Prasad's work in arithmetic theory of algebraic groups

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Prasad's contributions to arithmetic theory of algebraic groups and related areas include:

- Proof of strong approximation over global fields of positive characteristic
- Investigation of congruence subgroup problem (particularly, computation of metaplectic kernel)
- Kneser-Tits conjecture
- Volume formula for S-arithmetic quotients and its applications
- Weakly commensurable Zariski-dense subgroups and applications to isospectral locally symmetric spaces

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Weakly commensurable Zariski-dense subgroups and applications to isospectral locally symmetric spaces

Let

- G be a linear algebraic group over a global field k
- S be a (nonempty, and usually finite) set of places of k
- $\mathbb{A}_k(S)$ be ring of S-adeles of k (i.e., adeles without components corresponding to places in S)

Definition

We say that G has strong approximation with respect to S **if** diagonal embedding $G(k) \hookrightarrow G(\mathbb{A}_k(S))$ has *dense* image.

Informally, one should think of this property as a far-reaching generalization of Chinese Remainder Theorem to algebraic groups.

While SA for k-split simply connected groups (like SL_n) is easy to establish using unipotent root subgroups, proving SA for k-anisotropic simply connected groups is **hard**.

Various cases were studied by Eichler, Shimura, Weil, Kneser, ...

Platonov (1969) found a uniform argument over number fields. His argument used p-adic Lie theory which is **not** available in characteristic $p > 0$.

Solution of problem of SA over global fields of positive characteristic was obtained independently by Prasad and Margulis using ergodic-theoretic considerations 15 years later!

G. Prasad, *Strong approximation for semi-simple groups over function fields*, Ann. Math. **105**(1977), no. 3, 553-572

Congruence subgroup problem for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ was considered by Fricke and Klein.

For each $n \geqslant 1$, we have congruence subgroup of level n:

$$\Gamma(n) := \ker\left(\mathrm{SL}_2(\mathbb{Z}) \longrightarrow \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})\right),$$

which is a *normal subgroup of finite index.*

Congruence subgroup problem is the following question:

Does every finite index normal subgroup of Γ *contain a suitable* $\Gamma(n)$*?*

Fricke and Klein observed that that for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ the answer is no.

In 1960-70, it was shown that for other groups (such as $\mathrm{SL}_{n\geqslant 3}(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z}[p^{-1}])$) answer to a similar question is yes.

Serre introduced congruence kernel C that measures deviation from positive solution.

- $C(\mathrm{SL}_2(\mathbb{Z}))$ is a free profinite group of countable rank
- $C(\mathrm{SL}_{n\geqslant 3}(\mathbb{Z})) = C(\mathrm{SL}_2(\mathbb{Z}[p^{-1}])) = \{1\}$
- $C(\mathrm{SL}_{n\geqslant 3}(\mathbb{Z}\left[\sqrt{-1}\right])) \simeq \mathbb{Z}/4\mathbb{Z}$

So, CSP becomes problem of computing C

Two aspects: 1) proving that in the higher rank situation C is finite (equivalently, central)

2) computing C precisely (equivalently, computing metalplectic kernel)

Prasad's work has contributed to *both* aspects.

In fact, metaplectic kernel has been computed in all cases relevant for CSP.

Basically, result is that metaplectic kernel is either trivial or is isomorphic to group of roots of unity in base field.

1. G. Prasad, M.S. Raghunathan, *On the congruence subgroup problem: determination of the "metaplectic kernel,"* Invent. math. **71**(1983), no. 1, 21-42.

2. ———, *Topological central extensions of semisimple groups over local fields, I, II,* Ann. Math. **119**(1984), no. 1, 143-201; no. 2, 203-268.

3. ———, *Topological central extensions of $SL_1(D)$,* Invent. math. **92**(1988), no. 3, 645-689.

4. G. Prasad, A.S. Rapinchuk, *Computation of the metaplectic kernel,* Publ. math. IHES **84**(1996), 91-187.

5. ———, *On the congruence kernel for simple groups,* Proc. Steklov Inst. Math. **292**(2016), 216-246.

One of results of classical reduction theory (Borel - Harish Chandra) is that *for a semi-simple algebraic $\mathbb{Q}$-group G, quotient $G(\mathbb{R})/G(\mathbb{Z})$ has finite Haar measure.*

More generally, let K be a number field, S be a finite set of places of K containing all archimedean ones. Set

$$G_S := \prod_{v \in S} G(K_v).$$

Then *for any semi-simple K-group G and any S-arithmetic subgroup $\Gamma \subset G(K)$ quotient G_S/Γ has finite Haar measure.*

It is important to know exact value of volume (w.r.t. certain canonical measures) as it carries significant arithmetic and topological information, in particular about Euler-Poincare characteristic $\chi(\Gamma)$.

Volume formula for (principal) S-arithmetic quotients in general situation (including global fields of positive characteristic) was found in

G. Prasad, *Volumes of S-arithmetic quotients of semi-simple groups,* Publ. math. IHES **69**(1989), 91-117.

Volume is expressed as a product of factors that depend on field (like *discriminant*), on root system, and also of *local factors.*

To indicate nature of local factors, let us consider

Example.

Let $G = \mathrm{SL}_2$. Explicit computation shows that

$$\mathrm{vol}(G(\mathbb{R})/G(\mathbb{Z})) = \frac{\pi^2}{6} = \zeta(2) = \prod_p \frac{1}{1-p^{-2}}.$$

In this example local factors are precisely local factors of ζ-function.

In general, local factors are products of local factors of some ζ- or L-functions.

So, volume formula can be efficiently analyzed using number-theoretic techniques.

Volume formula was used by many mathematicians (including Belolipetski, Emery, Golsefidy, Lubotzky, ...) to obtain very explicit results.

E.g., to identify lattices of minimal covolume, to determine growth of number of lattices as a function of covolume etc.

Fake projective planes (G. Prasad, S.-K. Yeung)

A fake projective plane is a smooth projective complex surface that is not $\mathbb{CP}^2$ but has same Betti numbers as $\mathbb{CP}^2$.

First fake projective plane was constructed by Mumford in 1979.
Until 2006, only 3 additional examples were found.

It was proved that any fake projective plane is a locally symmetric space of $\mathrm{PU}(2,1)$ and that its fundamental group is *arithmetic*.

So, classification of fake projective planes reduces to classification of torsion-free *cocompact arithmetic lattices* $\Gamma \subset \mathrm{PU}(2,1)$ with Euler characteristic $\chi(\Gamma) = 3$.

Using volume formula, Prasad and Yeung identified all such lattices.

Their analysis produced 28 new families of fake projective planes.

Using these techniques and employing computer, Cartwright and Steger gave a *complete list* of fake projective planes.

The list consists of 50 items up to diffeomorphism, and each item has two complex structures.

Subsequently, Prasad and Yeung were able to analyze fake forms of some other projective varieties.

1. G. Prasad, S.-K. Yeung, *Fake projective planes,* Invent. math. **168**(2007), no. 2, 321-370.

2. ———, *Arithmetic fake projective spaces and fake Grassmanians,* Amer. J. Math. **131**(2009), no. 2, 379-407.

3. ———, *Nonexistence of arithmetic fake compact Hermitian symmetric spaces of type other than* A_n $(n \leqslant 4)$, J. Math. Soc. Japan **64**(2012), no. 3, 683-731.

M. Kac, Amer. Math. Monthly, **73**(1966), 1-23

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

"La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Before I explain the title and introduce the theme of the lecture I should like to state that my presentation will be more in the nature of a leisurely excursion than of an organized tour. It will not be my purpose to reach a specified destination at a scheduled time. Rather I should like to allow myself on many occasions the luxury of stopping and looking around. So much effort is being spent on streamlining mathematics and in rendering it more efficient, that a solitary transgression against the trend could perhaps be forgiven.

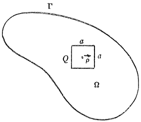

FIG. 1

1. And now to the theme and the title.

It has been known for well over a century that if a membrane Ω, held fixed along its boundary Γ (see Fig. 1), is set in motion its displacement (in the direction perpendicular to its original plane)

$$F(x, y; t) \equiv F(\vec{\rho}; t)$$

obeys the wave equation

$$\frac{\partial^2 F}{\partial t^2} = c^2 \nabla^2 F,$$

where c is a certain constant depending on the physical properties of the membrane and on the tension under which the membrane is held.

I shall choose units to make $c^2 = \frac{1}{2}$.

1

Classical rigidity

For $i = 1, 2$, let $\mathcal{G}_i$ be a semi-simple Lie group,

let $\Gamma_i \subset \mathcal{G}_i$ be a lattice (or some other "large" subgroup)

Then (under appropriate assumptions):

a homo/isomorphism $\phi \colon \Gamma_1 \longrightarrow \Gamma_2$ (virtually) *extends* to a homo/isomorphism of Lie groups $\tilde{\phi} \colon \mathcal{G}_1 \longrightarrow \mathcal{G}_2$.

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{\tilde{\phi}} & \mathcal{G}_2 \\ \bigcup & & \bigcup \\ \Gamma_1 & \xrightarrow{\phi} & \Gamma_2 \end{array}$$

Consequence: let

- $\Gamma_1 = \mathrm{SL}_n(\mathbb{Z}) \quad (n \geqslant 3)$,
- $\Gamma_2 = G(\mathcal{O})$, G is an *absolutely almost simple* algebraic group over a *number field* K with ring of integers $\mathcal{O}$.

If Γ_1 and Γ_2 are virtually isomorphic, **then**

- $K = \mathbb{Q}$ (hence $\mathcal{O} = \mathbb{Z}$), and
- $G \simeq \mathrm{SL}_n$ over $\mathbb{Q}$.

Thus,

structure of a (higher rank) arithmetic group *determines*

field of definition & ambient algebraic group over this field.

• *Structural* approach to rigidity does **not** extend to

arbitrary Zariski-dense subgroups

as these may, for example, be **free** groups.

• **However**, one should be able to recover such data as

field of definition & ambient algebraic group

from *any* Zariski-dense subgroup **if** instead of

structural information

one uses information about the *eigenvalues of elements.*

• We call this phenomenon *eigenvalue rigidity*.

• How do we match the eigenvalues of elements of two Zariski-dense subgroups?

Note that the subgroups may be represented by matrices of *different sizes*,

hence their elements may have *different numbers* of eigenvalues.

• Why do we care about the eigenvalues?

We will address these issues in next section.

Let F be a field of characteristic zero.

Definition.

(1) Let $\gamma_1 \in \mathrm{GL}_{n_1}(F)$ and $\gamma_2 \in \mathrm{GL}_{n_2}(F)$ be *semi-simple* matrices,

let

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2} \quad (\in \overline{F})$$

be their eigenvalues. Then γ_1 and γ_2 are *weakly commensurable*

if $\exists\ a_1, \ldots, a_{n_1}\ ,\ b_1, \ldots, b_{n_2} \in \mathbb{Z}$ such that

$$\lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} \neq 1.$$

Let $G_1 \subset \mathrm{GL}_{n_1}$ and $G_2 \subset \mathrm{GL}_{n_2}$ be reductive F-groups, $\Gamma_1 \subset G_1(F)$ and $\Gamma_2 \subset G_2(F)$ be Zariski-dense subgroups.

(2) Γ_1 and Γ_2 are *weakly commensurable* if

every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order

is weakly commensurable to

some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order,

and vice versa.

Let M be a Riemannian manifold.

- $\mathcal{E}(M)$ = *spectrum* of *Laplace - Beltrami* operator

  (eigenvalues with multiplicities)

- $L(M)$ = (weak) *length spectrum*

  (lengths of closed geodesics w/o multiplicities)

- M_1 and M_2 are commensurable if they have a common *finite-sheeted* cover:

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ M_1 & & M_2 \end{array}$$

Question: *Are* M_1 *and* M_2 *necessarily isometric / commensurable if*

(1) $\mathcal{E}(M_1) = \mathcal{E}(M_2)$, i.e. M_1 and M_2 are *isospectral*;

Can one hear the shape of a drum? (M. Kac)

(2) $L(M_1) = L(M_2)$, i.e. M_1 and M_2 are *iso-length-spectral*;

(3) $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$, i.e. M_1 and M_2 are *length-commensurable*.

- There exist examples of isospectral and iso-length spectral manifolds that are **not** isometric.

- Constructions were proposed by M.-F. Vignéras and T. Sunada.

- *Both* constructions produce *commensurable* manifolds.

- There are *noncommensurable* *isospectral* manifolds (Lubotzky et al.);

nevertheless one *expects* to prove the commensurability of isospectral and iso-length spectral manifolds in *many* situations.

- Prior to our work, this was done only for arithmetically defined Riemann surfaces (A. Reid) and hyperbolic 3-manifolds (A. Reid et al.).

Prasad & A.R. proved commensurability of *many* arithmetically defined isospectral and iso-length spectral locally symmetric spaces.

Tool: connecting isospectrality to weak commensurability.

Notations

G - absolutely simple real algebraic group, $\mathcal{G} = G(\mathbb{R})$

$\mathcal{K}$ - maximal compact subgroup, $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$ (symmetric space)

For a discrete torsion-free subgroup $\Gamma \subset \mathcal{G}$, let $\mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$ (locally symmetric space)

$\mathfrak{X}_\Gamma$ is *arithmetically defined* **if** Γ is arithmetic.

- For compact locally symmetric spaces:

(1) (isospectrality) $\Rightarrow$ (2) (iso-length spectrality)

(proof uses the *trace formula*)

- $\mathcal{E}(M)$, $L(M)$ **change** when M is replaced by a *commensurable* manifold

$\Rightarrow$ Conditions (1) & (2) are **not** invariant under passing to a commensurable manifold.

So, we proposed length-commensurability:

(3) $\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2)$

$\mathbb{Q} \cdot L(M)$ - *rational* length spectrum (invariant of commensurability class)

Thus, for compact locally symmetric spaces:

$$(1) \Rightarrow (2) \Rightarrow (3)$$

Theorem

Let $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *be locally symmetric spaces having* finite volume, *of absolutely simple real algebraic groups* G_1 *and* G_2*.*

If $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are* *length-commensurable,* **then** Γ_1 *and* Γ_2 *are* *weakly commensurable.*

The proof relies on results and conjectures from *transcendental number theory.*

- $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable **iff** Γ_1 and Γ_2 are commensurable up to an isomorphism between G_1 and G_2.

- For geometric applications:

When does weak commensurability of Γ_1 and Γ_2 imply their commensurability?

- This is the case for many *arithmetic* Γ_1 and Γ_2 (below)

- Remarkably, weak commensurability has strong consequences for *arbitrary* Zariski-dense subgroups

(leading to the concept of *eigenvalue rigidity* ...)

Let

- F – a field of characteristic zero
- G_1 and G_2 – absolutely almost simple algebraic F-groups
- $\Gamma_i \subset G_i(F)$ – finitely generated Zariski-dense subgroup, $i = 1, 2$

Theorem 1

If Γ_1 *and* Γ_2 *are weakly commensurable,* **then** *either* G_1 *and* G_2 *have same Killing-Cartan type, or one of them is of type* B_ℓ *and the other of type* C_ℓ $(\ell \geqslant 3)$.

For a Zariski-dense subgroup $\Gamma \subset G(F)$, let

$$K_\Gamma = \text{ subfield of } F \text{ generated by } \mathrm{tr}\,(\mathrm{Ad}\,\gamma), \quad \gamma \in \Gamma$$

(*trace field*).

E.B. Vinberg: $K = K_\Gamma$ *is the minimal field of definition of* $\mathrm{Ad}\,\Gamma$

Algebraic hull: $\mathcal{G} :=$ Zariski-closure of $\mathrm{Ad}\,\Gamma$ in $\mathrm{GL}(\mathfrak{g})$, where

$\mathfrak{g}$ is the Lie algebra of G

- $\mathcal{G}$ is a K-defined algebraic group (in fact, an F/K-form of $\overline{G}$)
- $\mathcal{G}$ is an *important characteristic* of Γ; it *determines* Γ if it is arithmetic

Theorem 2

If Γ_1 *and* Γ_2 *are weakly commensurable,* **then** $K_{\Gamma_1} = K_{\Gamma_2}$.

Finiteness conjecture.

Let

- G_1 *and* G_2 *be absolutely simple algebraic* F*-groups,* char $F = 0$;
- $\Gamma_1 \subset G_1(F)$ *be a finitely generated Zariski-dense subgroup,* $K_{\Gamma_1} = K$.

Then there exists a finite collection $\mathcal{G}_2^{(1)}, \ldots, \mathcal{G}_2^{(r)}$ *of* F/K*-forms of* G_2 *such that* **if**

$\Gamma_2 \subset G_2(F)$ *is a finitely generated Zariski-dense subgroup weakly commensurable to* Γ_1,

then Γ_2 can be conjugated into some $\mathcal{G}_2^{(i)}(K)$ $(\subset G_2(F))$.

Example. Let A be a central simple K-algebra, $G = \mathrm{PSL}_{1,A}$.

Fix a f. g. Zariski-dense subgroup $\Gamma \subset G(K)$ with $K_\Gamma = K$.

FINITENESS CONJECTURE $\Rightarrow$ There are only finitely many c.s.a. A' such that for $G' = \mathrm{PSL}_{1,A'}$,

$$\exists \text{ f.g. Zariski-dense subgroup } \Gamma' \subset G'(K)$$

weakly commensurable to Γ.

- Similar consequences for orthogonal groups of quadratic forms etc.

The finiteness conjecture is known in the following cases:

- K a number field (although Γ_1 does not have to be arithmetic)

- G_1 is an inner form of type A_ℓ over K

(so, previous example is already a theorem ...)

Note that these two cases cover all lattices in simple real Lie groups.

General case is work in progress ...

V.I. Chernousov, A.R., I.A. Rapinchuk, *Simple algebraic groups with the same maximal tori, weakly commensurable Zariski-dense subgroups, and good reduction,* arxiv:2112.04315.

Finiteness conjecture for algebraic hulls of weakly commensurable subgroups was reduced to Finiteness conjecture for forms with good reduction.

A.R., I.A. Rapinchuk, *Linear algebraic groups with good reduction,* Res. Math. Sci. **7**(2020), article 28.

Theorem 3 (G. Prasad, A.R.)

Let

- G_1 *and* G_2 *be absolutely almost simple* F*-groups,* char $F = 0$*;*
- $\Gamma_i \subset G_i(F)$ *be a Zariski-dense* S*-arithmetic subgroup,* $i = 1, 2$*.*

(1) *Assume* G_1 *and* G_2 *are of* **same type***, different from* A_n, D_{2n+1} $(n > 1)$, *and* E_6.

If Γ_1 *and* Γ_2 *are* weakly commensurable, *then they are* commensurable.

(2) *In* **all** *cases,* S*-arithmetic* $\Gamma_2 \subset G_2(F)$ *weakly commensurable to a given* S*-arithmetic* $\Gamma_1 \subset G_1(F)$*, form* finitely many *commensurability classes.*

(cont.)

(3) **If** Γ_1 *and* Γ_2 *are weakly commensurable,* **then**

$$\Gamma_1 \text{ contains nontrivial unipotents} \Leftrightarrow \Gamma_2 \text{ does.}$$

(4) (arithmeticity theorem) *Let now* F *be a locally compact field, and let* $\Gamma_1 \subset G_1(F)$ *be an* S*-arithmetic lattice.*

If $\Gamma_2 \subset G_2(F)$ *is a lattice weakly commensurable to* Γ_1, **then** Γ_2 *is also* S*-arithmetic.*

Theorem 4

Let (as above)

- $\mathfrak{X}_{\Gamma_1}$ *be an* arithmetically defined *locally symmetric space,*
- $\mathfrak{X}_{\Gamma_2}$ *be a locally symmetric space of* finite volume.

- **If** $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are length-commensurable,* **then**

(1) $\mathfrak{X}_{\Gamma_2}$ *is arithmetically defined;*

(2) $\mathfrak{X}_{\Gamma_1}$ *is compact* $\Leftrightarrow$ $\mathfrak{X}_{\Gamma_2}$ *is compact.*

- *The set of* $\mathfrak{X}_{\Gamma_2}$*'s length-commensurable to* $\mathfrak{X}_{\Gamma_1}$ *is a union of finitely many commensurability classes.*

It consists of single commensurability class if G_1 *and* G_2 *are of same type different from* A_n, D_{2n+1} $(n > 1)$, *or* E_6.

Corollary

Let M_1 and M_2 be arithmetically defined hyperbolic d-manifolds where $d \neq 3$ is even or $\equiv 3 \pmod 4$.

If *M_1 and M_2 are length-commensurable,* **then** *they are commensurable.*

- Hyperbolic manifolds of different dimensions are **not** length-commensurable.

(In fact, their length spectra are *very* different ...)

- A *complex* hyperbolic manifold cannot be length-commensurable to a *real* or *quaternionic* hyperbolic manifold, etc.

Theorem 5

Let $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *be compact isospectral locally symmetric spaces.*

- **If** $\mathfrak{X}_{\Gamma_1}$ *is arithmetically defined,* **then** *so is* $\mathfrak{X}_{\Gamma_2}$.

- $G_1 = G_2 =: G$, *hence* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *have same* universal cover.

- *Assume that at least one of* Γ_1 *and* Γ_2 *is arithmetic.*

If G *is of type different from* A_n, D_{2n+1} $(n > 1)$, *and* E_6, **then** $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are commensurable.*

1. G. Prasad, A.S. Rapinchuk, *Weakly commensurable arithmetic groups and isospectral locally symmetric spaces,* Publ. math. IHES **109**(2009), 113-184.

2. ———, *Local-global principles for embedding of fields with involution into simple algebras with involution,* Comment. Math. Helv. **85**(2010), no. 3, 583-645.

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5. ———, *Generic elements in Zariski-dense subgroups and isospectral locally symmetric spaces,* Thin groups and superstrong approximation, 211-252, Math. Sci. Res. Inst. Publ. **61**, Cambridge Univ. Press 2014.