathrm{Nrd}_{D/K}(T)$, and, if $d = \mathrm{Nrd}_{D/K}(\tau^{-1})$, $\tau \in T$, then $\mathrm{Nrd}_{D/K}(\tau t) \in \mathrm{Nrd}_{D/K}(K(f))$.

Equations (3) and (4) are equivalent to the following:

(3′) $\quad (\delta, d)_v = (\delta, \mathrm{Nrd}_{D/K}(t))_v \quad \forall v \in V;$

(4′) $\quad (\alpha, d)_v = 1 \quad \forall v \in V.$

By our choice of the set V_1, it suffices to prove that for any $v \in V$ there exists $d_v \in K_v^*$ such that

(3″) $\quad (\delta, d_v)_v = (\delta, \mathrm{Nrd}_{D/K}(t))_v;$

(4″) $\quad (\alpha, d_v)_v = 1.$

For $v \in V$ satisfying $(\delta, \mathrm{Nrd}_{D/K}(t))_v = 1$, we set $d_v = 1$. But if $(\delta, \mathrm{Nrd}_{D/K}(t))_v = -1$, then for such v we may set $d_v = \mathrm{Nrd}_{D/K}(t)$. In fact, by construction we have $(\delta, \mathrm{Nrd}_{D/K}(t))_v = 1$ for $v \in V_1$, and for $v \in V_0$ this follows from the positivity of $\mathrm{Nrd}_{D_v/K_v}(t)$; hence $(\alpha, d_v)_v = 1$. Lemma 3 is proved.

Proof of Theorem 1. Let $z \in SL(1, D)$. By Lemma 2,

$$z = x^{-1}y^{-1}xy, \quad x \in SL(1, D), \quad y \in GL(1, D).$$

In accordance with Lemma 3 we set

$$F = (xa^{-1}b^{-1}ab \mid a,\ b \in SL(1, D)), \quad T = K(x), \quad t = y$$

and we choose $\tau \in T^*$ and $f = xg \in F$ satisfying

$$\mathrm{Nrd}_{D/K}(\tau y) \in \mathrm{Nrd}_{D/K}(K(f)).$$

Then

$$z = x^{-1}y^{-1}xy = x^{-1}(\tau y)^{-1}x(\tau y) = gg^{-1}x^{-1}(\tau y)^{-1}xgg^{-1}(\tau y)$$
$$= gf^{-1}(\tau y)^{-1}f(\tau y)(\tau y)^{-1}g^{-1}(\tau y).$$

There exists $\omega \in K(f)$ such that $\mathrm{Nrd}_{D/K}(\omega) = \mathrm{Nrd}_{D/K}(\tau y)^{-1}$; hence

$$\omega\tau y \in SL(1, D) \quad \text{and} \quad z = g[f^{-1}(\omega\tau y)^{-1} \cdot f(\omega\tau y)][(\tau y)^{-1}g^{-1}(\tau y)].$$

The expressions in brackets are obviously commutators in $SL(1, D)$; hence z can be represented as a product of three commutators in $SL(1, D)$. Finally, if $SL(1, D) = [SL(1, D), SL(1, D)]$, then $D_v \simeq M_2(K_v)$ for all $v \in V_f$.

In fact, otherwise, for some $v_0 \in V_f$, D_{v0} would be a division ring and $SL(1, D_{v0})$ would be a compact pro-solvable group, and, in particular,

$$D_{v_0}) \neq [SL(1, D_{v_0}), SL(1, D_{v_0})],$$

so that it would follow from the denseness of $SL(1, D)$ in $SL(1; D_{v0})$ that

$$SL(1, D) \neq [SL(1, D), SL(1, D)].$$

Theorem 1 is proved.

In the proof of Theorem 3 the following assertion, which gives a refinement of Lemma 2 modulo a normal subgroup H, plays an essential role.

Lemma 4. *Let T be any maximal subfield in D. Then any coset gH of the group $SL(1, D)/H$ has a representative of the form $x^{-1}y^{-1}xy$, where $x \in T$ and $y \in SL(1, D)$.*

Theorems 1 and 3 show that the conjecture on simplicity of $SL(1, D)/\{\pm 1\}$ reduces, for example, to the question of the existence of finite simple groups having certain natural properties. The authors do not know whether or not there exist finite simple groups G with the following properties: 1) all of the characters of G are real; 2) G has an outer automorphism relative to which the characters are invariant. Despite the apparent difficulty of the last problem, this reduction is of independent interest.

Institute of Mathematics
Academy of Sciences of the Belorussian SSR

Received 30/MAR/79

BIBLIOGRAPHY

1. V. P. Platonov, Proc. Internat. Congr. Math. (Vancouver, 1974), Vol. 1, Canad. Math. Congr., Montréal, 1975, p. 471; English transl. in Amer. Math. Soc. Transl. (2) 109 (1977).

2. ———, Izv. Akad. Nauk SSSR Ser. Mat. 40 (1976), 227; English transl. in Math. USSR Izv. 10 (1976).

3. Martin Kneser, J. Reine Angew. Math. **196** (1956), 213.
4. Jean A. Dieudonné, *La géométrie des groupes classiques*, 3rd ed., Springer-Verlag, Berlin and New York, 1971.
5. Gopal Prasad, Ann. of Math. (2) **105** (1977), 553.
6. R. C. Thompson, Trans. Amer. Math. Soc. **101** (1961), 16.
7. N. Bourbaki, *Algèbre*, Chap. 8, Actualités Sci. Indust., no. 1261, Hermann, Paris, 1958.
8. J. W. S. Cassels and A. Fröhlich, editors, *Algebraic number theory* (Proc. Instructional Conf., Brighton, 1965), Academic Press, London; Thompson, Washington, D. C., 1967.

Translated by N. KOBLITZ