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Finiteness theorems for algebraic tori over function fields

Andrei S. Rapinchuk[*, a] **and Igor A. Rapinchuk**[b]

[a] Department of Mathematics, University of Virginia, Charlottesville, VA 22904-4137, USA

[b] Department of Mathematics, Michigan State University, East Lansing, MI 48824, USA

E-mails: asr3x@virginia.edu, rapinchu@msu.edu

Abstract. We present a number of finiteness results for algebraic tori (and, more generally, for algebraic groups with toric connected component) over two classes of fields: finitely generated fields and function fields of algebraic varieties over fields of type (F), as defined by J.-P. Serre.

Résumé. Nous présentons plusieurs résultats de finitude pour les tores (et, plus généralement, pour les groupes algébriques dont la composante connexe est un tore) définis sur les corps de type fini et les corps de fonctions des variétés algébriques définies sur les corps satisfaisant la condition (F) de Serre.

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The goal of this note is to formulate a series of finiteness results of a local-global nature for algebraic tori over two classes of fields that are more general than classical local and global fields. First, we will consider fields that are finitely over their prime subfields (these will be referred to simply as *finitely generated fields*). For our purposes, it is important to point out that any such field K comes equipped with an almost canonical set V of discrete valuations (places) called *divisorial*. More precisely, one can choose a model $\mathfrak{X}$ for K (i.e., a normal separated irreducible scheme of finite type over $\mathbb{Z}$ or over a finite field such that K is the function field of $\mathfrak{X}$), and then V consists of the discrete valuations corresponding to the prime divisors of $\mathfrak{X}$. We recall that any two divisorial sets of places of K corresponding to different models are commensurable (cf. [11, 5.3]). Second, we will also consider function fields $K = k(X)$ of normal geometrically integral varieties X over a base field k. Again, K is naturally equipped with an almost canonical set V of discrete valuations that correspond to the prime divisors of X; we will call such a set V *geometric*. For most of our finiteness results to hold in this set-up, we need to assume that k is of *type* (F). This notion goes back to Serre [13, Ch. III, 4.2], who defined a profinite group $\mathcal{G}$ to be of type (F) if it satisfies the following condition:

(F) For every integer n, the group $\mathcal{G}$ has only finitely many open subgroups of index n.

* Corresponding author.

Then a field k is said to be of type (F) if its absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$, where k^{sep} is a separable closure, is of type (F). (It should pointed out that Serre also requires a field of type (F) to be perfect, but this requirement is not necessary in our context.) We note that finitely generated fields of positive characteristic can be treated as function fields over a finite field, which of course is a field of type (F) (cf. Section 1 below).

1. Tori with good reduction

Let G be a (connected) reductive algebraic group over a field K. Given a discrete valuation v of K, we let K_v and $\mathcal{O}_v \subset K_v$ denote the corresponding completion and valuation ring, respectively. We say that G has *good reduction* at v if there exists a reductive group scheme $\mathcal{G}$ over $\mathcal{O}_v$ whose generic fiber $\mathcal{G} \times_{\mathcal{O}_v} K_v$ is isomorphic to $G \times_K K_v$. A recent finiteness conjecture predicts that given a reductive group G over a finitely generated field K equipped with a divisorial set of places V, the set of K-isomorphism classes of K-forms that have good reduction at all $v \in V$ is finite provided that the characteristic of K is "good" for the type of G (cf. [11, Conj. 5.7]). For algebraic tori over finitely generated fields of characteristic zero, this is confirmed by the following Hermite–Minkowski type theorem.

Theorem 1. *Let K be a finitely generated field of characteristic 0, and let V be a divisorial set of places of K. Then for any integer $d \geq 1$, the set of K-isomorphism classes of d-dimensional K-tori that have good reduction at all $v \in V$ is finite.*

This result essentially reduces the proof of the finiteness conjecture over fields of characteristic zero to semi-simple groups. On the other hand, Theorem 1 is no longer valid in positive characteristic without additional assumptions, as one can see by considering Artin–Schreier extensions already of global fields. The following theorem describes appropriate assumptions in the more general context of function fields over fields k of type (F). We let p denote the characteristic exponent of k, i.e. $p = 1$ if k is of characteristic zero and $p = \mathrm{char}\, k$ otherwise.

Theorem 2. *Let $K = k(X)$ be the function field of a normal geometrically integral variety X defined over a field k of type* (F)*, and let V be the corresponding geometric set of places of K.*

(i) *If X is complete, then for each $d \geq 1$, the set of K-isomorphism classes of d-dimensional K-tori that have good reduction at all $v \in V$ is finite.*
(ii) *In the general case, for each $d \geq 1$, the set of K-isomorphism classes of d-dimensional K-tori T that have good reduction at all $v \in V$ and for which the degree $[K_T : K]$ of the minimal splitting field is prime p is finite.*

Any finitely generated field K of characteristic $p > 0$ can be realized as the function field $k(X)$ of a geometrically integral normal variety X over a finite field k. The choice of such a realization gives rise to a divisorial/geometric set V of discrete valuations of K. If X is chosen to be complete, the corresponding V is also called *complete*. Since finite fields are of type (F), we obtain the following.

Corollary 3. *Let K be a finitely generated field of characteristic $p > 0$, and let V be a divisorial set of places of K.*

(i) *If V is complete, then for any $d \geq 1$, the set of K-isomorphism classes of d-dimensional K-tori that have good reduction at all $v \in V$ is finite.*
(ii) *In the general case, for any $d \geq 1$, the set of K-isomorphism classes of d-dimensional K-tori T that have good reduction at all $v \in V$ and for which the degree $[K_T : K]$ of the minimal splitting field is prime to p is finite.*

The proofs of Theorems 1 and 2 rely on the information of when the étale fundamental group of a flat scheme of finite type over $\mathbb{Z}$ / a variety, or its maximal prime-to p quotient, is a group of type (F) (see [5], [7, Exp. IX and X], [8], [12]).

2. Class sets and Condition (T)

Let K be a field equipped with a set V of discrete valuations, and let G be a linear algebraic K-group with a fixed matrix realization $G \subset \mathrm{GL}_n$. For each $v \in V$, we set $G(\mathcal{O}_v) = G(K_v) \cap \mathrm{GL}_n(\mathcal{O}_v)$ and then define the corresponding adelic group as

$$G(\mathbb{A}(K,V)) = \left\{ (g_v) \in \prod_{v \in V} G(K_v) \;\middle|\; g_v \in G(\mathcal{O}_v) \text{ for almost all } v \in V \right\}.$$

The product $G(\mathbb{A}^\infty(K,V)) = \prod_{v \in V} G(\mathcal{O}_v)$ is called the *subgroup of integral adeles*. Let us now assume that V satisfies the following condition (which holds automatically for all divisorial (resp., geometric) sets of places of finitely generated (resp., function) fields):

$(*)$ For any $a \in K^\times$, the set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite.

Then the group $G(\mathbb{A}(K,V))$ does not depend on the choice of a faithful representation $G \hookrightarrow \mathrm{GL}_n$ (while the group $G(\mathbb{A}^\infty(K,V))$ does). Furthermore, in this case there is a diagonal embedding $G(K) \hookrightarrow G(\mathbb{A}(K,V))$, whose image is called the *subgroup of principal adeles* and which will still be denoted simply by $G(K)$. The set of double cosets

$$\mathrm{cl}(G,K,V) := G(\mathbb{A}^\infty(K,V)) \backslash G(\mathbb{A}(K,V)) / G(K)$$

is called the *class set* of G. It was proved by A. Borel [1] that for any algebraic group G over a number field K and the set V of all nonarchimedean places of K, the class set is always finite, and then its cardinality is called the *class number*. The reader can find some interpretations and computations of the class number in this case in [10, Ch. VIII]. Over fields more general than global fields, the class set may be infinite. For example, for the 1-dimensional split torus $G = \mathbb{G}_m$, the class set can be identified with the Picard group $\mathrm{Pic}(V)$, defined as the quotient of the free abelian group on the set V by the subgroup of "principal divisors", which can be considered in view of condition $(*)$ (cf. [3, §2]). Then it is easy to find examples of finitely generated fields K and their divisorial sets of places V for which $\mathrm{Pic}(V)$ is infinite, although in this case it is known to be finitely generated [9]. When G is commutative, the class set has a group structure and is then called the *class group*, and one can again ask if/when it is finitely generated; however, in the general case, no natural group structure can be introduced. So, in [4] we proposed a weaker finiteness condition (which holds automatically if either the class set is finite or G is commutative and the class group is finitely generated) that turned out to be instrumental in the analysis of unramified cohomology.

Condition (T). There exists a finite subset $S \subset V$ such that $|\mathrm{cl}(G,K,V \setminus S)| = 1$.

We note that in the presence of $(*)$, the fact whether or not Condition (T) holds for a given algebraic K-group G does not depend on the choice of its matrix realization. Referring to [11, §6] for a general discussion of Condition (T), in this note we will limit ourselves to the following statement that treats the case of groups whose connected component is a torus and will play a key role in the analysis of the global-to-local map for Galois cohomology in the next section.

Theorem 4. *Let K be a finitely generated field and V be a divisorial set of places of K. Then any algebraic K-group D whose connected component D° is a torus satisfies Condition* (T).

One reduces the general case to the case where D is a torus, where the required fact is derived from a result of B. Kahn [9].

3. Global-to-local map in Galois cohomology: finitely generated fields

Let G be an algebraic group over a field K, and let L/K be a Galois extension. As usual, the 1-cohomology set $H^1(\mathrm{Gal}(L/K), G(L))$ will be denoted $H^1(L/K, G)$. If G is commutative, the groups $H^i(L/K, G)$ are defined for all $i \geq 1$. We will write $H^i(K, G)$ for $H^i(K^{\mathrm{sep}}/K, G)$. Suppose now that K is equipped with a set V of discrete valuations. Then one considers the global-to-local map

$$\lambda^i_{G,V,L/K}\colon H^i(L/K, G) \longrightarrow \prod_{v\in V} H^i(L_w/K_v, G), \quad w|v,$$

which will be denoted $\lambda^i_{G,V}$ when $L = K^{\mathrm{sep}}$. One says that the Hasse principle holds if $\lambda^i_{G,V,L/K}$ is injective (this term is particularly used for $i = 1$, cf. [10, Ch. 6], [13, Ch. III, §4]). This may or may not be the case, but Borel and Serre [2] showed that when K is a number field and V consists of almost all places of K, the map $\lambda^1_{G,V}$ is *proper* (i.e., has finite fibers) for any algebraic K-group G. We note that when G is commutative, then the properness of $\lambda^1_{G,V}$ is equivalent to the finiteness of the Tate–Shafarevich group $\mathrm{Ш}(G, V) := \ker\lambda^1_{G,V}$. Currently, it is expected that $\lambda^1_{G,V}$ should be proper for any reductive group G over a finitely generated field K for any divisorial set V of places of K (cf. [11, §6]). The results below show that this is indeed the case for algebraic groups whose connected component is a torus.

Theorem 5. *Let K be a finitely generated field and V be a divisorial set of places of K. Then for any linear algebraic K-group D whose connected component D° is a torus and any finite Galois extension L/K, we have:*

(i) *if D is commutative, then the map $\lambda^i_{D,V,L/K}$ is proper for any $i \geq 1$;*
(ii) *in the general case, the map $\lambda^1_{D,V,L/K}$ is proper.*

For the proof, one first re-interprets the statement in terms of adelic groups and then uses Theorem 4 to complete the argument. It should be noted that the standard proof of the finiteness of the Tate–Shafarevich group for tori over global fields (cf. [15, 11.3]) makes use of Tate–Nakayama duality, while our argument in this case only requires the finiteness of the class number and the finite generation of the unit group.

Theorem 6. *For K, V, and D as in Theorem 5, the map $\lambda^1_{D,V}$ is proper. In particular, for any K-torus T, the Tate–Shafarevich group $\mathrm{Ш}(T, V)$ is finite.*

Here one begins by considering the case of a finite group D. This enables one to reduce the proof to the statement proved in Theorem 5 (ii). We also observe that for a semi-simple K-group G, the truth of the finiteness conjecture for the K-forms of G having good reduction at all $v \in V \setminus S$ for an arbitrary finite subset $S \subset V$ (see Section 1) automatically implies the properness of $\lambda^1_{G,V}$ for the corresponding adjoint group $\overline{G}$ (cf. [11, §6]); for tori, however, these two finiteness properties do not appear to be related.

As an application, we obtain the following finiteness result on the local-global conjugacy problem for maximal tori in any reductive group.

Theorem 7. *Let G be a connected reductive group over a finitely generated field K, and let V be a divisorial set of places of K. Fix a maximal K-torus T of G and let $\mathcal{C}(T)$ be the set of all maximal K-tori T' of G such that T and T' are $G(K_v)$-conjugate for all $v \in V$. Then $\mathcal{C}(T)$ consists of finitely many $G(K)$-conjugacy classes.*

It was pointed out to us by the anonymous referee that over global fields, this result is due to P. Gille and L. Moret-Bailly [6, Thm. 7.9]; it was used by E. Ullmo and A. Yafaev [14] in their work on the André–Oort conjecture. Yet another consequence of Theorem 6 is the following.

Theorem 8. *Let T be a K-torus, and let $\mathrm{Ш}^2(T, V) = \ker\lambda^2_{T,V}$. Then for any integer $\ell > 0$, the ℓ-torsion subgroup ${}_\ell\mathrm{Ш}^2(T, V)$ is finite.*

4. Global-to-local map in Galois cohomology: function fields

In this section, we take K to be the function field $k(X)$ of a normal geometrically integral variety X defined over a field k, and V will be the associated set of geometric places. We also denote by p the characteristic exponent of k.

Theorem 9. *If k has characteristic zero and is of type* (F)*, then for any K-torus T the Tate–Shafarevich group $Ш(T,V)$ is finite.*

The proof uses finiteness theorems for étale cohomology and purity results. We also have the following properness result for finite Galois modules that does not require any assumptions on the base field k.

Proposition 10. *Let Ω be a finite (but not necessarily commutative) étale K-group. Then in each of the following situations*

(1) $\dim X \geq 2$,
(2) *X is a projective curve,*
(3) *X is an arbitrary curve, but the order of Ω is prime to p,*

the map

$$H^1(K,\Omega) \longrightarrow \prod_{v\in V} H^1(K_v,\Omega)$$

is proper.

This proposition has the following consequence in the spirit of Theorem 7.

Proposition 11. *Let G be a connected reductive K-group. Fix a maximal K-torus T of K and let $\mathcal{C}(T)$ be the set of all maximal K-tori T' of G such that T and T' are $G(K_v)$-conjugate for all $v \in V$. Then, with the exception of the following case*

(•) *X is an affine curve and the order of the Weyl group $W(G,T)$ is divisible by p,*

$\mathcal{C}(T)$ consists of finitely many K-isomorphism classes.

Using Artin–Schreier extensions of the field $K = k(x)$ of rational functions on $X = \mathbb{A}^1_k$ over an algebraically closed field k of characteristic $p > 0$, one can construct an infinite family of pairwise non-isomorphic maximal K-tori of the group $G = \mathrm{SL}_p$ such that any two of these are $G(K_v)$-conjugate for all v in the corresponding geometric set of places V. So, the situation in (•) is an honest exception in Proposition 11.

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