Galois Grothendieck Seminar - 2021/2022

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Contents

Chapter 1

Some Category Theory

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1.1 Lecture 1 (8/31/2021)

Main idea: Category theory gives a convenient way to describe mathematical objects with functions between them. It allows general organizing principles together with general results that apply to many examples.

Definition 1.1.1. A **category** $\mathcal{C}$ consists of a collection of objects $\mathrm{ob}(\mathcal{C})$, and for any pair of **objects** (A, B), a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of **morphisms** $A \to B$, together with,

- for every object A, an identity morphism $id_A : A \to A$, and
- a composition law

$$\mathrm{Hom}_{\mathcal{C}}(A,B) \times \mathrm{Hom}_{\mathcal{C}}(B,C) \to \mathrm{Hom}_{\mathcal{C}}(A,C)$$
$$(f,g) \mapsto g \circ f$$

satisfying axioms

- (unitarity) given any $f : A \to B$, $f \circ id_A = f = id_B \circ f$
- (associativity) for any $f : A \to B, g : B \to C, h : C \to D$,

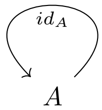
$$(h \circ g) \circ f = h \circ (g \circ f)$$

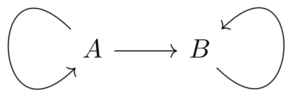
E.g.

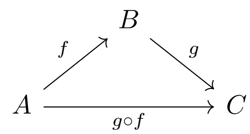
1) sets and functions ($\mathcal{S}et$)
2) groups and group homomorphisms ($\mathcal{G}p$)

3) topological spaces and continuous maps ($\mathcal{T}op$)

4) vector spaces and linear maps

5) [0] : 

, [1] : 

, [2] : 

6) Any group G induces a category with one object X, and $\mathrm{Hom}(X, X) = G$.

7) Any equivalent relation on a set X defines a category with objects, the elements of X and a single morphism $x \to y$ whenever $x \sim y$.

Examples (5)-(7) are **small**, in that $\mathrm{ob}(\mathcal{C})$ is a set. (1)-(4) are **large**, i.e. $\mathrm{ob}(\mathcal{C})$ is a proper class.

We can also take **subcategories**, by taking subcollections of objects and morphisms, in a compatible way.

E.g.

8) Finite sets and functions is a subcategory of $\mathcal{S}et$.

9) Groups and isomorphisms forms a subcategory of $\mathcal{G}p$.

Example(8) is a **full subcategory**: $\mathrm{Hom}_{\mathcal{F}in\mathcal{S}et}(X, Y) \cong \mathrm{Hom}_{\mathcal{S}et}(X, Y)$

Example(9) is not full, but it is **wide**, in that it contains all objects.

Definition 1.1.2. An **isomorphism** is a morphism $f : A \to B$ such that, $\exists g : B \to A$ such that $g \circ f = id_A$ and $f \circ g = id_B$.

E.g.

- bijections between sets
- group isomorphisms
- homeomorphisms

Examples of categories in which all morphisms are isomorphisms are [0] from (5), and (6), (7), (9). We call them **groupoids**.

Definition 1.1.3. An **initial object** in a category $\mathcal{C}$ is an element $\varnothing$ such that for any object C of $\mathcal{C}$, $\exists!$ morphism $\varnothing \to C$.

Dually, a **terminal object** is an object $*$ such that for any C, there is a unique $C \to *$.

E.g.

- In $\mathcal{S}et$, the empty set is initial, and any singleton $\{x\}$ is terminal.

- In $\mathcal{G}p$, the trivial group $\{e\}$ is both initial and terminal, we call it a **zero object**.
- In a group G regarded as a category, the single object is neither initial nor terminal, unless $G = \{e\}$.
- The category $0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \ldots$ has an initial object, but no terminal objects.

Definition 1.1.4. A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ assigns to any object A of $\mathcal{C}$ an object $F(A)$ of $\mathcal{D}$ and to any morphism $f : A \to B$, a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$, satisfying

- $F(id_A) = id_{F(A)}$
- for $f : A \to B, g : B \to C, F(g \circ f) = F(g) \circ F(f)$.

A **contravariant functor** F takes $f : A \to B$ to $F(f) : F(B) \to F(A)$.

E.g.

- Fundamental group $\pi_1 : \mathcal{T}op_* \to \mathcal{G}p$ is covariant.
- Homology groups $H_q : \mathcal{T}op_* \to \mathcal{A}b$ are covariant.
- Cohomology groups $H^q : \mathcal{T}op \to \mathcal{A}b$ are contravariant.

Definition 1.1.5. Given a category $\mathcal{C}$, its opposite category $\mathcal{C}^{op}$ has the same objects as $\mathcal{C}$, but

$$\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(B, A)$$

.

Fact: Contravariant functors $\mathcal{C} \to \mathcal{D} \iff$ Covariant functors $\mathcal{C}^{op} \to \mathcal{D}$.

E.g. The **forgetful functor** $U : \mathcal{G}p \to \mathcal{S}et$ takes a group to its underlying set.

The **free functor** $F : \mathcal{S}et \to \mathcal{G}p$ takes a set to the free group on the set.

These functors are **adjoint**, in that $\mathrm{Hom}_{\mathcal{G}p}(FX, G) \cong \mathrm{Hom}_{\mathcal{S}et}(X, UG)$ naturally (Functions out of a free group are determined by where the generators go), i.e.

For any $G \to H$,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{G}p}(FX, G) & \longrightarrow & \mathrm{Hom}_{\mathcal{G}p}(FX, H) \\ \downarrow \cong & & \cong \downarrow \\ \mathrm{Hom}_{\mathcal{S}et}(X, UG) & \longrightarrow & \mathrm{Hom}_{\mathcal{S}et}(X, UH) \end{array}$$

commutes.

For any $X \to Y$,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{G}p}(FX, G) & \longrightarrow & \mathrm{Hom}_{\mathcal{G}p}(FY, G) \\ \downarrow \cong & & \cong \downarrow \\ \mathrm{Hom}_{\mathcal{S}et}(X, UG) & \longrightarrow & \mathrm{Hom}_{\mathcal{S}et}(Y, UG) \end{array}$$

commutes.

We write this as $F : \mathcal{S}et \rightleftharpoons \mathcal{G}p : U$ or (F, U) or $F \dashv U$. F is the **left adjoint** and U is the **right adjoint**.

Observe that $F(U(G)) \neq G$ and $U(F(X)) \neq X$. But there do exist unit and counit maps $X \to UFX$ and $FUG \to G$ for any set X and group G.

We can state this in terms of natural transformations.

Definition 1.1.6. Given $F, G : \mathcal{C} \to \mathcal{D}$, a **natural transformation** $\alpha : F \Rightarrow G$ consists of

for any object A of $\mathcal{C}$, $\alpha_A : F(A) \to G(A)$ in $\mathcal{D}$, s.t.

For any $f : A \to B$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ \Big\downarrow{\scriptstyle F(f)} & & \Big\downarrow{\scriptstyle F(g)} \\ F(B) & \xrightarrow[\alpha_B]{} & G(B) \end{array}$$

commutes.

So, for any adjoint pair (F, U), there are natural transformations $FU \Rightarrow id_{\mathcal{D}}$ (counit) and $id_{\mathcal{C}} \Rightarrow UF$ (unit).

E.g. Consider the forgetful functor $U : \mathcal{T}op \to \mathcal{S}et$.

It has a left adjoint F, taking a set to a discrete space $\mathrm{Hom}_{\mathcal{T}op}(FX, Y) \cong \mathrm{Hom}_{\mathcal{S}et}(X, UY)$.

It also has a right adjoint R, taking a set to an indiscrete space $\mathrm{Hom}_{\mathcal{S}et}(UY, X) \cong \mathrm{Hom}_{\mathcal{T}op}(Y, RX)$.

E.g. (Non-examples)

The forgetful functor $U : \mathcal{F}ield \to \mathcal{S}et(\mathcal{A}b/\mathcal{R}ing)$ doesn't have a left adjoint.

Nontrivial field homomorphisms only exist between fields of the same characteristic, but can have set functions or group/ring homomorphisms between fields of different characteristics.

E.g. The left adjoint to the forgetful functor

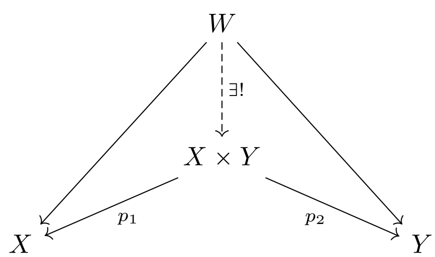
$$\mathcal{A}b \to \mathcal{A}bMon$$

(abelian monoid) is the group completion/Grothendieck group functor K_0.

1.2 Lecture 2 (9/7/2021)

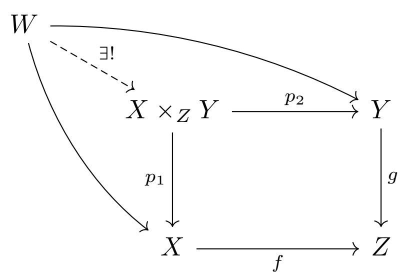
Last time we talked about adjoint functors $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$, where $\mathrm{Hom}_{\mathcal{D}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, GY)$ for objects X of $\mathcal{C}$ and Y of $\mathcal{D}$.

Definition 1.2.1. Given objects X and Y, then the **product** $X \times Y$, with morphisms $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$, is defined by a universal property: for any W and any morphisms $W \to X$ and $W \to Y$, there exists a unique morphism $W \to X \times Y$, s.t. the following diagram commutes



E.g. Usual cartesian product of sets, spaces, groups.

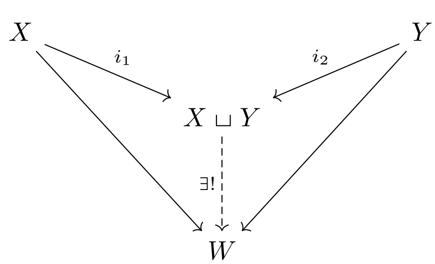
Definition 1.2.2. Given objects X, Y, Z and two morphisms $f : X \to Z$ and $g : Y \to Z$, then the **pullback** $X \times_Z Y$, with morphisms $p_1 : X \times_Z Y \to X$ and $p_2 : X \times_Z Y \to Y$, is defined by a universal property: for any W and any morphisms $W \to X$ and $W \to Y$, there exists a unique morphism $W \to X \times_Z Y$, s.t. the following diagram commutes



E.g. In $\mathcal{S}et$, $X \times_Z Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$. We can think of this diagram $X \to Z \leftarrow Y$ as a functor from the category $\bullet \longleftarrow \bullet \longrightarrow \bullet$ to $\mathcal{C}$.

The pullback is "universal cone": anything else mapping to this diagram factors through it.

Definition 1.2.3. Dually (to product), the **coproduct** $X \sqcup Y$ satisfies the universal property corresponding to the following commutative diagram



E.g. Disjoint union of sets, spaces, free products of groups, direct sum of modules.

Definition 1.2.4. Dually (to pushback), the **pushout** $X \sqcup_Z Y$ satisfies the universal property corresponding to the following commutative diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "i_2"] \arrow[ddr, bend left] & \\ X \arrow[r, "i_1"'] \arrow[drr, bend right] & X \sqcup_Z Y \arrow[dr, dashed, "\exists!"] & \\ & & W \end{tikzcd}$$

E.g. In $\mathcal{S}et$ or $\mathcal{T}op$, $X \sqcup_Z Y = X \sqcup Y / \sim$, where $f(z) \sim g(z), \forall z \in Z$.

E.g. Amalgamated free product of groups: $G *_A H$.

We can think of the diagram $X \xleftarrow{f} Z \xrightarrow{g} Y$ as a functor from $\bullet \longleftarrow \bullet \longrightarrow \bullet$ to $\mathcal{C}$.

The pushout is a "universal cocone", maps out of the diagram factor through it.

Want to describe similar phenomena for more complicated diagrams.

Let ℓ be a small category, $\mathcal{C}$ an arbitrary category, and $F : \ell \to \mathcal{C}$ a functor, thought of as a diagram in $\mathcal{C}$. We have a category $\mathcal{C}^\ell$ of such functors: objects are functors, morphisms are natural transformations, e.g. $\ell = \underset{a}{\bullet} \longrightarrow \underset{c}{\bullet} \longleftarrow \underset{b}{\bullet}$

$$\begin{tikzcd} F(a) \arrow[r] \arrow[d] & F(c) \arrow[d] & F(b) \arrow[l] \arrow[d] \\ G(a) \arrow[r] & G(c) & G(b) \arrow[l] \end{tikzcd}$$

Consider the constant diagram functor $\Delta : \ell \to \mathcal{C}^\ell$, $X \mapsto \Delta X$, e.g. $X \xrightarrow{id} X \xleftarrow{id} X$. If it exists, its right adjoint $\lim_\ell : \mathcal{C}^\ell \to \mathcal{C}$, $F \mapsto \lim_\ell F$ takes a diagram to its **limit**.

Check what's happening for $\ell = (\bullet \longrightarrow \bullet \longleftarrow \bullet)$.

$$\Delta : \ell \leftrightarrows \mathcal{C}^\ell : \lim_\ell .$$

$$\mathrm{Hom}_{\mathcal{C}^\ell}(\Delta X, F) \cong \mathrm{Hom}_{\mathcal{C}}(X, \lim_\ell F).$$

$$\begin{tikzcd} X \arrow[r, "id"] \arrow[d] & X \arrow[d] & X \arrow[l, "id"'] \arrow[d] \\ F(a) \arrow[r] & F(c) & F(b) \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[d] & \\ & \lim_\ell F \arrow[dl] \arrow[d] \arrow[dr] & \\ F(a) \arrow[r] & F(c) & F(b) \arrow[l] \end{tikzcd}$$

Its left adjoint, if it exists, takes a diagram to its colimit

$$\mathrm{colim}_\ell : \mathcal{C}^\ell \leftrightharpoons \ell : \Delta$$

$$\mathrm{Hom}_\ell(\mathrm{colim}_\ell F, X) \cong \mathrm{Hom}_{\mathcal{C}^\ell}(F, \Delta X)$$

e.g. for $\ell = (\ \bullet \longleftarrow \bullet \longrightarrow \bullet\ )$,

$$\begin{array}{ccccccc} \mathrm{colim}_\ell F & \quad & F(a) & \longleftarrow & F(c) & \longrightarrow & F(b) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X & & X & \xleftarrow{id} & X & \xrightarrow{id} & X \end{array}$$

In many our standart examples, small limits and colimits (i.e. where ℓ is small) all exist. But they need not, in general.

E.g. In $\mathcal{S}et$, the colimit of $\{1\} \hookrightarrow \{1,2\} \hookrightarrow \{1,2,3\} \hookrightarrow \cdots$ is $\{1,2,3,...\}$. In $\mathcal{F}in\mathcal{S}et$, this colimit doesn't exist.

Fact: If ℓ has an initial object $\varnothing$, then the limit of any $F : \ell \to \mathcal{C}$ is (isomorphic to) $F(\varnothing)$. Dually, if ℓ has a terminal object $*$, then the colimit of any $F : \ell \to \mathcal{C}$ is $F(*)$.

Note: Limits and colimits, if they exists, are unique up to unique isomorphisms.

E.g. Consider the diagram $\ell = \left(\underset{a}{\bullet} \underset{f}{\overset{g}{\rightrightarrows}} \underset{b}{\bullet} \right)$. The limit of a functor $F : \ell \to \mathcal{C}$ is called an equializer.

Its colimit is called a coequalizer.

E.g. In $\mathcal{A}b$, $G \underset{e}{\overset{f}{\rightrightarrows}} H$ has equalizer $\ker(f)$ and coequalizer is $\mathrm{coker}(f)$.

Note: The qualizer of $G \underset{g}{\overset{f}{\rightrightarrows}} H$ is not the same as the pullback of $\begin{array}{ccc} & & G \\ & & \downarrow g \\ G & \xrightarrow{f} & H \end{array}$.

Proposition 1.2.5. A category $\mathcal{C}$ has all small limits (is **complete**) if and only if it has all products and equalizers. It has all small colimits (is **cocomplete**) if and only if that has coproducts and coequalizers.

Definition 1.2.6. A functor $\mathcal{C} \to \mathcal{S}et$ is **representable** if it is naturally isomorphic to one of the form $\mathrm{Hom}_{\mathcal{C}}(X, -)$ (for covariant functors) or $\mathrm{Hom}_{\mathcal{C}}(-, X)$ (for contravariant functors) for some object X of $\mathcal{C}$.

E.g. $id : \mathcal{S}et \to \mathcal{S}et$ is representable by a singleton set $\{a\}$, $id(X) = X = \mathrm{Hom}_{\mathcal{S}et}(\{a\}, X)$.

E.g. The forgetful functor $U : \mathcal{G}p \to \mathcal{S}et$ is representable by $\mathbb{Z}$. $\mathrm{Hom}_{\mathcal{G}p}(\mathbb{Z}, G) = UG$, the underlying set of G, since a morphism $\mathbb{Z} \to G$ is specified by where 1 goes.

E.g. The contravariant functor $\mathrm{Hom}(- \times A, B) : \mathcal{S}et \to \mathcal{S}et$ is represented by $\mathrm{Hom}_{\mathcal{S}et}(A, B) = B^A$.

$$\mathrm{Hom}_{\mathcal{S}et}(C \times A, B) \cong \mathrm{Hom}_{\mathcal{S}et}(C, \mathrm{Hom}_{\mathcal{S}et}(A, B)).$$

$$C \times A \to B \qquad C \to \mathrm{Hom}_{\mathcal{S}et}(A, B)$$

$$(c, a) \mapsto b_{c,a} \qquad c \mapsto b_{c,a}$$

Note: Representable functors preserve limits, so not all functors are representable.

Lemma 1.2.7. (Yoneda) Given a functor $F : \mathcal{C} \to \mathcal{S}et$ and an object X of $\mathcal{C}$, there is a natural bijection (in F and X)

$$\text{NatTrans}(\text{Hom}_{\mathcal{C}}(X, -), F) \cong F(X)$$

$$\{\alpha_Y : \text{Hom}_{\mathcal{C}}(X, Y) \to F(Y)\} \mapsto \alpha_X(id_X) \in F(X)$$

E.g. For any functor $F : \mathcal{G}p \to \mathcal{S}et$, the set of natural transformations $U \to F$ is given by

$$\text{NatTrans}(\text{Hom}_{\mathcal{G}p}(\mathbb{Z}, -), F) \cong F(\mathbb{Z}).$$

Yoneda Embedding: Let $\mathcal{C}$ be a small category. Then the functor

$$\begin{array}{ccc} \mathcal{C} \longrightarrow \mathcal{S}et^{\mathcal{C}^{op}} & and \mathcal{C}^{op} \longrightarrow \mathcal{S}et^{\mathcal{C}} \\ x \longmapsto \text{Hom}(-, x) & x \longmapsto \text{Hom}(x, -) \end{array}$$

are fully faithful.

Definition 1.2.8. A functor $F : \mathcal{C} \to \mathcal{D}$ is **full** if for any objects X, Y of $\mathcal{C}$, the map $\text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective. It is **faithful** if this map is injective. F is **essentially surjective** if for any object Z of $\mathcal{D}$, there exists an object X of $\mathcal{C}$ such that $F(X) \cong Z$.

Definition 1.2.9. $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence** if it is fully faithful and essentially surjective.

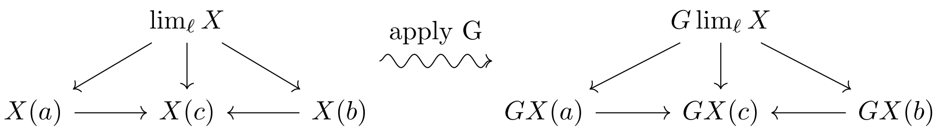
E.g. $\mathcal{C} = (\bullet)$ and $\mathcal{D} = \left(\bullet \rightleftarrows \bullet \right)$

E.g. $\mathcal{C} = \mathcal{F}inSet$, $\mathcal{D}$ is the category of sets $\varnothing, \{1\}, \{1, 2\}, ..., \{1, ..., n\}, ...$ We sometimes call $\mathcal{D}$ a **skeletal** subcategory of $\mathcal{D}$, since there are no non-identity isomorphisms.

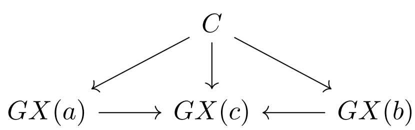
1.3 Lecture 3 (9/14/2021)

Theorem 1.3.1. *Right adjoints preserve limite, and left adjoints preserve colimits.*

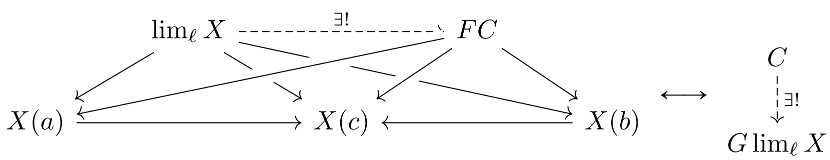
Idea of proof: Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjoint pair, $X : \ell \to \mathcal{D}$ with ℓ small, with limit $\lim_\ell X$. e.g.,



If C is another cone of this diagram:



Apply F:



Dual argument for colimits. □

Can also write as isomorphisms:

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{C}^\ell}(\Delta G, GX) &\cong \operatorname{Hom}_{\mathcal{D}^\ell}(F\Delta C, X) \\
&\cong \operatorname{Hom}_{\mathcal{D}^\ell}(\Delta FC, X) \\
&\cong \operatorname{Hom}_{\mathcal{D}}(FC, \lim_\ell X) \\
&\cong \operatorname{Hom}_{\mathcal{C}}(C, G\lim_\ell X)
\end{aligned}$$

Last topic: Abelian Categories.

Idea: formalize nice properties of the category ${}_R\mathcal{M}od$ of $R-$modules

First properties:

- enriched in $\mathcal{A}b$: $\operatorname{Hom}(M, N)$ is an abelian group
- has a zero object (the 0 module) that is both initial and terminal
- has all binary products $(M \times N)$ and coproducts $(M \oplus N\ )$.

Any category with these properties is called **additive**.

Note: A consequence is that finite products agree with finite coproducts, e.g., $M \times N \cong M \oplus N$.

But there's more structure in ${}_R\mathcal{M}od$, intuitively given by the existence of short exact sequencec, i.e., monomorphisms, epimorphisms, kernels and cokernels.

Definition 1.3.2. A morphism $f : X \to Y$ in a category $\mathcal{C}$ is:

- a **monomorphism** if for any $h, k : W \to X$, if $fh = fk$, then $h = k$; and
- an **epimorphism** if for any $h, k : Y \to Z$, if $hf = kf$, then $h = k$.

Definition 1.3.3. Let $\mathcal{C}$ be an additive category, and $f : X \to Y$ a morphism

- the **kernel** of f is the equalizer of $X \underset{0}{\overset{f}{\rightrightarrows}} Y$;and
- the **cokernel** of f is the coequalizer of $X \underset{0}{\overset{f}{\rightrightarrows}} Y$.

Definition 1.3.4. An **abelian category** is an additive category $\mathcal{C}$ such that every morphism has a kernel and a cokernel, every monomorphism arises as a kernel, and every epimorphism arises as a cokernel.

E.g. ${}_R\mathcal{M}od$, $\mathcal{C}h(R)$: chain complexes of $R-$modules.

Definition 1.3.5. A functor $\mathcal{C} \to \mathcal{D}$ between abelian categories is **exact** if it preserves exact sequences.

Theorem 1.3.6. *(Freyd-Mithcell Embedding Theorem) For any small abelian category $\mathcal{C}$, there exists a ring R and an exact, fully faithful functor $\mathcal{C} \to {}_R\mathcal{M}od$ that embeds $\mathcal{C}$ as a full subcategory.*

Chapter 2

Introduction to spectral sequences

Peter Abramenko

For this set of notes, R is a ring with 1 and $\mathcal{C} = {}_R\mathbf{Mod}$ is the category of left R-modules.

2.1 Filtered Differential Modules

Definition 2.1.1. A *decreasing filtration* of $M \in \mathcal{C}$ is a sequence $(F^pM)_{p\in\mathbb{Z}}$ if submodules with $F^{p+1}M \leqslant F^pM$ for all $p \in \mathbb{Z}$ and $M = \bigcup_{p\in\mathbb{Z}} F^pM$. The filtration is called *finite* if there exist $p_1, p_2 \in \mathbb{Z}$ such that $F^{p_1}M = M$ and $F^{p_2}M = 0$.

With any filtration (F^pM) of M, we associate a *graded module*

$$\operatorname{Gr} M := (\operatorname{Gr}^p M)_{p\in\mathbb{Z}}$$

where $\operatorname{Gr}^p M := F^pM/F^{p+1}M$. The module $\operatorname{Gr} M$ is sometimes identified with $\bigoplus_{p\in\mathbb{Z}} \operatorname{Gr}^p M$.

Example 2.1.2. *Suppose* $M = \bigoplus_{p\in\mathbb{Z}} M^p$, $F^pM = \bigoplus_{i\geqslant p} M^i$. *Then* $\operatorname{Gr}^p M = M^p$ *for all* $p \in \mathbb{Z}$.

Lemma 2.1.3. *If* (F^pM) *and* (F^pM') *are finite filtrations of* $M, M' \in \mathcal{C}$ *and* $f : M \to M'$ *is a homomorphism of filtered modules (i.e.* $f(F^pM) \subseteq F^pM'$ *for all* p*) such that the map induced by* f, $\operatorname{Gr} f : \operatorname{Gr} M \to \operatorname{Gr} M'$, *is an isomorphism, then* f *is also an isomorphism.*

Proof. Without loss of generality, we may shift the indices of the filtrations so that

$$M = F^0M \geqslant F^1M \geqslant \ldots \geqslant F^nM = 0$$
$$M' = F^0M' \geqslant F^1M' \geqslant \ldots \geqslant F^nM' = 0$$

Note that we are applying *the same shift* to each filtration. By assumption, we know that the maps $\operatorname{Gr}^p f : F^pM/F^{p+1}M \to F^pM'/F^{p+1}M'$ are isomorphisms for all p. We will show that f is an isomorphism by induction on n.

We can easily reduce to the case where $n = 2$. That is, we are given

$$M \geqslant F^1 M \geqslant 1 \quad \text{and} \quad M' \geqslant F^1 M' \geqslant 0$$

where $f_1 := \mathrm{Gr}^1 f : F^1 M \to F^1 M'$ and $\tilde{f} := \mathrm{Gr}^0 f : M/F^1 M \to M'/F^1 M'$ are isomorphisms.

To check that f is injective, note that for all $x \in \ker f$, $\tilde{f}(x + F^1 M) = 0$. Since $\tilde{f}$ is injective, we have that $x + F^1 M = F^1 M$ and $x \in F^1 M$. Therefore, $f_1(x) = f(x) = 0$ and by injectivity of f_1, $x = 0$. Thus f is injective.

We now check that f is surjective. Let $y \in M'$. Now, since $\tilde{f}$ is surjective, there exists an $m \in M$ such that $\tilde{f}(x + F^1 M) = y + F^1 M'$. Ergo, $y \in f(x) + F^1 M'$. Since f_1 is surjective, $F^1 M' = f(F^1 M)$. Ergo, $y \in f(x) + f(F^1 M) = f(x + F^1 M) \subseteq f(M)$. Thus f is surjective as desired. □

Definition 2.1.4. A *differential module* is a pair (M, d) with $M \in \mathcal{C}$ and $d \in \hom_R(M, M)$ such that $d^2 = 0$ (that is, $\operatorname{im} d \subseteq \ker d$). We set

$$\begin{aligned} B := B(M) &:= \operatorname{im} d && \text{(coboundaries)} \\ Z := Z(M) &:= \ker d && \text{(cocycles)} \\ H(M) &:= Z/B && \text{(cohomology)} \end{aligned}$$

A filtration of (M, d) is a filtration $(F^p M)_{p \in \mathbb{Z}}$ of M such that $d(F^p M) \subseteq F^p M$ for all p.

Example 2.1.5. *Take* $\mathbf{C} = (C^n, d^{\cdot} C^n \to C^{n+1})$ *to be a cochain complex. Set* $M := \bigoplus_n C^n$*, and* $d := \bigoplus_n d^n : M \to M$*. Then* (M, d) *is a differential module. If additionally, each* C^n *has a filtration* $(F^p C^n)_{p \in \mathbb{Z}}$ *such that* $d^n(F^p C^n) \subseteq F^p C^{n+1}$ *for all* p*, then* $(F^p M = \bigoplus_n F^p C^n)_{p \in \mathbb{Z}}$ *is a filtration of* (M, d)*.*

A filtration of $(F^p M)_{p \in \mathbb{Z}}$ of the differential modules (M, d) induces a filtration $(H(M)^p)_{p \in \mathbb{Z}}$ on the cohomology $H(M)$.

$$H(M)^p := (Z \cap F^p M)/(B \cap F^p M) \hookrightarrow Z/B = H(M).$$

On the other hand, each $F^p M$ becomes a differential modules with respect to $d_{F^p} = D_{1F^p} : F^p M \to F^p M$. The cohomology of $(F^p M, d_{F^p})$ is

$$H(F^p M) = \ker d_{F^p} / \operatorname{im} d_{F^p} = (Z \cap F^p M)/d_{F^p}(F^p M).$$

Note that $d_{F^p}(F^p M) = d(F^p M) \subseteq B \cap F^p M$ but $\operatorname{im} d_{F^p}$ need not equal $B \cap F^p M$ in general.

Thus, for each p, there exists a homomorphism $\zeta^p : H(F^p M) = (Z \cap F^p M)/d_{F^p}(F^p M) \to Z/B = H(M)$ with image $(Z \cap F^p M)/(B \cap F^p M) = G(M)^p = \{z + B | z \in Z \cap F^p M\}$.

The filtration $(H(M)^p)_{p \in \mathbb{Z}}$ of $H(M)$ induces a graded module $\mathrm{Gr}\, H(M) = (H(M)^p / H(M)^{p+1})_{p \in \mathbb{Z}}$.

We may now talk about the purpose of the spectral sequence: to approximate $E_\infty^p := H(M)^p / H(M)^{p+1}$ by a sequence $(E_r^p)_{r \in \mathbb{N}_0}$ starting with $E_0^p = F^p M / F^{p+1} M$ with $p \in \mathbb{Z}$.

2.2 Constructing the Spectral Sequence

Definition 2.2.1. For $r \in \mathbb{N}_0$, a *cochain complex* C *with differential* d *of degree* r is a direct sum $C = \bigoplus_{p\in\mathbb{Z}} C^p$ with a homomorphism $d : C \to C$ where $d^2 \equiv 0$ and $d^p : C^p \to C^{p+r}$ for all p. That is, $d^p \circ d^{p-r} = 0$. This also yields a cohomology module $H^p(C) = \ker d^p / \operatorname{im} d^{p-r}$ and $H(C) := \bigoplus_{p\in\mathbb{Z}} H^p(C)$.

The case with $r = 1$ gives us classical cochain complexes. An example of the $r = 0$ case is as follows.

Example 2.2.2. *Let (M, d) be a filtered differential module and $C = \operatorname{Gr} M$. Take $d_0^p : F^pM/F^{p+1}M \to F^pM/F^{p+1}M$ to be the map induced by $d^p = d|_{F^pM} : F^pM \to F^pM$. Then $H(\operatorname{Gr} M)$ will become $E_1 = \bigoplus_{p\in\mathbb{Z}} H^p(\operatorname{Gr} M) = \bigoplus_{p\in\mathbb{Z}} H(F^pM/F^{p+1}M)$.*

We now move towards constructing the spectral sequence $(E_r^p)_{p\in\mathbb{Z}}$ which we identify with $\bigoplus_{p\in\mathbb{Z}} E_r^p$, a cochain complex of degree $r \in \mathbb{N}_0$.

For the moment, we only have the differential $d : M \to M$ satisfying $d^2 = 0$ from our filtered differential module (M, d). Using this d we define for $r, p \in \mathbb{Z}$:

$$\begin{aligned} Z_r^p &:= \{x \in F^pM | dx \in F^{p+r}M\}, & Z_\infty^p &:= Z \cap F^pM \\ B_r^p &:= d(F^{p-r}M) \cap F^pM, & B_\infty^p &:= B \cap F^pM \end{aligned}$$

Observation 2.2.3. *The following observations will be useful.*

1. *For $r \geqslant 0$, we have $F^pM \subseteq F^{p+r}M$, $Z_r^p = F^pM$, and $B_r^p \subseteq F^{p-r}M$.*

2. *For fixed p we have that:*

$$\begin{matrix} d(F^pM) = & B_0^p & \subseteq & B_1^p & \subseteq & \dots & \subseteq & B_s^p & \subseteq & B_{s+1}^p & \subseteq & \dots & \subseteq & B_\infty^p & \subseteq Z_\infty^p \\ Z_\infty^p = & F^pM \cap Z & \subseteq & \dots & \subseteq & Z_{r+1}^p & \subseteq & Z_r^p & \subseteq & \dots & \subseteq & Z_1^p & \subseteq & Z_0^p & = F^pM \end{matrix}$$

3. *The "(r, p)-boundary identity": $Z_r^{p-r} = \{x \in F^{p-r}M | dx \in F^pM\}$ and thus, $d(Z_r^{p-r}) = d(F^{p-r}M) \cap F^pM = B_r^p$.*

4. *The "Z-jump identities": $F^{p+1}M \subseteq F^pM$ and $Z_{r-1}^{p+1} \subseteq Z_r^p$, $Z_\infty^{p+1} \subseteq Z_\infty^p$.*

Definition 2.2.4. For $r \geqslant 0$ and $p \in \mathbb{Z}$ we define $E_r^p := Z_r^p/(B_{r-1}^p + Z_{r-1}^{p+1})$ and $E_\infty^p := Z_\infty^p/(B_\infty^p + Z_\infty^{p+1})$.

Note: $B_{-1}^p \subseteq F^{p+1}M = Z_{-1}^{p+1}$ and $Z_0^p = F^pM$ implies that $E_0^p = Z_0^p/(B_{-1}^p + Z_{-1}^{p+1}) = F^pM/F^{p+1}M$.

Lemma 2.2.5. $E_\infty^p = Z_\infty^p/(B_\infty^p + Z_\infty^{p+1}) \cong H(M)^p/H(M)^{p+1}$

Proof. We have that $Z_\infty^p = Z \cap F^pM$ projects onto $H(M)^p = (Z \cap F^pM)/(B \cap F^pM) \cong \{z + B | z \in Z_\infty^p\} \subseteq Z/B = H(M)$. Thus, we get a projection $\pi_\infty^p : Z_\infty^p \to H(M)^p/H(M)^{p+1}$.

Now note that $\pi_\infty^p(z) = 0$ iff $z + B \in H(M)^{p+1}$ iff $z = z_1 + b$ for some $z_1 \in Z \cap F^{p+1}M = Z_\infty^{p+1}$ and some $b \in B$. Moreover $Z_\infty^{p+1} \subseteq Z_\infty^p$ and thus $b = z - z_1 \in Z_\infty^p \subseteq F^pM$. Ergo, $b \in B \cap F^pM = B_\infty$. Hence, $z + B \in H(M)^{p+1}$ iff $z \in B_\infty^p + Z_\infty^{p+1}$.

This implies that $\ker \pi^p_\infty = B^p_\infty + z^{p+1}_\infty$ and thus

$$Z^p_\infty/(B^p_\infty + Z^{p+1}_\infty) \cong H(M)^p/H(M)^{p+1}$$

as desired. □

We now introduce *differentials of degree* r. Note that $d(Z^p_r) \subseteq F^{p+r}M \cap B \subseteq F^{p+r}M \cap Z = Z^{p+r}_\infty \subseteq Z^{p+r}_r$. Furthermore, since $d^2 = 0$, $d(B^p_{r-1}) = 0$ and $d(Z^{p+1}_{r-1}) \subseteq B^{p+r}_{r-1}$ be definition Z^{p+1}_{r-1}. Thus, $d(B^p_{r-1} + Z^{p+1}_{r-1}) \subseteq B^{p+r}_{r-1}$ and d induces a homomorphism $d^p_r : E^p_r = Z^p_r/(B^p_{r-1}+Z^{p+1}_{r-1}) \to Z^{p+r}_r/(B^{p+r}_{r-1}+Z^{p+r+1}_{r-1}) = E^{p+r}_r$. Again since $d^2 = 0$, $d^p_r d^{p-r}_r = 0$ for all p. Thus $E_r := \bigoplus_{p\in\mathbb{Z}} E^p_r$ is a cochain complex with differential d_r of degree $r \in \mathbb{N}_0$.

Next we need to determine the cohomology $H(E_r) = \bigoplus_{p\in\mathbb{Z}} H^p(E_r)$. For this, we need the following.

Lemma 2.2.6. *1.* $\ker d^p_r = (Z^p_{r+1} + Z^{p+1}_{r-1})/(B^p_{r-1} + Z^{p+1}_{r-1})$.

2. $\operatorname{im} d^{p-r}_r = (B^p_r + Z^{p+1}_{r-1})/(B^p_{r-1} + Z^{p+1}_{r-1})$.

Proof. (1): Consider $d^p_r : Z^p_r/(B^p_{r-1} + Z^{p+1}_{r-1}) \to Z^{p+r}_r/(B^{p+r}_{r-1} + Z^{p+r+1}_{r-1})$. We first show that $\ker d^p_r \supseteq (Z^p_{r+1} + Z^{p+1}_{r-1})/(B^p_{r-1} + Z^{p+1}_{r-1})$. Recall that $d(Z^{p+1}_{r-1}) \subseteq B^{p+r}_{r-1}$ as observed earlier and $d(Z^p_{r+1}) \subseteq F^{p+r+1}M \cap B = B^{p+r+1}_\infty \subseteq Z^{p+r+1}_{r-1}$. Thus, $(Z^{p+1}_{r-1} + Z^p_{r+1})/(B^p_{r-1} + Z^{p+1}_{r-1}) \subseteq \ker d^p_r$.

Now we show that $\ker d^p_r \subseteq (Z^p_{r+1} + Z^{p+1}_{r-1})/(B^p_{r-1} + Z^{p+1}_{r-1})$. Let $x \in Z^p_r$ such that the corresponding coset $\overline{d^p_r(x)} = 0$, i.e. $d(x) \in B^{p+r}_{r-1} + Z^{p+r+1}_{r-1} = d(Z^{p+1}_{r-1}) + Z^{p+r-1}_{r-1}$ by Observation 2.2.3.3. Equivalently, we have $dx = dy + z$ for some $y \in Z^{p+1}_{r-1}$ and $z \in Z^{p+r+1}_{r-1}$. Observation 2.2.3.4 implies that $Z^{p+1}_{r-1} \subseteq Z^p_r \subseteq F^pM$. Now, set $u := x - y \in F^pM$. Then $dx = dy + z = dy + du$ and $z = du \in Z^{p+r+1}_{r-1}$. Ergo, $du \in F^{p+r+1}M$. Since $u \in F^pM$ and $du \in F^{p+r+1}M$ we have that $u \in Z^p_{r+1}$. Ergo, $x = u + y$ with $u \in Z^p_{r+1}$ and $y \in Z^p_{r-1}$. Thus proving the desired inclusion.

(2): Note $E^{p-r}_r = Z^{p-r}_r/(B_{r-1} + Z^{p-r+1}_{r-1})$ and $E^p_r = Z^p_r/(B^p_{r-1} + Z^{p+1}_{r-1})$. Thus, $d^{p-r}_r(E^{p-1}_r) = (d(Z^{p-r}_r) + B^p_{r-1} + Z^{p+1}_{r-1})/(B^p_{r-1} + Z^{p+1}_{r-1})$. Recall that $d(Z^{p-r}_r) = B^p_r$ by Observation 2.2.3.3, so

$$\operatorname{im} d^{p-r}_r = (B^p_r + B^p_{r-1} + Z^{p+1}_{r-1})/(B^p_{r-1} + Z^{p+1}_{r-1}) = (B^p_r + Z^{p+1}_{r-1})/(B^p_{r-1} + Z^{p+1}_{r-1})$$

where the last equality follow from Observation 2.2.3.2 (as $B^p_r \supseteq B^p_{r-1}$). □

Corollary 2.2.7. $H^p(E_r) = \ker d^p_r/\operatorname{im} d^{p-r}_r \cong (Z^p_{r+1} + Z^{p+1}_{r-1})/(B^p_r + Z^{p+1}_{r-1})$.

We can translate this further using two easy isomorphisms.

Fact. *For any* R*-modules* A, B, Z, $(A + Z)/(B + Z) \cong A/(A \cap (B + Z))$.

Applying this to the corollary yields that

$$H^p(E_r) \cong Z^p_{r+1}/(Z^p_{r+1} \cap (B^p_r + Z^{p+1}_{r-1})).$$

Now recall from Observation 2.2.3.2 that $B^p_r \subseteq Z^p_{r+1}$. So we can apply

Fact. $A \cap (B + C) = B + (A \cap C)$ *if* $B \subseteq A$

to get that

$$H^p(E_r) \cong Z^p_{r+1}/(B^p_r + (Z^p_{r+1} \cap Z^{p+1}_{r-1})).$$

It follows from the definition of Z^p_r that $Z^p_{r+1} \cap Z^{p+1}_{r-1} = Z^{p+1}_r$. Hence,

$$H^p(E_r) \cong Z^p_{r+1}/(B^p_r + Z^p_r) = E^p_{r+1}$$

by definition.

Proposition 2.2.8. *If $(E_r = \bigoplus_{p\in\mathbb{Z}} E^p_r)_{r\in\mathbb{N}_0}$ is the spectral sequence associated to the filtered differential module (M, d), then $H(E_r) \cong E_{r+1}$ for all $r \geqslant 0$. I.e. $H^p(E_r) \cong E^p_{r+1}$ for all $r \geqslant 0$ and all $p \in \mathbb{Z}$. In particular, since $E^p_0 = F^pM/F^{p+1}M$ for all p, $H(\operatorname{Gr} M) \cong E_1$.*

So we start with $\operatorname{Gr} M$ associated to the filtration of M and by repeatedly taking cohomology, $E_1 \cong H(E_0)$, $E_2 \cong H(E_1)$, ..., $E_{r+1} \cong H(E_r)$, we approximate E_∞, the graded module of the cohomology $H(M)$.

Remark: If the filtration $(F^pM)_{p\in\mathbb{Z}}$ of M is finite, then there exists an $r \in \mathbb{N}$ such that $E_r = E_\infty$.

2.3 Spectral Sequences for Filtered Cochain Complexes

In the previous section we built spectral sequences for filtered differential modules. Now we start with a filtered cochain complex and build its spectral sequence.

2.3.1 Building Blocks

Definition 2.3.1. A *filtered cochain complex* (C, d) is a cochain complex of degree 1 together with a decreasing filtration $(F^pC)_{p\in\mathbb{Z}}$ such that $\bigcup_{p\in\mathbb{Z}} F^pC = C$, $\bigcap_{p\in\mathbb{Z}} F^pC = 0$, and also satisfy conditions (1) and (2) below.

Recall that we consider a cochain complex of degree 1 as a direct sum $C = \bigoplus_{n\in\mathbb{Z}} C^n$ of R-modules together with a differential $d = \bigoplus_{n\in\mathbb{Z}} d^n : C \to C$ with $d^n(C^n) \subseteq C^{n+1}$ and $d \circ d = 0$, i.e. $d^{n+1}d^n = 0$ for all $n \in \mathbb{Z}$.

1. The filtration and d are compatible, i.e. $d(F^pC) \subseteq F^pC$ for all p.
2. The filtration and grading are compatible, i.e. $F^pC = \bigoplus_{n\in\mathbb{Z}}(F^pC \cap C^n)$. That is, for each $n \in \mathbb{Z}$ we have a filtration $(F^pC^n)_{p\in\mathbb{Z}}$ of C^n and $F^pC = \bigoplus_{n\in\mathbb{Z}} F^pC^n$.

The traditional notation here is $C^{p,n-p} := F^pC^n = F^pC \cap C^n$.

It follows that $C^{p,q} = C^{p+q} \cap F^pC$ for all $p, q \in \mathbb{Z}$ where $p+q$ is the *degree* and p is the *index of the filtration*. We can then write $F^pC = \bigoplus_{q\in\mathbb{Z}} C^{p,q} = \bigoplus_{n\in\mathbb{Z}} C^{p,n-p}$. In this notation, compatibility with d means

$$\begin{aligned} d(C^{p,n-p}) &\subseteq C^{p,n+1-p} \quad \forall n, \\ \text{or} \quad d(C^{p,q}) &\subseteq C^{p,q+1} \quad \forall q. \end{aligned}$$

For each pair (p, q), the differential d induces a map $d^{p+q} : C^{p,q} \to C^{p,q+1}$. Furthermore for each pair (p, n), we have a restriction $d_{C^{p,n-p}:C^{p,n-p}\to C^{p,n+1-p}}$. If we set

$$d_{F^p} := \bigoplus_{n\in\mathbb{Z}} d_{C^{p,n-p}} : F^pC \to F^pC$$

then $d^2 = 0$ implies $(d_{F^p})^2 = 0$ and so (F^pC, d_{F^p}) is a cochain complex of degree 1 for all p.

The filtration $(F^pC)_{p\in\mathbb{Z}}$ gives rise to a graded module $\operatorname{Gr} C = \bigoplus_{p\in\mathbb{Z}} \operatorname{Gr}^p C$ with $\operatorname{Gr}^p C := F^pC/F^{p+1}C = \bigoplus_{q\in\mathbb{Z}} C^{p,q}/\bigoplus_{q\in\mathbb{Z}} C^{p+1,q-1} \cong \bigoplus(C^{p,q})/C^{p+1,q-1})$. We define $\operatorname{Gr}^{p,q} C := C^{p,q})/C^{p+1,q-1}$. This tells us that $\operatorname{Gr} C$ is a bigraded module (which will become E_0).

$$\operatorname{Gr} C = \bigoplus_{p,q\in\mathbb{Z}} \operatorname{Gr}^{p,q} C = \bigoplus_{p,n\in\mathbb{Z}} \operatorname{Gr}^{p,n-p} C.$$

The differential $d_{C^{p,n-p}} : C^{p,n-p} \to C^{p,n+1-p}$ induces a homomorphism

$$d_0^{p,n-p} : C^{p,n-p}/C^{p+1,n-1-p} \to C^{p,n+1-p}/C^{p+1,n-p}$$

so with $d_0^p := \bigoplus_{n\in\mathbb{Z}} d_0^{p,n-p} : \operatorname{Gr}^p C = \bigoplus_{n\in\mathbb{Z}} \operatorname{Gr}^{p,n-p} C \to \operatorname{Gr}^p C$, $(\operatorname{Gr}^p C, d_0^p)$ is also a cochain complex for all $p \in \mathbb{Z}$. Thus, $\operatorname{Gr} C = \bigoplus_{p\in\mathbb{Z}} \operatorname{Gr}^p C$ is a cochain complex with differential $d_0 := \bigoplus_{p\in\mathbb{Z}} d_0^p$ of degree 0. This gives rise to a cohomology $H(\operatorname{Gr} C) = \bigoplus_{p\in\mathbb{Z}} H(\operatorname{Gr}^p C)$ (which will become E_1), where

$$H(\operatorname{Gr}^p C) = \bigoplus_{n\in\mathbb{Z}} H^n(\operatorname{Gr}^p C) = \bigoplus_{n\in\mathbb{Z}} H^n(F^pC/F^{p+1}C).$$

Recall that $H^n(F^pC/F^{p+1}C) = \ker d_0^{p,n-p}/\operatorname{im} d_0^{p,n-p-1}$. Using $q = n - p$ instead of n, we can also write

$$H(\operatorname{Gr} C) = \bigoplus_{p,q\in\mathbb{Z}} H^{p+q}(F^pC/F^{p+1}C) =: \bigoplus_{p,q\in\mathbb{Z}} H(\operatorname{Gr} C)^{p,q}$$

as a bigraded module.

Now, $C = \bigoplus_{n\in\mathbb{Z}} C^n$ has cohomology $H(C) = \bigoplus_{n\in\mathbb{Z}} H^n(C)$ with $H^n(C) = \ker d^n/\operatorname{im} d^{n+1}$ (this is what we're interested in). $F^pC = \bigoplus_{n\in\mathbb{Z}} C^{p,n-p}$ has cohomology $H(F^pC) = \bigoplus_{n\in\mathbb{Z}} H^n(F^pC)$ with $H^n(F^pC) = \ker d_{C^{p,n-p}}/\operatorname{im} d_{C^{p,n-1-p}}$. Since $F^pC \hookrightarrow C$ is a cochain map, we get an induced homomorphism of cohomology $H(F^pC) \to H(C)$. Denote the image of this homomorphism by $H(C)^p$ so that $H(F^pC) \twoheadrightarrow H(C)^p \leqslant H(C)$. So for each $n \in \mathbb{Z}$, we have a homomorphism $H^n(F^pC) \twoheadrightarrow H(C)^{p,n-p} := H^n(C) \cap H(C)^p$; in other words: $H(C)^{p,n-p}$ is the image of the homomorphism $H^n(F^pC) \to H^n(C)$.

So we obtain a filtration $(H(C)^p)_{p\in\mathbb{Z}}$ of $H(C)$ and for each $n \in \mathbb{Z}$, a filtration $(H(C)^{p,n-p})_{p\in\mathbb{Z}}$ of $H^n(C)$.

Definition 2.3.2.

$$\begin{aligned} \operatorname{Gr}^p H(C) &:= H(C)^p/H(C)^{p+1} \\ \operatorname{Gr} H(C) &:= \bigoplus_{p\in\mathbb{Z}} \operatorname{Gr}^p H(C) \qquad \text{(will become } E_\infty) \end{aligned}$$

Note: $\bigcup_{p\in\mathbb{Z}} H(C)^{p,n-p} = H^n(C)$ but in general we do not have that $\bigcap_{p\in\mathbb{Z}} H(C)^{p,n-p} = 0$ (if the latter is satisfied, then we call the filtration "regular").

Using not only the filtration of $H(C)$ but also the grading of C, we can see that also $H(C)$ is bigraded.

$$\begin{aligned}\operatorname{Gr}^p H(C) = H(C)^p/H(C)^{p+1} &= \bigoplus_{n\in\mathbb{Z}} H(C)^{p,n-p} / \bigoplus_{n\in\mathbb{Z}} H(C)^{p+1,n-p-1} \\ &\cong \bigoplus_{n\in\mathbb{Z}} (H(C)^{p,n-p}/H(C)^{p+1,n-p-1}) \\ &=: \bigoplus_{n\in\mathbb{Z}} \operatorname{Gr}^{p,n-p} H(C)\end{aligned}$$

So with $\operatorname{Gr}^{p,q} H(C) = H(C)^{p,q}/H(C)^{p+1,q-1}$, we have

$$\operatorname{Gr}^p H(C) = \bigoplus_{q\in\mathbb{Z}} \operatorname{Gr}^{p,q} H(C),$$

and

$$\operatorname{Gr} H(C) = \bigoplus_{p\in\mathbb{Z}} \operatorname{Gr}^p H(C) = \bigoplus_{p,q\in\mathbb{Z}} \operatorname{Gr}^{p,q} H(C)$$

bigraded, and

$$\operatorname{Gr} H^n(C) = \bigoplus_{p\in\mathbb{Z}} \operatorname{Gr}^{p,n-p} H(C).$$

The term $\operatorname{Gr}^{p,n-p} H(C)$ will become $E_\infty^{p,n-p}$.

The spectral sequence $(E_r)_{r\geqslant 0}$ will be a sequence of bigraded modules which are also cochain complexes of degree r starting with

$$E_0 = \operatorname{Gr} C,\ E_1 = H(\operatorname{Gr} C),\ \ldots$$

and approximating $E_\infty = \operatorname{Gr} H(C)$.

2.3.2 Defining the Spectral Sequence

We now move to define the spectral sequence$(E_r)_{r\geqslant 0}$ for the filtered cochain complex (C,d) where $C = \bigoplus_{n\in\mathbb{Z}} C^n$, $F^pC = \bigoplus_{q\in\mathbb{Z}} C^{p,q} = \bigoplus_{n\in\mathbb{Z}} C^{p,n-p}$. We do this by comparing with the easier, already discussed case of a filtered differential module (M,d).

(M,d)	(C,d)
$Z_r^p := \{x \in F^pM \mid dx \in F^{p+r}M\}$	$Z_r^{p,q} := \{x \in C^{p+q} \cap F^pC \mid dx \in C^{p+q+1} \cap F^{p+r}C\}$
$B_r^p := \{x \in F^pM \mid \exists y \in F^{p-r}M, x = dy\}$	$B_r^{p,q} := \{x \in C^{p+q} \cap F^pC \mid \exists y \in C^{p+q-1} \cap F^{p-r}C, x = dy\}$
$Z_\infty^p := Z \cap F^pM$	$Z_\infty^{p,q} := \{x \in C^{p+q} \cap F^pC \mid dx = 0\}$
$B_\infty^p := B \cap F^pM$	$B_\infty^{p,q} := \{x \in C^{p+q} \cap F^pC \mid \exists y \in C^{p+q-1}, x = dy\}$

Observation 2.3.3. *We have the following easy properties.*

1. $B_r^{p,q} = dZ_r^{p-r,q+r-1}$

2. $Z_{r-1}^{p+1,q-1} \subseteq Z_r^{p,q}$

3. $B_{r-1}^{p,q} \subseteq B_r^{p,q} \subseteq Z_r^{p,q}$

4. $Z_\infty^{p+1,q-1} \subseteq Z_\infty^{p,q}$

5. $B_\infty^{p,q} \subseteq Z_\infty^{p,q}$

Note that (1) and (3) imply that $dZ_r^{p,q} \subseteq Z_r^{p+r,q-r+1}$ and $dZ_{r-1}^{p+1,q-1} \subseteq Z_{r-1}^{p+r+1,q-r}$. So if we define $E_r^{p,q} := Z_r^{p,q}/(B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1})$, the the differential d induces a homomorphism $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$. We also define

$$E_r^p := Z_r^p/(B_{r-1}^p + Z_{r-1}^{p+1}) = \bigoplus_{q\in\mathbb{Z}} E_r^{p,q}$$
$$E_\infty^p := Z_\infty^p/(B_\infty^p + Z_\infty^{p+1}) = \bigoplus_{q\in\mathbb{Z}} E_\infty^{p,q}$$

where $Z_r^p := \bigoplus_{q\in\mathbb{Z}} Z_r^{p,q}$, $B_r^p := \bigoplus_{q\in\mathbb{Z}} B_r^{p,q}$, $Z_\infty^p := \bigoplus_{q\in\mathbb{Z}} Z_\infty^{p,q}$, $B_\infty^p := \bigoplus_{q\in\mathbb{Z}} B_\infty^{p,q}$, and $E_\infty^{p,q} := Z_\infty^{p,q}/(B_\infty^{p,q} + Z_\infty^{p+1,q-1})$. With similar computations as for (M,d), one establishes the following.

Proposition 2.3.4.
$$H^{p,q}(E_r) := \ker d_r^{p,q}/\operatorname{im} d_r^{p-r,q+r-1} \cong E_{r+1}^{p,q}$$

If we sum over q, we get a differential $d_r^p := \bigoplus_{q\in\mathbb{Z}} d_r^{p,q} : E_r^p \to E_r^{p+r}$. So $E_r = \bigoplus_{p\in\mathbb{Z}} E_r^p$ is a graded complex with differential $d_r = \bigoplus_{p\in\mathbb{Z}} d_r^p$ of degree r. For the cohomology of E_r we obtain

Corollary 2.3.5. $H^p(E_r) := \ker d_r^p/\operatorname{im} d_r^{p-r} \cong E_{r+1}^p$.

Summarizing these and observations made about E_0 and E_∞, we get the following.

Theorem 2.3.6. *Given a filtered and graded complex (C,d) with differential d of degree 1 and filtration $(F^pC)_{p\in\mathbb{Z}}$ compatible with the grading, there is a spectral sequence $(E_r)_{r\in\mathbb{N}_0\cup\{\infty\}}$ which satisfies:*

1. *$E_0 = \operatorname{Gr} C$ and $E_1 \cong H(\operatorname{Gr} C)$. This means that $E_0^{p,q} = C^{p,q}/C^{p+1,q-1}$ and $E_1^{p,q} \cong H^{p,q}(F^pC/F^{p+1}C)$ for all $p,q \in \mathbb{Z}$.*

2. *Each $E_r = \bigoplus_{p,q} E_r^{p,q}$ is a bigraded module, and with $E_r^p = \bigoplus_{q\in\mathbb{Z}} E_r^{p,q}$, $H^p(E_r) = \bigoplus_{q\in\mathbb{Z}} H^{p,q}(E_r) \cong \bigoplus_{q\in\mathbb{Z}} E_{r+1}^{p,q} = E_{r+1}^p$ so that $H(E_r) = \bigoplus_{p\in\mathbb{Z}} H^p(E_r) \cong \bigoplus_{p\in\mathbb{Z}} E_{r+1}^p = E_{r+1}$.*

3. *$E_\infty \cong \operatorname{Gr}(H(C))$, which means that $E_\infty^{p,q} \cong \operatorname{Gr}^{p,q} H(C) = H(C)^{p,q}/H(C)^{p+1,q-1}$ for all p,q and so $\operatorname{Gr}(H^n(C)) \cong \bigoplus_{p\in\mathbb{Z}} E_\infty^{p,n-p}$.*

Note that for all $r \geqslant 0$, the differentials $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ induced by d for $p,q \in \mathbb{Z}$ have bidegree $(r, 1-r)$ and that $H^{p,q}(E_r) \cong E_{r+1}^{p,q}$.

An easy consequence here is that whenever $E_r^{p,q} = 0$, then we have that $E_s^{p,q} = 0$ for all $s \geqslant r$

Recall: The foal of the spectral sequence $(E_r)_{r\in\mathbb{N}_0}$ is to "approximate" E_∞, which yields information about the cohomology $H(C)$.

So there should be a relation between $E_\infty^{p,q}$ and $E_r^{p,q}$ for "sufficiently big r." Ideally, we would like to have an $r_0 \in \mathbb{N}$ such that $E_r^{p,q} = E_\infty^{p,q}$ for all $r \geqslant r_0$ and $p, q \in \mathbb{Z}$. This is usually too much to ask. So we will use the following notion of *convergence of a spectral sequence* $(E_r)_{r \geqslant 0}$: for every pair $(p,q) \in \mathbb{Z}^2$, there exists an $r_0 = r_0(p,q)$ such that $E_r^{p,q} = E_\infty^{p,q}$ for all $r \geqslant r_0$.

Here are some special cases where some statements regarding the convergence of spectral sequences can be made.

2.3.3 First Quadrant Spectral Sequences

Definition 2.3.7. The filtration $(F^pC)_{p \in \mathbb{Z}}$ of (C,d) is called *positive* if $F^pC = C$ for all $p \leqslant 0$. The filtration is called "*canonically cobounded*" (one might also call it a *first quadrant filtration*) if it is positive and $C^{p,q} = 0$ for all $q < 0$ (equivalently $C^{p,n-p} = 0$ for all (p,n) with $p > n$).

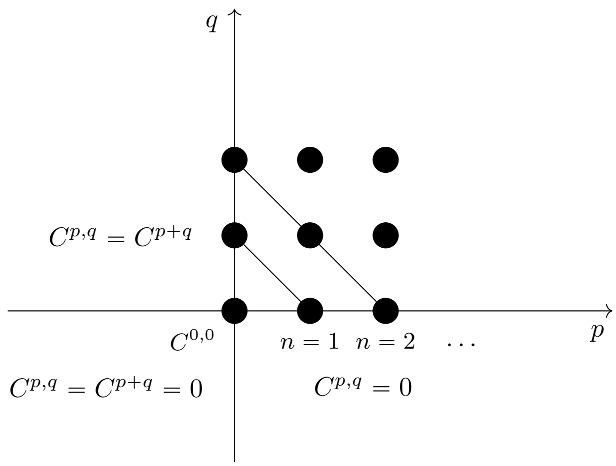


Figure 2.1: A visualization of a canonically cobounded filtration.

Side-Remark: For $n \leqslant 0$, the only options are $F^pC^n = C^n$ (if $p \leqslant 0$) or $F^pC^n = 0$ (if $n = p + q$ with $p \geqslant 0$ and hence $q < 0$).

Proposition 2.3.8. *If $(F^pC)_{p \in \mathbb{Z}}$ is canonically cobounded, then $E_r^{p,q} = 0$ for all $p < 0$, $q \in \mathbb{Z}$, $r \in \mathbb{N}_0 \cup \{\infty\}$ and $E_r^{p,q} = 0$ for all $q < 0$, $p \in \mathbb{Z}$, $r \in \mathbb{N}_0 \cup \{\infty\}$ (i.e. (E_r) is a first quadrant spectral sequence) and $E_r^{p,q} = E_\infty^{p,q}$ for all r, p, q with $r > \max\{p, q+1\}$.*

The assumptions in this proposition can be weakened as follows:

Definition 2.3.9. The filtration $(F^pC)_{p \in \mathbb{Z}}$ of (C,d) is called *bounded* if there exist functions $\mu, \nu : \mathbb{Z} \to \mathbb{Z}$ such that $\nu(n) \leqslant \mu(n)$ for all $n \in \mathbb{Z}$ and:

1. $C^{p,n-p} = C^n$ for all $p \leqslant \nu(n)$;
2. $C^{p,n-p} = 0$ for all $p > \mu(n)$.

Proposition 2.3.10. *If the filtration is bounded with functions μ, ν, then:*

1. *$E_r^{p,n-p} = 0$ for all r, p, q with $p < \nu(n)$ or $p > \mu(n)$;*

2. *and* $E_r^{p,q} = E_\infty^{p,q}$ *for all* r, p, q *with* $r > \max\{p - \nu(p+q-1), \mu(p+q+1) - p\}$.

Remark: The previous proposition is a special case of this one with $\nu(n) = 0$ for all n and $\mu(n) = n$ for all n.

What about $E_\infty^{p,q}$? Recall that $E_\infty^{p,q} := Z_\infty^{p,q}/(B_\infty^{p,q} + Z_\infty^{p+1,q-1})$ where $Z_\infty^{p,q} = \{x \in C^{p,q} | dx = 0\}$ and $B_\infty^{p,q} = C^{p,q} \cap d(C^{p+q-1})$. Then we have the following.

1. If (F^pC) is positive, then $C^{p,q} = C^{p+1,q-1} = C^{p+q}$ for all $p \leqslant -1$. Ergo, $Z_\infty^{p,q} = Z_\infty^{p+1,q-1}$ and $E_\infty^{p,q} = 0$ for all $q < 0$.
2. Similarly, $C^{p,q} = 0$ for all $q < 0$ implies $E_\infty^{p,q} = 0$ for all $q < 0$.

Side-Remark: These easy observations only use $Z_\infty^{p,q}$ and $Z_\infty^{p+1,q-1}$, not $B_\infty^{p,q}$.

Now assume that $(E_r)_{r \in \mathbb{N}_0}$ is a first quadrant spectral sequence (i.e. $E_r^{p,q} = 0$ whenever $p < 0$ or $q < 0$). Let $p, q \in \mathbb{N}_0$ be given. When does the sequence $(E_r^{p,q})_{r \in \mathbb{N}_0}$ stabilize?

Recall that $E_{r+1}^{p,q} \cong \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q+r-1}$. Since $(E_r)_r$ is a first quadrant spectral sequence we have the following.

1. Then we have that $\operatorname{im} d_r^{p-r,q+r-1} = 0$ whenever $E_r^{p-r,q+r-1} = 0$. This occurs when $r > p$ (or $p - r < 0$).
2. We have that $\ker d_r^{p,q} = E_r^{p,q}$ whenever $E_r^{p+r,q-r+1} = 0$, which occurs when $r > q+1$ (or $q - r + 1 < 0$).

Corollary 2.3.11. *If* $(E_r)_{r \in \mathbb{N}_0}$ *is a first quadrant spectral sequence, then for any* $(p, q) \in \mathbb{N}_0^2$*, the sequence* $(E_r^{p,q})_{r \geqslant 0}$ *stabilizes for* $r > \max(p, q+1) =: r_0$*, i.e.* $E_r^{p,q} = E_{r_0+1}^{p,q}$ *for all* $r \geqslant r_0 + 1$.

Question: Is this limit (for fixed (p, q)) equal to $E_\infty^{p,q}$?

Observation 2.3.12. 1. *If* $C^{p,q} = 0$ *for all* $q < 0$ *and* $r > q + 1$*, then* $Z_r^{p,q} = Z_\infty^{p,q}$*. This follows from*

$$Z_r^{p,q} = \{x \in C^{p,q} | dx \in C^{p+r,q-r+1}\} = \{x \in C^{p,q} | dx = 0\} = Z_\infty^{p,q}$$

since $C^{p+r,q-r+1} = 0$ *if* $r > q + 1$.

2. *If* (F^pC) *is positive and* $r \geqslant p$*, then* $B_r^{p,q} = B_\infty^{p,q}$*. This follows from*

$$B_r^{p,q} = \{x \in C^{p,q} | x \in d(C^{p-r,q+r-1})\} = \{x \in C^{p,q} | x \in d(C^{p+q-1})\} = B_\infty^{p,q}$$

since $C^{p-r,q+r-1} = C^{p+q-1}$ *if* $r \geqslant p$*. Thus,* $r > p$ *implies that* $B_{r-1}^{p,q} = B_\infty^{p,q}$.

This leads us to the following corollary.

Corollary 2.3.13. *Assume that 1)* (F^pC) *is positive and 2)* $C^{p,q} = 0$ *for all* $q < 0$*. Then for all* $r > \max(p, q+1)$

$$E_r^{p,q} = Z_r^{p,q}/(B_{r-1}^{p,q} + Z_{r-1}^{p+1,q-1}) = Z_\infty^{p,q}/(B_\infty^{p,q} + Z_\infty^{p+1,q-1}) = E_\infty^{p,q}.$$

(Note: $r > q + 1$ *implies that* $r - 1 > (q - 1) + 1$*).*

Thus, we have proved the following result.

Proposition 2.3.14. *If (F^pC) is "canonically cobounded" (i.e. satisfies (1) and (2) from the corollary), then $E_r^{p,q} = 0$ for all r, p, q with $p < 0$ or $q < 0$ (including $r = \infty$) and $E_r^{p,q} = E_\infty^{p,q}$ for all r, p, q with $r > \max(p, q + 1)$.*

Remark: Similar (albeit more technically involved, but not really more difficult) arguments yield the proof of Proposition 2.3.10 about general bounded filtrations, of which those in Proposition 2.3.8 are an important special case.

2.4 Spectral Sequence of a Double Complex

The canonically cobounded filtrations of the previous section naturally arise from first quadrant double complexes.

Definition 2.4.1. A *double (cochain) complex* is a doubly-graded complex $C = (C^{p,q})_{p,q\in\mathbb{Z}}$ with two differentials $d_I^{p,q} : C^{p,q} \to C^{p+1,q}$, $d_{II}^{p,q} : C^{p,q} \to C^{p,q+1}$ such that $d_I^{p+1,q} d_I^{p,q} = 0$, $d_{II}^{p,q+1} d_{II}^{p,q} = 0$, and the following diagram commutes for all p, q (i.e. $d_{II}^{p+1,q} d_I^{p,q} = d_I^{p,q+1} d_{II}^{p,q}$). We say that the *degree* of $C^{p,q}$ is $p + q$

$$\begin{array}{ccc} C^{p,q+1} & \xrightarrow{d_I^{p,q+1}} & C^{p+1,q+1} \\ {\scriptstyle d_{II}^{p,q}}\uparrow & & \uparrow{\scriptstyle d_{II}^{p+1,q}} \\ C^{p,q} & \xrightarrow[d_I^{p,q}]{} & C^{p+1,q} \end{array}$$

We will mainly deal with *first quadrant double complexes* where $C^{p,q} = 0$ if $p < 0$ or $q < 0$.

Example 2.4.2. *Let $\mathbf{C} = (C^p)$ be a cochain complex of right R-modules and $\mathbf{D} = (D^q)$ be a cochain complex of left R-modules. Denote the differentials by $\gamma^p : C^p \to C^{p+1}$ and $\delta^q : D^q \to D^{q+1}$. Define $C^{p,q} := C^p \otimes_R D^q$. Observe that $(C^{p,q})$ is a double complex of abelian groups since $(1_{C^{p+1}} \otimes \delta^q)(\gamma^p \otimes 1_{D^q}) = \gamma^p \otimes \delta^q = (\gamma^p \otimes 1_{D^{q+1}})(1_{C^p} \otimes \delta^q)$ which makes the following diagram commute.*

$$\begin{array}{ccc} C^p \otimes D^{q+1} & \xrightarrow{\gamma^p \otimes 1_{D^{q+1}}} & C^{p+1} \otimes D^{q+1} \\ {\scriptstyle 1_{C^p} \otimes \delta^q}\uparrow & & \uparrow{\scriptstyle 1_{C^{p+1}} \otimes \delta^q} \\ C^p \otimes D^q & \xrightarrow[\gamma^p \otimes 1_{D^q}]{} & C^{p+1} \otimes D^q \end{array}$$

Definition 2.4.3. If $\mathbf{C} = (C^{p,q})$ is a double complex, we define the associated *total complex* $TC = (TC^n)_{n\in\mathbb{Z}}$ by $TC^n := \bigoplus_{p+q=n} C^{p,q}$ with total differential $d_T^n = \bigoplus_{p+q=n} (d_I^{p,q} + (-1)^p d_{II}^{p,q}) : TC^n \to TC^{n+1}$.

In more detail, for each (p, q) we have a homomorphism $d_I^{p,q} + (-1)^p d_{II}^{p,q} : C^{p,q} \to C^{p+1,q} \bigoplus C^{p,q+1} \leqslant TC^{n+1}$. d_T^n is then the direct sum over all (p, q) with $p + q = n$ of the homomorphisms $d_I^{p,q} + (-1)^p d_{II}^{p,q}$. The sign $(-1)^p$ is introduced in order to obtain that d_T is a differential, i.e. (TC, d_T) is a cochain complex.

Lemma 2.4.4. $d_T^{n+1} d_T^n = 0$ *for all n.*

Proof. It suffices to show $d_T^{n+1} d_T^n(C^{p,q}) = 0$ for all p, q with $p + q = n$. For $c^{p,q} \in C^{p,q}$ we obtain

$$\begin{aligned} d_T^{n+1} d_T^n(c^{p,q}) =& d_T^{n+1}(\underbrace{d_I^{p,q}(c^{p,q})}_{\in C^{p+1,q}} + (-1)^p \underbrace{d_{II}^{p,q}(c^{p,q})}_{\in C^{p,q+1}}) \\ =& \underbrace{d_I^{p+1,q} d_I^{p,q}}_{=0}(c^{p,q}) + (-1)^{p+1} \underbrace{d_{II}^{p+1,q} d_I^{p,q}}_{=d_I^{p,q+1} d_{II}^{p,q}}(c^{p,q}) + d_I^{p,q+1} d_{II}^{p,q}(c^{p,q}) + (-1)^{2p} \underbrace{d_{II}^{p,q+1} d_{II}^{p,q}}_{=0}(c^{p,q}) \\ =& (-1)^{p+1} d_I^{p,q+1} d_{II}^{p,q} + (-1)^p d_I^{p,q+1} d_{II}^{p,q}(c^{p,q}) = 0 \end{aligned}$$

□

There are now two canonical filtrations of (TC, d_T):

1. $F_I^p TC^n := \bigoplus_{s \geqslant p} C^{s,n-s}$. This yields $F_I^p TC = \bigoplus_{n \in \mathbb{Z}} F_I^p TC^n = \bigoplus_{s \geqslant p, n \in \mathbb{Z}} C^{s,n-s} = \bigoplus_{s \geqslant p, q \in \mathbb{Z}} C^{s,q}$.
2. $F_{II}^q TC := \bigoplus_{t \geqslant q} C^{n-t,t}$. This yields $F_{II}^q TC = \bigoplus_{n \in \mathbb{Z}} F_{II}^q TC^n = \bigoplus_{n \in \mathbb{Z}, t \geqslant q} C^{n-t,t} = \bigoplus_{p \in \mathbb{Z}, t \geqslant q} C^{p,t}$.

Let's concentrate on F_I^p for now.

Observation 2.4.5. *1. If $(C^{p,q})$ is a first quadrant double complex, then $TC^n = 0$ for all $n < 0$; $F_I^p TC^n = TC^n$ for all $p \leqslant 0$ (i.e. F_I^p is a positive filtration); and $n - p < 0$ (or $p > n$) implies that $F_I^p TC^n = \bigoplus_{s \geqslant p \geqslant n} C^{s,n-s} = 0$. So in this case the filtration F_I^p (and similarly F_{II}^q) is "canonically cobounded".*

*2. **Warning**: Note that $C^{p,q}$ has a different meaning here than in the general discussion of spectral sequences. In the general set up we have $C^{p,q} = F^p C^{p+q} = F^p C \cap C^{p+q}$ and $C^{p,n-p} = F^p C^n = F^p C \cap C^n$. Here we specify $C = TC$ and*

$$C^{p,n-p} = \bigoplus_{s \geqslant p} C^{s,n-s} / \bigoplus_{s \geqslant p+1} C^{s,n-s} = F_I^p TC^n / F_I^{p+1} TC^n = TC^{p,n-p} / TC^{p+1,n-p-1} = \mathrm{Gr}^{p,n-p} TC.$$

So we have $C^{p,q} = \mathrm{Gr}^{p,q} TC = {}^I E_0^{p,q}$ in the case of $F^p = F_I^p$.

Remark: In principal, we get the same bigraded module $E_0^{p,q} = \mathrm{Gr}^{p,q} TC$ if we use the filtration F_{II}^q of TC. However, if we stick with the convention that in $E_0^{*,*}$ the first index refers to the filtration index, then ${}^{II}E_0^{q,p} = C^{p,q}$ or ${}^{II}E_0^{p,q} = C^{q,p}$ for all p, q.

Next we want to identify the levels ${}^I E_1$ and ${}^I E_2$ associated with the first filtration $F_I^p TC$. We are using that the next level is obtained from the previous one by taking cohomology, see Theorem 2.3.6.

In particular, $E_1 = H(E_0)$. In order to make this more explicit, we need to consider the differentials ${}^I d_0^{p,q} : C^{p,q} \to C^{p,q+1}$. Recall that all the differentials $d_r^{p,q}$ are induced by d_T, the total differential of TC. So we have $TC^n = \bigoplus_{p+q=n} C^{p,q}$, and

$$d_T^n|_{C^{p,q}} = d_I^{p,q} + (-1)^p d_{II}^{p,q} : C^{p,q} \to \underbrace{C^{p+1,q} \oplus C^{p,q+1}}_{\leqslant TC^{n+1}}.$$

Since ${}^I d_0^{p,q} : C^{p,q} \to C^{p,q+1}$ is induced by d_T, we must have ${}^I d_0^{p,q} = (-1)^p d_{II}^{p,q}$. This can also be seen using the definition $E_0^{p,q} = Z_0^{p,q}/(B_{-1}^{p,q} + Z_{-1}^{p+1,q-1})$ with

$$Z_0^{p,q} = \{x \in TC^{p+q} \cap F_I^p TC | d_T x \in TC^{p+q+1} \cap F_I^p TC\} = \bigoplus_{s \geqslant p} C^{s,p+q-s},$$
$$Z_{-1}^{p,q} = \{x \in TC^{p+q} \cap F^{p+1} TC | d_T x \in C^{p+q+1} \cap F^p TC\} = \bigoplus_{s \geqslant p+1} C^{s,p+q-s}.$$

So ${}^I d_0^{p,q}$ is determined by the restriction of d_T to $C^{p,q}$ (since $Z_{-1}^{p+1,q-1}$ is modded out), and since the image is in $E_0^{p,q+1}$, where $Z_{-1}^{p,q+1} = \bigoplus_{s \geqslant p+1} C^{s,p+q+1-s}$ is modded out, the map induced by d_T on $E_0^{p,q}$ is the same as the one induced by $(-1)^p d_{II}^{p,q}$.

Result: If for fixed $p \in \mathbb{N}_0$, C^p denotes the cochain complex $(C^{p,q}, (-1)^p d_{II}^{p,q})_{q \in \mathbb{N}_0}$, then ${}^I E_1^{p,q} = H^q(C^p)$ is the "vertical cohomology of $(C^{p,q})$". Of course, we get the same cohomology if we use the differential $d_{II}^{p,q}$ without the sign $(-1)^p$.

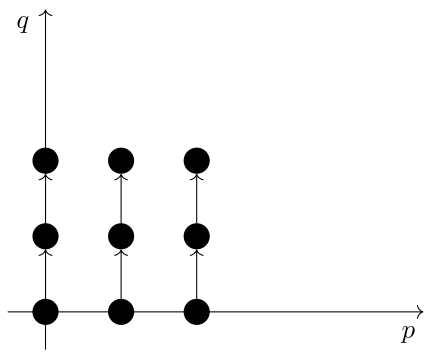


Figure 2.2: E_0 with ${}^I d_0$

We now turn towards the differentials ${}^I d_1$. Recall that it has bidegree $(1,0)$ (as $r = 1$), so ${}^I d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$. Our first observation refers to $Z_1^{p,q}$.

Lemma 2.4.6. $Z_1^{p,q} = Z_0^{p+1,q-1} + (C^{p,q} \cap Z_1^{p,q})$

Proof. By definition, $Z_1^{p,q} = \{x \in \bigoplus_{s \geqslant p} C^{s,p+q-s} | dx \in x \in \bigoplus_{s \geqslant p+1} C^{s,p+q-s}\}$ and

$$Z_0^{p+1,q-1} = \{x \in \bigoplus_{s \geqslant p+1} C^{s,p+q-s} | dx \in x \in \bigoplus_{s \geqslant p+1} C^{s,p+q+1-s} = \bigoplus_{s \geqslant p+1} C^{s,p+q-s}.$$

This last equation follows since the filtration is compatible with the grading, and so $d_T(F^{p+1}TC) \subseteq F^{p+1}TC$ is always true. It follows that

$$Z_1^{p,q} = (C^{p,q} \cap Z_1^{p,q}) + \bigoplus_{s \geqslant p+1} C^{s,p+q-s} = (C^{p,q} \cap Z_1^{p,q}) + Z_0^{p+1,q-1}.$$

□

Consequence: Every element of $E_1^{p,q} = Z_1^{p,q}/(Z_0^{p+1,q-1} + B_0^{p,q})$ can be represented by an element of $C^{p,q} \cap Z_1^{p,q}$.

Now assume that $c = c^{p,q} \in C^{p,q} \cap Z_1^{p,q}$. Then by definition of $d = d_T$ and $Z_1^{p,q}$

$$dc = \underbrace{d_I^{p,q}}_{\in C^{p+1,q}} + \underbrace{(-1)^p d_{II}^{p,q}(c)}_{\in C^{p,q+1}} \in \bigoplus_{s \geqslant p+1} C^{s,p+q-s},$$

which simplifies to $(-1)^p d_{II}^{p,q}(c) \in C^{p,q+1} \cap \bigoplus_{s\geqslant p+1} C^{s,p+q-s} = 0$. Hence, we have the following.

Corollary 2.4.7. $d|_{C^{p,q} \cap Z_1^{p,q}} = d_I^{p,q}$.

Result: ${}^I d_1^{p,q} : \underbrace{H^q(C^p)}_{=E_1^{p,q}} \to \underbrace{H^{q+1}(C^p)}_{=E_1^{p,q+1}}$ is induced by $d_I^{p,q}$. It follows that $E_2 = H(E_1)$ and $E_2^{p,q} = H^p(H^q(C^p))$ which is the "horizontal cohomology of the vertical cohomology of $(C^{p,q})$."

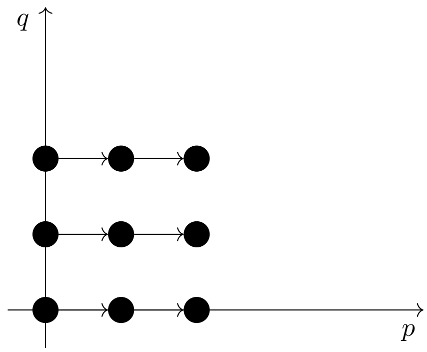


Figure 2.3: E_1 with ${}^I d_1$

More precisely: For a fixed q and variable p, $(H^q(C^p), d_I^{p,q})_{p\in\mathbb{N}_0}$ is a cochain complex with differential induced by $d_I^{p,q}$ and $E_2^{p,q}$ is obtained by taking the pth cohomology of this cochain complex.

Remark: The fact that $d_I^{p,q}$ induces a differential on $(H^q(C^p))_{p\in\mathbb{N}_0}$ can easily be verified without referring to the definition of $E_1^{p,q}$ and $Z_1^{p,q}$. We have that

$$\ker d_{II}^{p,q} / \operatorname{im} d_{II}^{p,q-1} = H^q(C^p) \xrightarrow{\tilde{d}_1^{p,q}} H^q(C^{p+1}) = \ker d_{II}^{p+1,q} / \operatorname{im} d_{II}^{p+1,q-1}.$$

Note that the inclusions $d_I^{p,q}(\ker d_{II}^{p,q}) \subseteq \ker d_{II}^{p+1,q-1}$ follows from the commutativity of

$$\begin{array}{ccc} C^{p,q+1} & \xrightarrow{d_I^{p+1,q}} & C^{p+1,q+1} \\ {\scriptstyle d_{II}^{p,q}}\uparrow & & \uparrow{\scriptstyle d_{II}^{p+1,q}} \\ C^{p,q} & \xrightarrow[d_I^{p,q}]{} & C^{p+1,q} \end{array}$$

and $d_I^{p,q}(\operatorname{im} d_{II}^{p,q-1}) \subseteq \operatorname{im} d_{II}^{p+1,q}$ follows from that of

$$\begin{array}{ccc} C^{p,q} & \xrightarrow{d_I^{p,q}} & C^{p+1,q} \\ {\scriptstyle d_{II}^{p,q-1}}\uparrow & & \uparrow{\scriptstyle d_{II}^{p+1,q}} \\ C^{p,q-1} & \xrightarrow[d_I^{p,q-1}]{} & C^{p+1,q-1} \end{array}$$

However, in order to verify that $\tilde{d}_I^{p,q} = {}^I d_1^{p,q}$, we had to go back to the definition of $E_1^{p,q}$ and $Z_1^{p,q}$. Putting things together, we get the following.

Theorem 2.4.8. *If $(C^{p,q})$ is a first quadrant double (cochain) complex, there exists a "canonically cobounded" filtration, $F_I^p TC$ of the associated total complex TC which yields a converging (in the sense of Proposition 2.3.8) spectral sequence with the following properties.*

1. ${}^I E_0^{p,q} = C^{p,q}$ *for all* p, q.
2. ${}^I E_1^{p,q} = H^q(C^p)$ *for all* p, q.
3. ${}^I E_2^{p,q} = H^p(H^q(C^p))$ *for all* p, q.

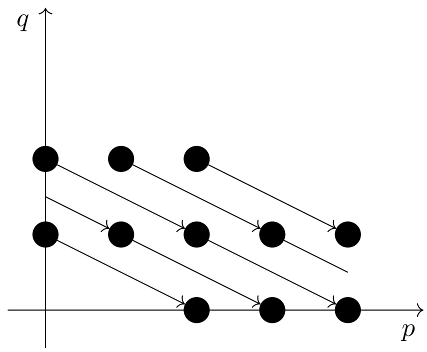


Figure 2.4: E_2 with ${}^I d_2$ of bidegree (2,-1).

Remark:

1. We explicitly determined the differentials ${}^I d_0$ and ${}^I d_1$ but not ${}^i d_2$, which is more involved. However, for certain applications the information collected in the theorem is sufficient to draw interesting conclusions.
2. Using the second filtration $F_{II}^q TC$, we obtain in a similar way a spectral sequence $({}^{II}E_r)$ with ${}^{II}E_0^{p,q} = C^{q,p}$ and ${}^{II}E_1^{p,q} = H^q(C^{*,p})$ where $C^{*,p} := (C^{q,p}, d_I^{q,p})_{q \in \mathbb{N}_0}$ for fixed $p \in \mathbb{N}_0$ (i.e. "horizontal cohomology") and ${}^{II}E_2^{p,q} = H^p(H^q(C^{*,p}))$ ("vertical cohomology of the horizontal cohomology").

Chapter 3

The Grothendieck Spectral Sequence

You Qi

3.1 Resolutions and derived functors

Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and $F : \mathcal{A} \to \mathcal{B}$ be and additive functor. Then F always sends s.e.s. (short exact sequence) in $\mathcal{A}$ to split s.e.s. in $\mathcal{B}$. But usually does not necessarily preserve the class of all s.e.s.'s.

Definition 3.1.1. F is called **left exact** if, given any short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, the induced sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is exact. F is called **right exact** if

$$F(A) \to F(B) \to F(C) \to 0$$

is exact. F is called **exact** if F is both left and right exact.

Because of $\mathcal{A} \longleftrightarrow \mathcal{A}^{op}$ duality, right exactness can be reduced to left exactness. However F being exact fails for many important functors.

E.g. One useful example to keep in mind: $\mathcal{A} = {}_R\mathcal{M}od$, $\mathcal{B} = {}_S\mathcal{M}od$ and M is an (R, S)-bimodule. Then

$$F = \mathrm{Hom}_R(M, -) : \mathcal{A} \to \mathcal{B}, {}_RN \mapsto \mathrm{Hom}_R(M_S, N)$$
$$G = M \otimes_S (-) : \mathcal{B} \to \mathcal{A}, {}_SL \mapsto M \otimes_S L$$

are adjoint functors. F is left exact and G is right exact.

More explicitly, consider $\mathcal{A} = \mathcal{B} = {}_{\mathbb{Z}}\mathcal{M}od$ and the functors

$$F = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n, -)$$
$$G = \mathbb{Z}_n \otimes (-)$$

applied to the s.e.s. $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n \to 0$.

The right derived functors of F give a functorial way to measure how F fails to be exact: i.e. functors $R^i F, i \in \mathbb{N}$ s.t.

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to ...$$

which is natural in s.e.s.

To properly define $R^iF, i \in \mathbb{N}$, we need the notion of injective resolutions.

Definition 3.1.2.

(1) An object P is called **projective** in $\mathcal{A}$ if $\mathrm{Hom}_{\mathcal{A}}(P, -) : \mathcal{A} \to {}_{\mathbb{Z}}\mathcal{M}od$ is exact. An object I is called injective if $\mathrm{Hom}_{\mathcal{A}}(-, I) : \mathcal{A}^{op} \to {}_{\mathbb{Z}}\mathcal{M}od$ is exact.

(2) $\mathcal{A}$ is said to **have enough projectives** if, $\forall M \in \mathcal{A}, \exists$ an epimorphism $P \to M \to 0$. $\mathcal{A}$ is said to **have enough injectives** if $\forall M \in \mathcal{A}, \exists$ a monomorphism $0 \to M \to I$.

E.g. As an illustration, consider $\mathcal{A} = {}_R\mathcal{M}od$ for some $\Bbbk$-algebra, where $\Bbbk$ is a field. Then any free module $R^{\oplus S}$ is projective (S can be infinity). $\forall M \in {}_R\mathcal{M}od, \exists$ surjection

$$R^{\oplus S} \twoheadrightarrow M$$

showing that ${}_R\mathcal{M}od$ has enough projectives. Any projective R-module is, in fact, a direct summand of a free R-module. Take a surjection $P \to M^* \to 0$ as right modules, and dualize to get

$$0 \to M^{**} \to P^*$$

Then:

(1) P^* is injective: $\mathrm{Hom}_R(-, P^*) \cong \mathrm{Hom}_{\Bbbk}(P \otimes_R, \Bbbk)$ is a composition of exact functors, and thus is exact.

(2) The canonical embedding $M \to M^{**} \to P^*$ embeds M in an injective R-module $\implies {}_R\mathcal{M}od$ has enough injectives.

Definition 3.1.3. Let $\mathcal{A}$ be an abelian category. An **injective resolution** $I^\bullet$ of M is a complex of injectives in $\mathcal{A}$

$$0 \to I^0 \xrightarrow{d_0} I^1 \xrightarrow{d_1} I^2 \xrightarrow{d_2} ...$$

s.t. it is exact everywhere except at I^0, where $\ker d_0 \cong M$.

Theorem 3.1.4. *Let $\mathcal{A}$ be an abelian category with enough injectives.*

(1) Any $M \in \mathcal{A}$ has an injective resolution $I^\bullet_M$.

(2) *Any* $f : M \to N$ *extends to* $f^\bullet : I_M^\bullet \to I_N^\bullet$. *Any such two extensions are homotopic to each other.*

The proof of theorem uses a very useful characterization of injectives: $\forall$ monomorphim i:

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{i} & N \\ & & f\big\downarrow & \swarrow_{\exists h \text{ s.t. } h\circ i = f} & \\ & & I & & \end{array}$$

As a consequence,

Corollary 3.1.5. Any two injective resolutions of M are homotopic to each other.

Lemma 3.1.6. (Horseshoe) In $\mathcal{A}$, an abelian category with enough injectives. Let $I_M^\bullet, I_N^\bullet$ be injective resolutions of M, N with

$$0 \to M \to K \to N \to 0$$

being exact, then there is an injective resolution $I_K^\bullet$ of K that fits into

$$0 \to I_M^\bullet \to I_K^\bullet \to I_N^\bullet \to 0$$

($I_K^\bullet$ must termwise split as $I_K^\bullet \cong I_M^i \oplus I_N^i, \forall i \geqslant 0$).

Definition 3.1.7. Let $\mathcal{A}$ be an abelian category with enough injectives and F be a left exact functor on $\mathcal{A}$. Then

$$R^i F(M) := H^i(F(I_M^\bullet))$$

Corollary 3.1.8.

(1) $R^i F(M) = 0, \forall i < 0$. $R^0(M) = F(M)$.

(2) $R^i F(I) = 0$ if I is injective and $i \neq 0$.

(3) $R^i F(M)$ is independent of choices of $I_M^\bullet$.

(4) Given any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, $\exists$ l.e.s.

$$\begin{array}{l} 0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow \cdots \\ \longrightarrow R^i F(A) \longrightarrow R^i F(B) \longrightarrow R^i F(C) \xrightarrow{\delta^i} \\ R^{i+1} F(A) \longrightarrow R^{i+1} F(B) \longrightarrow R^{i+1} F(C) \longrightarrow \ldots \end{array}$$

Proof. Only (4) needs some comment. Take injective resolutions of A, B, C as in the horseshoe lemma. Note that injectiveness of $I_A^\bullet$ shows that there is a termwise splitting $I_B^\bullet \cong I_A^\bullet \oplus I_C^\bullet$. Apply F we get

$$0 \to F(I_A^\bullet) \to F(I_B^\bullet) \to F(I_C^\bullet) \to 0$$

a termwise split sequence. Taking cohomology for this sequence gives the desired result. $\square$

E.g. $\mathrm{Ext}_R^i(M, N) = R^i \mathrm{Hom}_R(M, N) \cong H^i(\mathrm{Hom}_R(M, I_N^\bullet))$

3.2 Cartan-Eilenberg resolution

Instead of resolving a single object by injectives, we can also resolve certan complexes in $\mathcal{A}$, provided it has enough injectives.

Definition 3.2.1. Let $\mathcal{A}$ be an abelian category and $K^\bullet$ a complex bounded from below. ($K^p = 0$ for all $p << 0$). A **Cartan-Eilenberg resolution** of $K^\bullet$ is a double complex $I^{\bullet,\bullet}$ and a morphism of complexes $\epsilon : K^\bullet \to I^{\bullet,0}$, satisfying

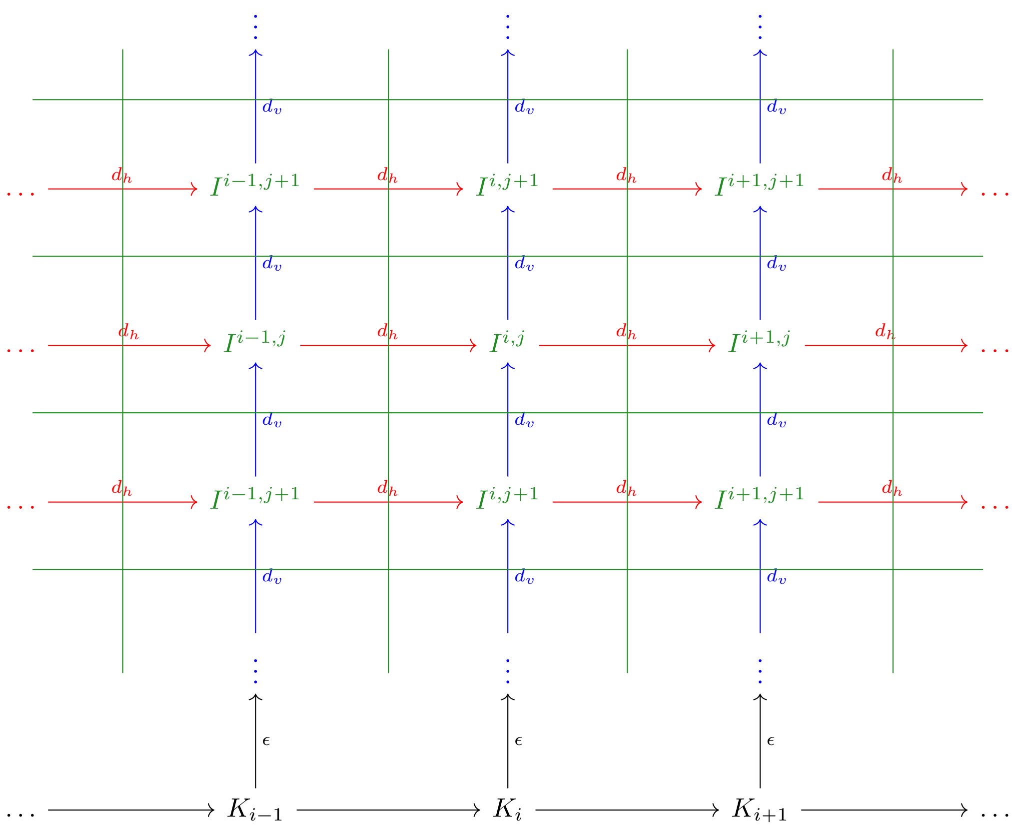
(1) $I^{p,\bullet} = 0$ for all $p << 0$, and $I^{\bullet,q} = 0$ if $q < 0$. (almost first quadrant).

(2) The complex $I^{p,\bullet}$ is an injective resolution of $K^p, \forall p$.

(3) The complex $(\ker d_h)^{p,\bullet}$ is an injective resolution of $\ker d_K^p$.

(4) The complex $(\text{Im } d_n)^{p,\bullet}$ is an injective resolution of $\text{Im } d_K^p$.

(5) The complex $H_h^p(I^{\bullet,\bullet})$ is an injective resolution of $H^p(K^\bullet)$.



Lemma 3.2.2. Let $\mathcal{A}$ be an abelian category with enough injectives, and $K^\bullet$ be a complex bounded from below. Then there exists a Cartan-Eilenberg resolution of $K^\bullet$.

Proof. Without loss of generality, assume $K^p = 0$ if $p < 0$. Let us break $K^\bullet$ into s.e.s. in $\mathcal{A}$

$$\begin{aligned} 0 \to Z^0 \to &K^0 \to B^1 \to 0 \\ 0 \to B^1 \to &Z^1 \to H^1 \to 0 \\ &\vdots \\ 0 \to Z^n \to &K^n \to B^{n+1} \to 0 \\ 0 \to B^{n+1} \to &Z^{n+1} \to H^{n+2} \to 0 \end{aligned}$$

Inductively, choose injective resolutions as in horseshoe lemma:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z^n & \longrightarrow & K^n & \longrightarrow & B^{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J_Z^{n,\bullet} & \longrightarrow & I^{n,\bullet} & \longrightarrow & J_B^{n+1,\bullet} & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^{n+1} & \longrightarrow & Z^{n+1} & \longrightarrow & H^{n+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J_B^{n+1,\bullet} & \longrightarrow & J_Z^{n+1,\bullet} & \longrightarrow & J_H^{n+1,\bullet} & \longrightarrow & 0 \end{array}$$

Take $d_h : I^{n,\bullet} \to I^{n+1,\bullet}$ to be the composition

$$I^{n,\bullet} \to J_B^{n+1,\bullet} \to J_Z^{n+1,\bullet} \to I^{n+1,\bullet}$$

The lemma follows. □

3.3 The Grothendieck spectral sequence

Definition 3.3.1. Let $\mathcal{A}$ be an abelian category with enough injectives, and F a left exact functor on $\mathcal{A}$. An object $A \in \mathcal{A}$ is called **F-acyclic** if

$$R^i F(A) = 0, \forall i > 0$$

E.g.

(1) Any injective object is F-acyclic.

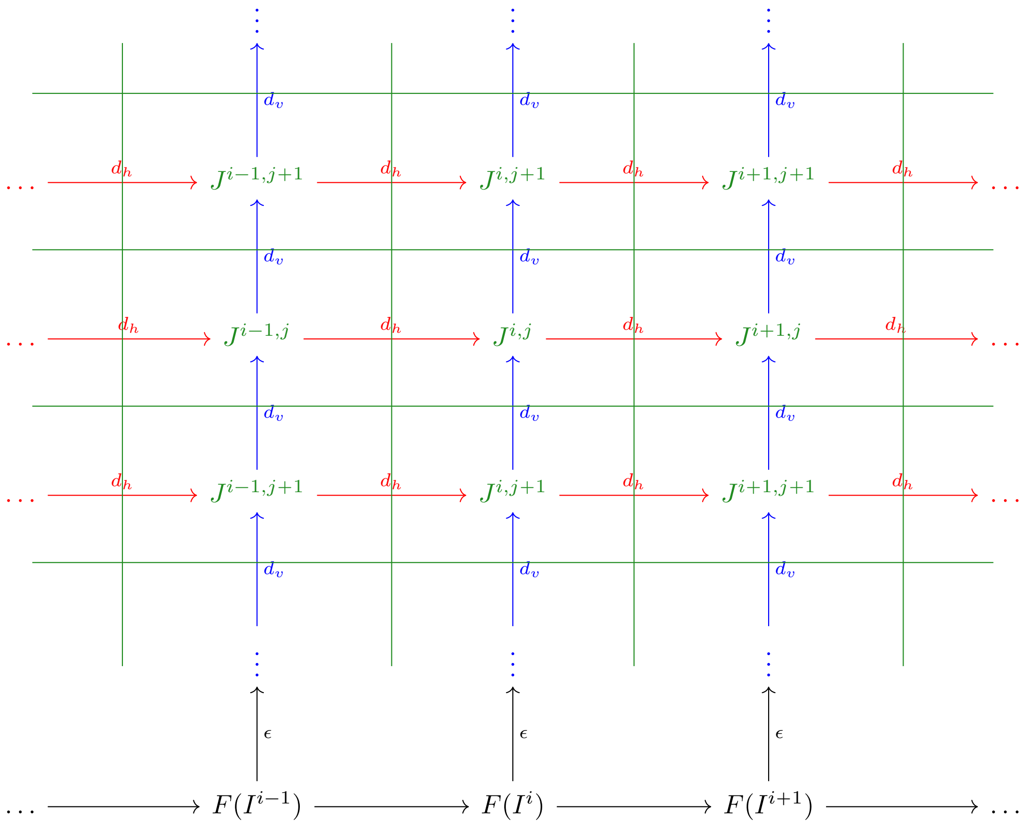
(2) If $F = \Gamma_X$ on a paracompact space X, then any sheaf admitting partition of 1 is F-acyclic.

Theorem 3.3.2. *If $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ are left exact functors between abelian categories such that $\mathcal{A}, \mathcal{B}$ have enough injectives. Suppose F takes injective objects to G-acyclic objects. Then, for any object $A \in \mathcal{A}$, $\exists$ spectral sequence*

$$E_2^{p,q} = R^p G(R^q F(A)) \Longrightarrow R^{p+q} F(A)$$

Proof. Take an injective resolution $0 \to A \to I^\bullet$. We then obtain $F(I^\bullet)$ is a complex of G-acyclic objects. Choose a Cartan-Eilenberg resolution

$$\varepsilon : F(I^\bullet) \to J^{\bullet,\bullet} :$$



Apply G to the bicomplex, we obtain two s.s. of a first quadrant bicomplex:

$$H_v^p H_h^q(G(J^{\bullet,\bullet})) \Longrightarrow H^{p+q}(\mathrm{Tot}(G(J^{\bullet,\bullet}))) \Longleftarrow H_h^q H_v^p(G(J^{\bullet,\bullet}))$$

The CE-resolution has each of its column an injective relosution of $F(I^\bullet)$, thus each $H_v^p(G(J^{\bullet,\bullet}))$ computes $R^pG(F(I^\bullet)) = 0$ if $p \geqslant 1$. Thus the second s.s. degenerates at E_2, and

$$\bigoplus_{p+q=k} H_h^p H_v^q(G(J^{\bullet,\bullet})) = H^k(GF(I^\bullet)) = R^kGF(A)$$

On the other hand, in each horizontal row, $J^{\bullet,\bullet}$ is built from split s.e.s. of injectives whose cohomology gives an injective relosution of $H^q(F(I^\bullet)) = R^qF(A)$, thus

$$H_v^p H_h^q(G(J^{\bullet,\bullet})) = H_v^p G(H_h^q(J^{\bullet,\bullet})) = R^pG(R^q(F(A)))$$

Comparing these two s.s. gives us the desired result. □

The proof of the Thm also implies the following.

Corollary 3.3.3. If $F : \mathcal{A} \to \mathcal{B}$ is left exact and $0 \to A \to K^\bullet$ is a resolution of A by F-acyclic objects, then

$$R^iF(A) \cong H^i(F(K^\bullet))$$

E.g. Acyclic resolutions are usually more handy. In algebraic topology we know that

$$H^\bullet_{sing}(X,\mathbb{C}) \cong R^\bullet\Gamma_X(\underline{\mathbb{C}})$$

where X is a reasonable topological space and $\underline{\mathbb{C}}$ is the constant sheaf.

When X is a smooth manifold, we have a resolution

$$0 \to \underline{\mathbb{C}} \to \Omega^\bullet_X \quad \text{(de Rham complex)}$$

of $\underline{\mathbb{C}}$ by Γ_X-acyclic sheaves ($\Omega^\bullet_X$ admits partition of 1). Thus the cohomology of the de Rham complex

$$R^\bullet\Gamma_X(\underline{\mathbb{C}}) \cong H^\bullet(\Gamma_X(\Omega^\bullet_X)) \cong H^\bullet_{sing}(X;\mathbb{C}).$$

3.4 Lyndon-Serre-Hochschild spectral sequence

Let G be a discrete group and K a normal subgroup. Suppose $A \in {}_{\Bbbk G}\mathcal{M}od$. Then the functor

$$(-)^G : A \mapsto A^G$$

is a left exact functor on ${}_{\Bbbk G}\mathcal{M}od$ ($\cong \mathrm{Hom}_G(\Bbbk,-)$, where $\Bbbk$ is equipped with the trivial G-action). Then there is a functor isomorphism

$$(-)^G = ((-)^K)^{G/K}$$

The GSS specializes in this situation to the Lyndon-Hochschild-Serre s.s.

$$H^p(G/K, H^q(K,M))\mathbb{R} \Longrightarrow H^{p+q}(G,M)$$

Lemma 3.4.1.

(1) The abelian category ${}_{\Bbbk G}\mathcal{M}od$ has enough injectives.

(2) $(-)^K$ takes injective $\Bbbk G$-modules to injective $\Bbbk(G/K)$-modules.

Proof. (sketch) Assume G is finite. Then $\Bbbk G$ is a self-injectie algebra. Any $M \in {}_{\Bbbk G}\mathcal{M}od$ admits an embedding $M \to \Bbbk G \otimes M$, $x \mapsto (\sum_{g\in G} g) \otimes x$. Further, as a $\Bbbk G$-module, there is an isomorphism

$$\Bbbk G \otimes M \xrightarrow{\cong} \Bbbk G \otimes M^{tr} \quad g \otimes x \mapsto g \otimes g^{-1}x$$

where M^{tr} denotes M with the trivial G-action. The inverse map is given by

$$\Bbbk G \otimes M^{tr} \xrightarrow{\cong} \Bbbk G \otimes M \quad h \otimes y \mapsto h \otimes hy$$

Thus $\Bbbk G \otimes M$ is injective and ${}_{\Bbbk G}\mathcal{M}od$ has enought injectives.

Now, by this discussion, any injective module is a direct summand of $\Bbbk G^{\oplus r}$, $r \in \mathbb{N} \cup \{\infty\}$. Thus it suffies to show that $(\Bbbk G)^K$ is an injective $\Bbbk G/K$-module. This is clear since $(\Bbbk G)^K = \Bbbk(G/K)$.

This proof relies on that $\Bbbk G \cong (\Bbbk G)^*$ as G-modules when G is finite. In general, when $|G| = \infty$, one replaces $\Bbbk G \otimes M \cong (\Bbbk G)^* \otimes M$ by $\mathrm{Hom}_{\Bbbk}(\Bbbk G, M)$. The proof can be adjusted accordingly. □

Chapter 4

Introduction to sheaves and their cohomology

Andrei Rapinchuk

4.1 Introduction to Sheaves

4.1.1 Presheaf

Let X be a topological space.

Definition 4.1.1 (Presheaf). A Presheaf of sets, $\mathcal{F}$ on X consists of the following data:

a) a set $\mathcal{F}(U)$ for each open set $U \subset X$.

b) a map of sets, $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ for each open set $V \subset U$ such that

- $\rho_U^U = id_{F(U)}$,
- $\rho_W^U = \rho_W^V \circ \rho_V^U$ whenever $W \subset V \subset U$.

The elements of $\mathcal{F}(U)$ are often called "sections of $\mathcal{F}$ over U". This terminology is justified by the fact that for a given presheaf $\mathcal{F}$ on X, one can construct a topological space $\mathcal{F}$ together with a local homeomorphism, $\phi : \mathcal{F} \to X$ (called the étale space of the presheaf $\mathcal{F}$) such that when $\mathcal{F}$ is a sheaf (section 4.1.2), the set $\mathcal{F}(U)$ can be naturally identified with the set of sections

$$\{s : U \to \mathcal{F} \mid s \text{ continuous and } \phi \circ s = id_U\}.$$

For an arbitrary presheaf, this provides one of the constructions of the sheafification of $\mathcal{F}$. In this realization the maps

$\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ are just the restriction maps. So, we will refer to ρ_V^U as the restriction maps in the completely general situation. The elements of $\mathcal{F}(X)$ are called global sections.

We now note that the definition of a presheaf can be reformulated using categorical language. Namely, given a topological space X, we let $Op(X)$ denote the category whose objects are all the open sets $U \subset X$, and whose morphisms are defined as follows: for $U, V \in Ob(Op(X))$,

$$Hom(V, U) := \begin{cases} \{i : V \to U\}(inclusion) & \text{if } V \subset U, \\ \phi & otherwise. \end{cases}$$

Then giving a presheaf on X is precisely equivalent to giving a contravariant functor on $Op(X)$ with values in the category of sets. From a more general perspective that we will use to discuss Grothendieck topologies, a presheaf is just a contravariant functor defined on an arbitrary category. In this case, the notion of a sheaf requires some important additional structures on that category, viz a system of coverings.

Another remark is that one can consider contravariant functors with values in the categories of abelian groups, rings etc. In this case we talk about presheaf of abelian groups, rings etc. In the situation of "classical" presheaves (i.e. contravariant functors on $Op(X)$), this amounts to the requirement that $\mathcal{F}(U)$ be an abelian group/ring for each open $U \subset X$, and the restriction maps $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ be group/ring homomorphisms.

Let $\mathcal{F}, \mathcal{G}$ be presheaves on a topological space X with the values in a category $\mathcal{C}$. We say that $\mathcal{G}$ is a sub-presheaf of $\mathcal{F}$ if for every open set $U \subset X$, $\mathcal{G}(U)$ is a subset of $\mathcal{F}(U)$, and for $V \subset U$, the restriction, $\rho(\mathcal{G})_V^U$ is the restriction of $\rho(\mathcal{F})_V^U$.

We typically assume that $\mathcal{F}(\phi)$ is a terminal object in our category; for example, for the category of sets, $\mathcal{F}(\phi)$ is a 1-element set, for the category of rings, it is the zero ring etc.

Example 4.1.2. *One of the most common examples, which actually serves as a prototype for many other example: the presheaf of continuous functions. More precisely, let X be a topological space, and for any open set $U \subset X$, we let $\mathcal{F}(U)$ denote the ring of all real (or complex) valued continuous functions $f : U \to \mathbb{R}$ (in fact, one can consider continuous functions with values in any topological space). The restriction of functions defines the restriction homomorphisms, $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$. By special definition, $\mathcal{F}(\phi) = \{0\}$.*

If X is an open set of $\mathbb{R}^n$, then quite similarly one defines the sheaf of smooth (differentiable) functions, and if X is an open subset of $\mathbb{C}$, one defines the sheaf of holomorphic functions, etc.

Example 4.1.3. *Constant presheaf: Fix an object C in our category $\mathcal{C}$ and define $\mathcal{F}(U) = C$ for every $U \subset X$ open, $U \neq \phi$, and $\mathcal{F}(\phi) = T$ (terminal object). Furthermore, we define $\rho_V^U = id_C$ if $V \neq \phi$ and ρ_ϕ^U to be the unique morphism $C \to T$. It is easy to check that this defines a presheaf which is called the constant presheaf on X with value C. One can think of the elements of $\mathcal{F} = C$ as constant functions $f : U \to C$.*

Example 4.1.4. *(Pre)sheaf of locally constant functions: The previous example has the following useful generalization/variation. Let $\mathcal{F}(U)$ be the set of locally constant functions $f : U \to C$ (where C is a fixed object, e.g. a fixed set). This means that every $x \in U$ has a neighborhood $U_x \ni x$ such that $f(U_x) = \{f(x)\}$ (i.e f is constant on U_x). Note that the restriction of a locally constant function to a smaller open set is also locally constant, so we can define ρ_V^U as the usual restriction maps, and get a presheaf. We note that locally constant functions are continuous if C is given any toplogy (e.g. the discrete one). So the presheaf of locally* *constant functions can be viewed as a sub-presheaf of the presheaf of constant functions. Another remark is that if C is an abelian group or a ring, then $\mathcal{F}(U)$ is also an abelian group or a ring (In particular, the sum of two locally constant functions is a locally constant function).*

Example 4.1.5. *Presheaf of sections: Let $\pi : Y \to X$ be a continuous map. For a nonempty open set U, a section of π over U is a continuous map $\phi : U \to Y$ such that $\pi \circ \phi = id_U$. Let $\mathcal{F}(U)$ be the set of all sections of π over U. In general, it may happen that there are no sections. So, to avoid dealing with the empty set of sections, we may want to assume that π admits a section over X. For ρ_V^U, we then take the maps given by the restrictions of sections.*

Example 4.1.6. *"Skyscraper" presheaf: Let X be a topological space, and fix a point $p \in X$. Let C be a set (or a group, ring etc.). Let us define*

$$\mathcal{F}(U) := \begin{cases} C & \text{if } p \in U, \\ \{*\} & \text{otherwise,} \end{cases}$$

here $\{*\}$ denotes a terminal object. Furthermore, for $V \subset U$, let us define

a) $\rho_V^U = id_C$ if $p \in V \subset U$,

b) $\rho_V^U =$ unique morphism $C \to \{*\}$ if $p \in U \backslash V$,

c) $\rho_V^U = id_{\{*\}}$ if $p \notin U$.

Intuitively, $\mathcal{F}$ is "concentrated" at p- this will have a more precise meaning/expression when we define the stalks.

Example 4.1.7. *A particularly important example for us comes from algebraic geometry. Let K be an algebraically closed field, $V \subset \mathbb{A}^n$ be an irreducible affine algebraic set, $K[V]$ be the ring of regular functions on V, i.e. $K[V] = K[x_1, x_2, \cdots, x_n]/I(V)$, where $I(V) \subset K[x_1, x_2, \cdots, x_n]$ is the ideal of all the polynomials that vanish on V, $K(V)$ be the field of rational functions on V (i.e. field of fraction of $K[V]$). We consider V as a topological space equipped with the Zariski topology. We say $f \in K(V)$ is regular at $p \in V$ if there exist a representation, $f = \dfrac{g_p}{h_p}$ with $g_p, h_p \in K[V]$ and $h_p(p) \neq 0$.*

For an open set $U \subset \mathbb{A}^n$, we let $\mathcal{F}(U)$ denote the set of all rational functions that are defined at all the points $p \in U$. Then for $V \subset U$ we have that $\mathcal{F}(U) \subset \mathcal{F}(V)$, so for ρ_V^U we just take the inclusion maps. This creates a presheaf which is historically known as the structure sheaf. It can be explicitly calculated at so called "principal" or "distinguished" open sets. More precisely, let $f \in K[V]$ and

$$D(f) := \{p \in V \mid f(p) \neq 0\},$$

then

$$\mathcal{F}(V) := K[V]_f \text{ (localization).}$$

This generalizes to arbitrary commutative rings. Let R be a commutative ring with 1. Let $X = Spec(R)$ be the set of all prime ideals of R (so called prime spectrum of R). One can equip X with a topology which is similar to the Zariski topology. For one thing, it admits basis consisting of distinguished open sets: for $f \in R$,

$$D(f) := \{\mathfrak{p} \in Spec(R) | \mathfrak{p} \not\ni f\}.$$

Then one can define a presheaf (actually, a sheaf) on X, called the structure sheaf of an affine scheme Spec(R), which is uniquely defined by

$$\mathcal{F}(D(f)) := A_f \text{ (localization).}$$

Definition 4.1.8. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on a topological space X with values in a category $\mathcal{C}$. A morphism of presheaves, $\phi : \mathcal{F} \to \mathcal{G}$ is a family of morphisms (in $\mathcal{C}$), $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$, one for each open set $U \subset X$ such that for open $V \subset U$, we get the following commutative diagram.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ {\scriptstyle \rho(\mathcal{F})^U_V}\downarrow & & \downarrow{\scriptstyle \rho(\mathcal{G})^U_V} \\ \mathcal{F}(V) & \xrightarrow{\phi_V} & \mathcal{G}(V) \end{array} \qquad (*)$$

Remark 1. Viewing presheaves as contravariant functors, $Op(X) \to \mathcal{C}$, we see that morphisms of presheaves are simply natural transformations of functors.

Presheaves on X and their morphisms form a category denoted by $\mathcal{P}sh(X)$.

Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups.then then commutativity of $(*)$ implies that for $V \subset U$, we have

$$\rho(\mathcal{F})^U_V(\ker \phi_U) \subset \ker \phi_V,$$

and

$$\rho(\mathcal{G})^U_V(\text{im}\phi_U) \subset \text{im}\phi_V.$$

This means that we obtain new presheaves $\mathcal{K}$ and $\mathcal{I}$ defined by $\mathcal{K}(U) = \ker \phi_U$ and $\mathcal{I}(U) = \text{im}\phi_U$ which are called the kernel presheaf and the image presheaf. Clearly, $\mathcal{K}$ is a subpresheaf of $\mathcal{F}$, and $\mathcal{I}$ is a subpresheaf of $\mathcal{G}$.

4.1.2 Sheaf

One of the features of continuous functions is that they can be glued from local information. In the simplest case, if $U = U_1 \cup U_2$ (U, U_1, U_2 open sets), then given any continuous function $f : U \to \mathbb{R}$ is equivalent to giving continuous functions $f_1 : U_1 \to \mathbb{R}$ and $f_2 : U_2 \to \mathbb{R}$ such that $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$. In other words, here the "local" data, subject to some natural compatibility conditions, can be glued into the global data. This kind of condition is precisely what distinguishes a sheaves from arbitrary presheaves.

Definition 4.1.9. A presheaf $\mathcal{F}$ on a topological space X is called a sheaf if every open set $U \subset X$ and every open covering $U = \cup_{\alpha \in I} U_\alpha$, the following condition holds.

a) If $s, t \in \mathcal{F}(U)$ and $\rho^U_{U_\alpha}(s) = \rho^U_{U_\alpha}(t)$ for all $\alpha \in I$, then $s = t$.

b) Given $s_\alpha \in \mathcal{F}(U_\alpha)$ for all $\alpha \in I$ such that $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta)$ then there exist an $s \in \mathcal{F}(U)$ such that $\rho^U_{U_\alpha}(s) = s_\alpha$ for all $\alpha \in I$.

(Note that according to a), the element $s \in \mathcal{F}(U)$ in b) is unique.)

If $\mathcal{F}$ is a presheaf of abelian groups, then $a)$ can be restated as follows: If $s \in \mathcal{F}(U)$ and $\rho^U_{U_\alpha}(s) = 0$ for all $\alpha \in I$, then $s = 0$. In this case, both the requirements can be stated together as the exactness of the following sequence.

$$0 \to \mathcal{F}(U) \to \prod_{\alpha \in I} \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta \in I} \mathcal{F}(U_\alpha \cap U_\beta).$$

The exactness is understood in the following sense: the map

$$\mathcal{F}(U) \to \prod_{\alpha \in I} \mathcal{F}(U_\alpha)$$

is given by the product of the restrictions,

$$\rho^U_{U_\alpha} : \mathcal{F}(U) \to \mathcal{F}(U_\alpha),$$

identifies $\mathcal{F}(U)$ with the subgroup consisting of those elements on which the two arrows coincide (equalizer). These arrows are given by

$$(f_\alpha) \mapsto \rho^{U_\alpha}_{U_\alpha \cap U_\beta}(f_\alpha)_{(\alpha,\beta)},$$

and

$$(f_\alpha) \mapsto \rho^{U_\beta}_{U_\alpha \cap U_\beta}(f_\beta)_{(\alpha,\beta)}.$$

In this form, the sheaf requirements easily generalize, for example, to the sheaves for Grothendieck topologies.

Terminology 1. Terminology Presheaves satisfying a) are called separated. Axiom b) is often called gluing axiom.

Example 4.1.10. *Let X and Y be topological spaces, and let $\mathcal{F}$ be the presheaf of Y-valued continuous functions i.e. for an open $U \subset X$,*

$$\mathcal{F}(U) = \{f : U \to Y \text{ continuous}\}$$

with ρ^U_V defined in terms of restrictions of functions. We previously noted that $\mathcal{F}$ is a presheaf, but actually it is a sheaf. Indeed, let $U = \cup_{\alpha \in I} U_\alpha$ be an open covering of an open set, $U \subset X$, and let $f, g : U \to Y$ be continuous functions. The fact that $\rho^U_{U_\alpha}(f) = \rho^U_{U_\alpha}(g)$ means that $f(x) = g(x)$ for all $x \in U_\alpha$. Since U_α's cover U, we obtain that $f(x) = g(x)$ for all $x \in U$ i.e. $f = g$. Now, given $f_\alpha : U_\alpha \to Y$ such that $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(f_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(f_\beta)$, we can define $f : U \to Y$ by letting $f(x) = f_\alpha(x)$ if $x \in U_\alpha$. The compatibility condition tells us that $f_\alpha(x) = f_\beta(x)$ if $x \in U_\alpha \cap U_\beta$, so f is well-defined. Furthermore, f is also continuous because for an open set $V \subset Y$ we have

$$f^{-1}(V) = \bigcup_{\alpha \in I} f_\alpha^{-1}(V),$$

and for each α, the set $f_\alpha^{-1}(V)$ is open in U_α, hence in U.

Example 4.1.11. *It is easy to see that sub-presheaves of sheaves are automatically separated but they may not satisfy the gluing axiom. For example, let $X = \mathbb{R}$ and $\mathcal{F}$ be the sheaf of continuous $\mathbb{R}$-valued functions. Let $\mathcal{G}$ be the sub-presheaf of bounded continuous functions i.e. for $U \subset \mathbb{R}$, open,*

$$\mathcal{G}(U) = \{f : U \to \mathbb{R} \,|\, f \text{ continuous and bounded}\}.$$

Consider the open covering of $U = \mathbb{R}$ *by* $U_n = (n - \frac{1}{3}, n + \frac{4}{3})$ *for* $n \in \mathbb{Z}$*, and let* $f_n : U_n \to \mathbb{R}$ *be given by* $f_n(x) = x$*. Clearly* $f_n \in \mathcal{G}(U_n)$ *and* $\rho^{U_n}_{U_n \cap U_m}(f_n) = \rho^{U_m}_{U_n \cap U_m}(f_m)$ *but there is NO bounded function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\rho^U_{U_n}(f) = f_n$ *for all* n*.*

Example 4.1.12. *Constant presheaf often fails to be a sheaf. Indeed, suppose* X *has two nonempty open sets* U_1, U_2 *such that* $U_1 \cap U_2 = \phi$*, and let* C *be a set of cardinality* $\geqslant 2$*. Pick* $c_1, c_2 \in C$ *such that* $c_1 \neq c_2$ *and consider* $f_i \in \mathcal{F}(U_i)$ *(where* $\mathcal{F}$ *is the constant presheaf) that corresponds to* c_i *i.e.* $f_i : U_i \to C$ *such that* $f_i(x) = c_i$ *for all* $x \in U_i$*. Since* $U_1 \cap U_2 = \phi$*, we have* $\mathcal{F}(U_1 \cap U_2) = \{*\}$ *(terminal element), and therefore*

$$\rho^{U_1}_{U_1 \cap U_2}(f_1) = \rho^{U_2}_{U_1 \cap U_2}(f_2).$$

On the other hand, there is now $f \in \mathcal{F}(U)$ *such that* $\rho^U_{U_i}(f) = f_i$ *because if the value of* f *on* U *is* c*, then the values on* U_1 *and* U_2 *will also be* c *(since the restriction maps are the identity maps). Then* $c_1 = c = c_2$*, a contradiction.*

Let now $\mathcal{F}$ *be the presheaf of locally constant functions. Then given an open subset* $U \subset X$*, an open covering* $U = \cup_{\alpha \in I} U_\alpha$*, and locally constant functions* $f_\alpha \in \mathcal{F}(U_\alpha)$ *such that*

$$\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(f_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(f_\beta) \qquad \text{for all } \alpha, \beta \in I.$$

As we have seen in the example of the (pre)sheaf of continuous functions, there exists a function, $f : U \to C$*, such that* $f|U_\alpha = f_\alpha$*. This function is automatically locally constant, for any* $x \in U$ *belongs to some* U_α*. Since* f_α *is locally constant, there exists* $x \in V_x \subset U_\alpha$ *such that* $f_\alpha(V_x) = \{f(x)\}$*. But then* V_x *is open in* U *and* $f(V_x) = f_\alpha(V_x) = \{f(x)\}$*. Thus, the presheaf of locally constant functions is a sheaf. A bit later we will describe the connection between the presheaves of constant and locally constant functions in more precise terms.*

Example 4.1.13. *The skyscraper presheaf: Recall the definition: Let* X *be a topological space, and fix a point* $p \in X$*. Let* C *be a set (or a group, ring etc.). Let us define*

$$\mathcal{F}(U) := \begin{cases} C & \text{if } p \in U, \\ \{*\} & \text{otherwise,} \end{cases}$$

here $\{*\}$ *denotes the terminal object. And, for* $V \subset U$*, let us define*

a) $\rho^U_V = id_C$ *if* $p \in V \subset U$*,*

b) $\rho^U_V =$ *unique morphism* $C \to \{*\}$ *if* $p \in U \backslash V$*,*

c) $\rho^U_V = id_{\{*\}}$ *if* $p \notin U$*.*

Let us show that $\mathcal{F}$ *is a sheaf. Let* $U \subset X$ *be an open set with an open covering* $U = \cup_{\alpha \in I} U_\alpha$*. Let* $s, t \in \mathcal{F}$ *be such that* $\rho^U_{U_\alpha}(s) = \rho^U_{U_\alpha}(t)$ *for all* $\alpha \in I$*.*
Case 1. $p \notin U$*. Then* $\mathcal{F}(U) = \{*\}$*, and for any* $\alpha \in I$*,* $\mathcal{F}(U_\alpha) = \{*\}$*, with* $\rho^U_{U_\alpha}$ *being the identity maps of* $\{*\}$*. Then* $\rho^U_{U_\alpha}(s) = \rho^U_{U_\alpha}(t)$ *implies that* $s = t$*.*
Case 2. $p \in U$*. Then* $p \in U_\alpha$ *for some* $\alpha \in I$*, in which case* $\rho^U_{U_\alpha}$ *is* id_C*. Again* $\rho^U_{U_\alpha}(s) = \rho^U_{U_\alpha}(t)$ *implies that* $s = t$*.*

Suppose we are given $f_\alpha \in \mathcal{F}(U_\alpha)$ *such that*

$$\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(f_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(f_\beta) \tag{4.1}$$

If $p \notin U$, *then* $\mathcal{F}(U) = \mathcal{F}(U_\alpha) = \mathcal{F}(U_\alpha \cap U_\beta) = \{*\}$ *for all* $\alpha, \beta \in I$ *with the restriction maps being the identity maps. So, (4.1) tells us that all* $f_\alpha \in \{*\}$ *are equal, and their common element gives a required element of* $\mathcal{F}(U) = \{*\}$.

Suppose now that $p \in U$. *Then* $p \in U_\alpha$ *for some* $\alpha \in I$. *Let* $f = f_\alpha \in C$. *It follows from (4.1) that* $f = f_\beta$ *for* <u>*any*</u> β *such that* $p \in U_\beta$. *This means that* f *is as required.*

Definition 4.1.14. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space X. A morphism, $\phi : \mathcal{F} \to \mathcal{G}$, is simply a morphism of the underlying presheaves. An isomorphism is a morphism that has a 2-sided inverse.

Sheaves and morphisms of sheaves on a topological space X with values in a category $\mathcal{C}$ form a category denoted by $\mathcal{S}h(X)$ or $\mathcal{S}h(X, \mathcal{C})$. It follows from the above definition that it is a full subcategory of the category $\mathcal{P}sh(X)$.

Some constructions on presheaves when applied to sheaves, result in sheaves and some do not. Here is one example when they do.

Lemma 4.1.15. *Let* $\phi : \mathcal{F} \to \mathcal{G}$ *be a morphism of sheaves of abelian groups on a topological space* X. *Then the kernel presheaf defined by* $\mathcal{K}(U) = \ker(\phi_U : \mathcal{F}(U) \to \mathcal{G}(U))$ *is a sheaf.*

Proof. Being a subpresheaf of a sheaf, $\mathcal{K}(U)$ is separated, i.e. satisfies axiom a) of a sheaf. Let us now verify axiom b). Let $U \subset X$ be an open set with an open covering $U = \cup_{\alpha \in I} U_\alpha$, and $k_\alpha \in \mathcal{K}(U_\alpha)$ be such that $\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(k_\alpha) = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(k_\beta)$ for all $\alpha, \beta \in I$. (Note that $\rho(\mathcal{K})^U_V = \rho(\mathcal{F})^U_V$ for any $V \subset U$.) Using the fact that $\mathcal{F}$ is a sheaf, we conclude that there exists $k \in \mathcal{F}(U)$ such that $\rho^U_{U_\alpha}(k) = k_\alpha$ for all α. We only need to show that $k \in \mathcal{K}(U)$, i.e. $\phi_U(k) = 0 \in \mathcal{G}(U)$. For this we observe that for any $\alpha \in I$ we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\
\rho(\mathcal{F})^U_{U_\alpha}\downarrow & & \downarrow\rho(\mathcal{G})^U_{U_\alpha} \\
\mathcal{F}(U_\alpha) & \xrightarrow{\phi_{U_\alpha}} & \mathcal{G}(U_\alpha)
\end{array}$$

Since $\rho^U_{U_\alpha}(k) = k_\alpha \in K(U_\alpha)$, we conclude that

$$\rho(\mathcal{G})^U_{U_\alpha}(\phi_U(k)) = 0.$$

This being true for all α, we see that $\phi_U(k) = 0$ since $\mathcal{G}$ is a sheaf. □

We will now examine the situation with the image/quotient.

Example 4.1.16. *Let* $X = S^1$ *(unit circle). Let* $\mathcal{F}$ *(resp.* $\mathcal{G}$*) be the presheaf on* X *of continuous functions with values in* $\mathbb{R}$ *(resp* X*) i.e. for* $U \subset X$, *open*

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \text{ continuous}\}$$

and

$$\mathcal{G}(U) = \{g : U \to X \ \ continuous\}$$

Note that these are sheaves of abelian groups (note that X has a natural group structure). Furthermore, let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves (of abelian groups) given by $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$, $\phi_U(f) = e^{2\pi i f(x)}$. The kernel sheaf is given by

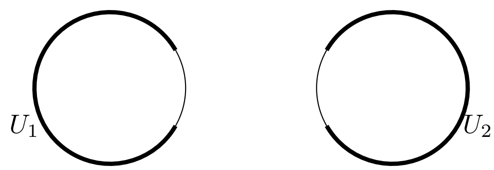
$$\mathcal{K}(U) = \{f : U \to \mathbb{Z} \ continuous\}.$$

Then image presheaf is given by

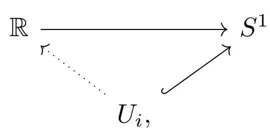
$$\mathcal{I}(U) = im\phi_U \simeq \mathcal{F}(U)/\mathcal{K}(U).$$

We note that given any presheaf of abelian groups and its subpresheaf, one can always from the quotient presheaf. In this example, we will show that image presheaf is NOT a sheaf, which will also show that quotient of a sheaf by its subpreheaf may not be a sheaf.

First, we observe that every function in $\mathcal{I}(X)$ defines a closed curve in X which is homotopic to a point. This implies that the identity function $g : X \to X$, $g(x) = x$, is not in $\mathcal{I}(X)$. On the other hand, let us consider the following open subsets,



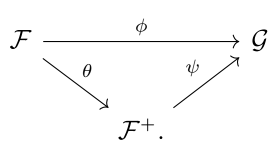
and let $g_i = g|U_i$. Since each U_i is simply connected, it can be lifted to the universal cover $\mathbb{R} \to S^1$:



which implies that $g_i \in \mathcal{I}(U_i)$. Clearly, g_1 and g_2 agree on $U_1 \cap U_2$ (because they are the restrictions of the same function g), and $U_1 \cup U_2 = X$, but there is no function in $\mathcal{I}(X)$ with the restrictions g_1 and g_2. Thus $\mathcal{I}$ is NOT a sheaf. (Philosophically, the reason is that "to belong to the image" is not a local property.)

This example seems to destroy completely any hopes to make the category of, say, the sheaves of the abelian groups on a topological spaces into an abelian category (which is needed to do homological algebra) as we do have kernels but not the cokernels. However, there is a way around this problem. More precisely, there is a natural procedure that attaches a sheaf to every presheaf; this procedure is called sheafification.

Theorem 1. For any presheaf $\mathcal{F}$ on a topological space X, there exists a sheaf $\mathcal{F}^+$ together with a morphism of presheaves, $\theta : \mathcal{F} \to \mathcal{F}^+$ such that for any sheaf $\mathcal{G}$ and a morphism $\phi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that the following diagram commutes.



The sheaf $\mathcal{F}^+$, which is defined uniquely up to an isomorphism, is called the sheaf associated with the presheaf $\mathcal{F}$, or the sheafification of $\mathcal{F}$.

Remark 2. 1) We will review two constructions of sheafification - one is based on the consideration of the etale space of a presheaf, which does not apply to the sheaves for Grothendieck topologies ; the other employs the Cech $\check{H}^0$, which does apply to the sheaves for Grothendieck topologies.

2) If $\mathcal{F}$ is a presheaf of abelian groups etc., then $\mathcal{F}^+$ is a sheaf of abelian groups etc., and θ is a morphism of presheaves of abelian groups.

3) There is one very important property, which does not follow from the universal property, but is a consequence of the construction(s). This property says that for every $x \in X$, we have an isomorphism of stalks: $\theta_x : \mathcal{F}_x \to \mathcal{F}^+_x$. We will define stalks in the next section. This property is fundamental for controlling $\mathcal{F}^+$, whose construction is not very tractable.

Another important point is that the construction of sheafification is functorial. More precisely, let $\mathcal{F}_1 \to \mathcal{F}_2$ be a morphism of presheaves. Consider the composition $\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_2^+$, which is a morphism of of $\mathcal{F}_1$ to the sheaf $\mathcal{F}_2^+$. Using the universal property, we obtain a morphism $\mathcal{F}_1^+ \to \mathcal{F}_2^+$ such that the following diagram commute.

$$\begin{array}{ccc} \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 \\ \downarrow & & \downarrow \\ \mathcal{F}_1^+ & \longrightarrow & \mathcal{F}_2^+ \end{array}$$

This implies that sheafification defines a functor $S : \mathcal{P}sh(X) \to \mathcal{S}h(X)$. Furthermore, the universal property implies that there is a bijection,

$$\mathrm{Mor}_{\mathcal{S}\mathrm{h(X)}}(\mathcal{F}^+, \mathcal{G}) \simeq \mathrm{Mor}_{\mathcal{P}\mathrm{sh(X)}}(\mathcal{F}, \mathcal{G}), \psi \leftrightarrow \phi.$$

Moreover, this bijection is natural in the sense that given a morphism of presheaves, $\alpha : \mathcal{F}_1 \to \mathcal{F}_2$, and the corresponding morphism of sheaves, $\alpha^+ : \mathcal{F}_1^+ \to \mathcal{F}_2^+$, the diagram

$$\begin{array}{ccc} \mathrm{Mor}_{\mathcal{S}\mathrm{h(X)}}(\mathcal{F}_1^+, \mathcal{G}) & \xrightarrow{\simeq} & \mathrm{Mor}_{\mathcal{P}\mathrm{sh(X)}}(\mathcal{F}_1, \mathcal{G}) \\ {\scriptstyle\alpha_*^+}\uparrow & & {\scriptstyle\alpha_*}\uparrow \\ \mathrm{Mor}_{\mathcal{S}\mathrm{h(X)}}(\mathcal{F}_2^+, \mathcal{G}) & \xrightarrow{\simeq} & \mathrm{Mor}_{\mathcal{P}\mathrm{sh(X)}}(\mathcal{F}_2, \mathcal{G}), \end{array}$$

$\alpha_*(f) := f \circ \alpha$, commutes. This can be rewritten as a natural bijection

$$\mathrm{Mor}_{\mathcal{S}h(X)}(S\mathcal{F}, \mathcal{G}) \simeq \mathrm{Mor}_{\mathcal{P}sh(X)}(\mathcal{F}, T\mathcal{G}),$$

where $S : \mathcal{P}sh(X) \to \mathcal{S}h(X)$ is the sheafification, and $T : \mathcal{S}h(X) \to \mathcal{P}sh(X)$ is the embedding. This means that the sheafification is the left adjoint of the embedding functor.

4.2 Limits and Stalks

We will deal with direct limits which are known as colimits in the category setting. We will begin with categorical definition, which is indispensable for Grothendieck topologies, and then specialize in our setting.

Let $\mathcal{I}$ and $\mathcal{C}$ be categories. Then to each $X \in Ob\mathcal{C}$, we can associate the constant functor, $c_X : \mathcal{I} \to \mathcal{C}$, that takes every object of $\mathcal{I}$ to X, and every morphism to identity morphism on X. Note that any morphism $X \to Y$ in $\mathcal{C}$ induces a natural transformation $c_X \to c_Y$.

Let $\mathcal{F} : \mathcal{I} \to \mathcal{C}$ be a functor. We can construct the following covariant functor.

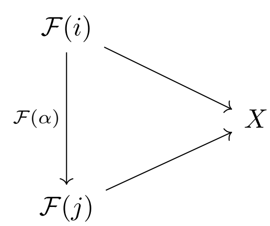
$$\begin{aligned} \mathcal{C} &\to Sets, \\ X &\mapsto \mathrm{Hom}(\mathcal{F}, c_X). \end{aligned}$$

If this functor is representable, then representing object is called the colimit (or direct limit) of $\mathcal{F}$ and denoted by $\varinjlim \mathcal{F}$. Thus, we are supposed to have a bijection

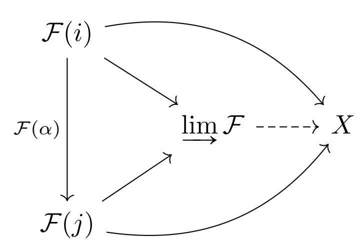
$$\mathrm{Hom}(\mathcal{F}, c_X) \simeq \mathrm{Hom}(\varinjlim \mathcal{F}, X), \tag{4.2}$$

for every object $X \in \mathcal{C}$, which is natural in X.

Let us unscramble this definition. To give a morphism (natural transformation), $\mathcal{F} \to c_X$ means to give a morphism (in $\mathcal{C}$), $\mathcal{F}(i) \to X$ for each $i \in Ob(\mathcal{I})$ so that for each morphism $\alpha : i \to j$ in $\mathcal{I}$, the diagram



commutes. In particular, taking $X = \varinjlim \mathcal{F}$ and the identity morphism in the right-hand side of (4.2), we see that for each $i \in \mathcal{I}$, there is a morphism $\mathcal{F}(i) \to \varinjlim \mathcal{F}$. Furthermore, for every morphism $\alpha : i \to j$ in $\mathcal{I}$, the diagram



commutes for any $X \in Ob(\mathcal{C})$ and any morphism $\mathcal{F} \to c_X$. When working with classical (pre)sheaves, we will only deal with the situation where $\mathcal{I}$ is a subcategory of $Op(X)$. In turn, $Op(X)$ is a particular case of categories attached to partially ordered sets.

So, let I be a partially ordered set with order (or preorder) relation $\leqslant$ (we only need reflexively and transitivity). When dealing with colimits (direct limits), we almost always assume that I is filtered or directed, which means that for every $i, j \in I$, there exists a $k \in I$ with the property $i, j \leqslant k$.

One can associate with I a category $\mathcal{I}$ by taking $Ob(\mathcal{I}) = I$, and for $i, j \in I$ we define

$$\mathrm{Hom}(i,j) = \begin{cases} \iota : i \to j & \text{if } i \leqslant j \\ \phi & \text{otherwise.} \end{cases}$$

We note that we recover $Op(X)$ from this construction by taking for I the set of all open subsets of X, and ordering I by reverse inclusion. Then to give a functor $\mathcal{F} : \mathcal{I} \to \mathcal{C}$ means to give an object $\mathcal{F}(i) =: A_i$ for each $i \in I$, and morphisms $\tau_i^j : A_i \to A_j$ whenever $i \leqslant j$, so so that the following properties hold: $\tau_i^i = \mathrm{id}_{A_i}$, and $\tau_i^k = \tau_j^k \circ \tau_i^j$ whenever $i \leqslant j \leqslant k$. This is called a direct (inclusion) system indexed by I.

We also give the following definition of morphisms of direct systems (that comes from functors $\mathcal{G} : I \to I'$ and natural transformations between $\mathcal{F}$ and $\mathcal{F}' \circ \mathcal{G}$): If $\{A'_{i'}\}_{i' \in I'}$, then a morphism of direct systems $\{A_i\}_{i \in I} \to \{A'_{i'}\}_{i' \in I'}$ consists of an order-preserving map $\psi : I \to I'$, and for each $i \in I$, a morphism $\psi_i : A_i \to A'_{\psi(i)}$ such that for all $i \leqslant j$, the following diagram commutes.

$$\begin{array}{ccc} A_i & \xrightarrow{\psi(i)} & A'_{\psi(i)} \\ {\scriptstyle \tau_i^j}\downarrow & & \downarrow{\scriptstyle \tau'^{\psi(j)}_{\psi(i)}} \\ A_i & \xrightarrow{\psi(j)} & A'_{\psi(i)} \end{array}$$

We will now restate the definition of colimit in this setup.

Definition 4.2.1. Let $\{A_i\}_{i \in I}$ be a direct system in a category $\mathcal{C}$. A direct limit of $\{A_i\}_{i \in I}$ consists of

a) an object $A = \varinjlim A$ of $\mathcal{C}$,

b) a family of morphisms $\sigma_i : A_i \to A$ for all $i \in I$ such that the following diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\tau_i^j} & A_j \\ & {\scriptstyle \sigma_i}\searrow \quad \swarrow{\scriptstyle \sigma_j} & \\ & A & \end{array}$$

commutes for all $i \leqslant j$, such that whenever we have an object B, and a family of morphisms $\theta_i : A_i \to B$ such that the following diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\tau_i^j} & A_j \\ & {\scriptstyle \theta_i}\searrow \quad \swarrow{\scriptstyle \theta_j} & \\ & B & \end{array}$$

commutes for all $i \leqslant j$, there exists a unique morphism $\theta : A \to B$ such that

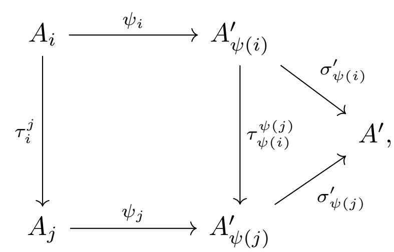
$$\begin{array}{ccc} A_i & \xrightarrow{\tau_i^j} & A_j \\ & {\scriptstyle \sigma_i}\searrow \quad \swarrow{\scriptstyle \sigma_j} & \\ {\scriptstyle \theta_i} & A & {\scriptstyle \theta_j} \\ & \downarrow{\scriptstyle \theta} & \\ & B & \end{array}$$

commutes.

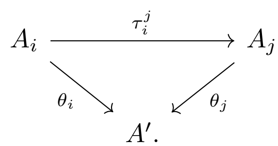
We will address the existence of the direct limit in some important situations below, but first we would like to make the following remark. Let $\{A_i, \tau_i^j\}_{i,j\in I}$ and $\{A'_i, {\tau'}_i^j\}_{i',j'\in I'}$ be two direct systems, and let $\{A_i\}_{i\in I} \to \{A'_i\}_{i'\in I'}$ be a morphism between these direct systems. If the direct limits $A = \varinjlim A_i$ and $A' = \varinjlim A'_i$ exist, then the above morphism of direct systems gives rise to a morphism $A \to A'$. Indeed, for each $i \in I$, we can consider

$$\theta_i : A_i \xrightarrow{\psi_i} A'_{\psi(i)} \xrightarrow{\sigma'_{\psi(i)}} A',$$

we then have the following commutative diagram.



which implies the commutativity of



Then the existence of $\theta : A \to A'$ follows from the universal property.

Before continuing our discussion of limits, we would like to point out how this construction applies to presheaves.

Let $\mathcal{F}$ be a presheaf on the topological space X, and $p \in X$. We let I_p denote thge set of all open subsets $U \subset X$ such that $p \in U$. We order I_p by reverse inclusion, i.e. $U \leqslant V$ iff $V \subset U$. In this case, there is the restriction map $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$. Set $\tau_U^V = \rho_V^U$. Then $\{\mathcal{F}(U), \tau_U^V\}_{U,V\in I_p}$ is a direct system. Its limit $\varinjlim \mathcal{F}(U)$, if exists, is called the stalk of $\mathcal{F}$ at p and denote by $\mathcal{F}_p$. Now, let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X, and fix $p \in X$. Then α obviously induces a morphism of the direct systems $\{\mathcal{F}(U)\}_{U\in I_p} \to \{\mathcal{G}(U)\}_{U\in I_p}$, and hence a morphism of stalks $\alpha_p : \mathcal{F}_p \to \mathcal{G}_p$.

Construction: First, let us discuss the existence of direct limits of direct systems of sets. Let $\{A_i\}_{i\in I}$ be a direct system of sets. Let

$$\mathcal{A} = \bigsqcup_{i\in I} A_i \qquad \text{(disjoint union)}.$$

Define a relation $\sim$ on $\mathcal{A}$ by declaring that $a \in A_i$ is equivalent to $b \in A_j$ if there exists $k \geqslant i, j$ such that

$$\tau_i^k(a) = \tau_j^k(b).$$

It is easy to check that $\sim$ is an equivalence relation. Indeed, we have $\tau_i^i(a) = \tau_i^i(a)$, i.e. $a \sim a$ (reflexive). If $a \sim b$, then there exists $k \geqslant i, j$ such that $\tau_i^k(a) = \tau_j^k(b)$, so $\tau_j^k(b) = \tau_i^k(a)$ and therefore $b \sim a$ (symmetric). Now, suppose $a \sim b$ and $b \sim c \in A_k$. This means that there exist $l \geqslant i, j$, and $m \geqslant j, k$ such that

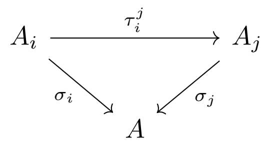
$$\tau_i^l(a) = \tau_j^l(b) \qquad \text{and} \qquad \tau_j^m(b) = \tau_k^m(c).$$

Since I is filtered, we can find $n \in I$ such that $n \geqslant l, m$. Applying τ_l^n to the first equality and τ_m^n to the second, we obtain

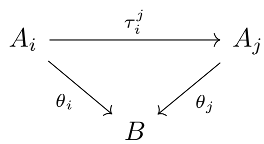
$$\tau_i^n(a) = \tau_j^n(b) = \tau_k^n(c),$$

implying that $a \sim c$.

Let $A = \mathcal{A}/\sim$. Define $\sigma_i : A_i \to \mathcal{A} \to A$. Then for any $a \in A_i$ and $j \geqslant i$, we have $a \sim \tau_i^j(a)$ by definition of our equivalence relation ($\tau_i^j(a) = \tau_j^j(\tau_i^j(a))$). This means that the diagram



commutes. Now, suppose there is a set B and maps $\theta_i : A_i \to B$ such that the diagram



commutes. Then, there is a unique map $\tilde{\theta} : \mathcal{A} \to B$ that restricts to θ_i on each A_i, and we only need to show that this map factors through the equivalence relation. So suppose $a \in A_i$ is equivalent to $b \in A_j$, i.e. there exists $k \geqslant i, j$ such that

$$\tau_i^k(a) = \tau_j^k(b)$$

$$\theta_i(a) = \theta_k(\tau_i^k(a)) = \theta_k(\tau_j^k(b)) = \theta_j(b).$$

It follows that $\tilde{\theta}$ factors through $\theta : A \to B$, verifying the universal property.

Next, let us show that direct limits exist in other categories such as the category of groups, the category of rings, etc. For example, let $\{A_i, \tau_i^j\}$ be a direct system in the category of groups. Let $A = \varinjlim A_i$ be the direct limit of the direct system of underlying sets. Let us endow A with a group structure as follows. Let $[a], [b] \in \mathcal{A}/\sim$ be two classes with $a \in A_i$ and $b \in A_j$. Find $k \in I$ such that $k \geqslant i, j$. Then $[a] = [\tau_i^k(a)], [b] = [\tau_j^k(b)]$. and we declare

$$[a][b] = [\tau_i^k(a)\tau_j^k(b)].$$

One checks that this operator is well-defined, i.e. is independent of the choice k (for this we need to use that τ_l^m are group homomorphisms) and makes A into a group.

Returning to stalks of presheaves, we see from the above construction that given a presheaf of sets $\mathcal{F}$ on a topological space X and a point $p \in X$, the stalk $\mathcal{F}_p$ consists of the equivalence classes of sections $f \in \mathcal{F}(U)$ over some open neighborhood U of p, and two sections $f_1 \in \mathcal{F}(U_1)$ and $f_2 \in \mathcal{F}(U_2)$ are considered equivalent if there exists $V \subset U_1 \cap U_2$ such that $\rho_V^{U_1}(f_1) = \rho_V^{U_2}(f_2)$. (If we think about the elements of $\mathcal{F}(U)$ as "functions" on U then the equivalence class is precisely what is known as the germ of a function.) Next, for every open neighborhood U of p there is a natural map $\mathcal{F}(U) \to \mathcal{F}_p$, and for $U \supset V \ni p$, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(U) & & \\ & \searrow & \\ {\scriptstyle \rho^U_V}\downarrow & & \mathcal{F}_p \\ & \nearrow & \\ \mathcal{F}(V) & & \end{array}$$

If $\mathcal{F}$ is a presheaf of groups, rings, etc., then $\mathcal{F}_p$ is also a group, ring, etc.

Now, let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on X, and $p \in X$. Then ϕ defines a morpsism of direct systems $\{\mathcal{F}(U)\}_{U \ni p} \to \{\mathcal{G}(U)\}_{U \ni p}$, which results in a map of stalks $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$. It follows from the construction of this map that for any $U \ni p$, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}_p & \xrightarrow{\phi_p} & \mathcal{G}_p. \end{array}$$

Let X be a topological space.

Example 4.2.2. *Constant presheaf: Here $\mathcal{F}(U) = C$ (a fixed set) for every open $U \subset X, U \neq \varnothing$. Then clearly, for any $p \in X$, the stalk $\mathcal{F}_p$ is naturally identified with C.*

Example 4.2.3. *Locally constant presheaf: Let $\mathcal{F}$ be the (pre)sheaf of locally constant functions on X with the value set C. We claim that in this case the stalk $\mathcal{F}_p$ at every point $p \in X$ is C. For this, let us first show that $f \in \mathcal{F}(U)$ and $g \in \mathcal{F}(V)$ where U, V are open neighborhoods of p, are equivalent, i.e. give the same element of $\mathcal{F}_p$, if and only if $f(p) = g(p)$. Indeed, if $f \sim g$ then $f|_W = g|_W$ for some open $W \subset U \cap V$; in particular, $f(p) = g(p)$. Conversely, suppose that $f(p) = g(p)$. Since f and g are locally constant, there exist $p \in W \subset U \cap V$ such that $f(W) = \{f(p)\} = \{g(p)\} = g(W)$. It follows that $f|_W = g|_W$, so $f \sim g$. Thus, the maps*

$$\mathcal{F}(U) \to C, \;\; f \mapsto f(p)$$

for open $U \subset X, U \ni p$, induce a well-defined map

$$\mathcal{F}_p = \varinjlim_{U \ni p} \mathcal{F}(U) \to C$$

that yields an identification $\mathcal{F}_p \simeq C$. Note that the stalks here are the same as for the constant presheaf - we will see that this is NOT a coincidence.

Example 4.2.4. *The skyscraper sheaf: Fix $x \in X$; then the corresponding skyscraper sheaf with the value C is defined by*

$$\mathcal{F}(U) = \begin{cases} C & \text{if } x \in U \\ \{*\} & \text{otherwise} \end{cases}$$

For $p = x$, we clearly have $\mathcal{F}_p = C$. Let us identify $\mathcal{F}_p$ at an arbitrary point p. There are two cases to consider.

Case 1. $p \in \overline{\{x\}}$. Then every neighborhood of p contains x, hence $\mathcal{F}(U) = C$. Clearly, in this case $\mathcal{F}_p = C$

Case 2. $p \notin \overline{\{x\}}$. *In this case, there exists a neighborhood $U \ni p$ that does not contain x, and hence $\mathcal{F}(U) = \{*\}$. In this case, any $f \in \mathcal{F}(V)$ for an open neighborhood $V \ni p$, is equivalent to $\{*\}$, so $\mathcal{F}_p = \{*\}$.*

In particular, if x is a closed point then $\mathcal{F}_x = C$ and $\mathcal{F}_p = \{*\}$ for any $p \neq x$.

Example 4.2.5. *Local rings of points: Let $V \subset K^n$ be an irreducible algebraic set, i.e. for a Zariski-open $U \subset V$, we have*

$$\mathcal{O}_V(U) = \{f \in K(V) \mid f \text{ is defined at every point } p \in U\}$$

Fix $p \in V$. Then $\mathcal{O}_{V,p} = \varinjlim \mathcal{O}_V(U)$ is simply the union (taken in $K(V)$) of all $\mathcal{O}_V(U)$ for all open neighborhoods U of p. By definition, $f \in K(V)$ is defined at p if there is a presentation

$$f = \frac{g_p}{h_p}$$

with $g_p, h_p \in K[V]$ and $h_p(p) \neq 0$. Clearly, if f is defined at p then f is defined on the neighborhood $D(h_p) = \{x \in V \mid h_p(x) \neq 0\}$. So, as a result, $\mathcal{O}_{V,p}$ = ring of functions defined at p. On the other hand, it follows immediately from this definition that the ring of functions defined at p is precisely the localization of $K[V]$ with respect to the maximal ideal $\mathfrak{m}_p \subset K[V]$ of functions that vanish at p. Thus, in this case, the stalk $\mathcal{O}_{V,p}$ coincides with what we call the local ring of of the point p; in particular, it is a local ring.

Some basic theorems. Let $\mathcal{F}$ be a presheaf on a topological space X. For any open $U \subset X$ and any $x \in U$, we have a natural map $\rho_x^U : \mathcal{F}(U) \to \varinjlim_{V \ni x} \mathcal{F}(V) = \mathcal{F}_x$ that sends every $f \in \mathcal{F}(U)$ to the corresponding equivalence class (i.e. its germ). As the next statement shows, sections of sheaves are determined by their germs.

Lemma 4.2.6. *Let $\mathcal{F}$ be a sheaf on a topological space X. Then for any open set $U \subset X$, the map*

$$\prod_{x \in U} \rho_x^U : \mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$$

is injective.

Proof. Let $s, t \in \mathcal{F}(U)$ such that $\rho_x^U(s) = \rho_x^U(t)$. Then for each $x \in U$, there exists an open neighborhood $x \in U_x$ such that $\rho_{U_x}^U(s) = \rho_{U_x}^U(t)$. Applying the separation axiom to the covering

$$U = \bigcup_{x \in U} U_x,$$

we obtain that $s = t$. □

Next, let us show that morphisms of sheaves are also determined by what they do on the stalks. Let us recall that given a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, for any $x \in X$ there is a morphism of stalks $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$. In fact, for any open $U \subset X$ and any $x \in U$, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ \rho(\mathcal{F})_x^U \downarrow & & \downarrow \rho(\mathcal{G})_x^U \\ \mathcal{F}_x & \xrightarrow{\phi_x} & \mathcal{G}_x \end{array} \qquad (*)$$

is commutative.

Lemma 4.2.7. *Let X be a topological space, and $\mathcal{F}$ be a presheaf of sets, and $\mathcal{G}$ be a sheaf of sets on X. If*

$$\phi_1, \phi_2 : \mathcal{F} \to \mathcal{G}$$

are two morpshisms of presheaves such that $\phi_{1,x} = \phi_{2,x}$ for all $x \in X$ (as morphism $\mathcal{F}_x \to \mathcal{G}_x$) then $\phi_1 = \phi_2$.

Proof. We need to show that $\phi_{1,U} = \phi_{2,U}$ as morphisms $\mathcal{F}(U) \to \mathcal{G}(U)$ for every open $U \subset X$. Let $s \in \mathcal{F}(U)$. Then it follows from $(*)$ that

$$\rho(\mathcal{G})_x^U(\phi_{1,U}(s)) = \phi_{1,x}(\rho(\mathcal{F})_x^U(s)) \text{ and}$$

$$\rho(\mathcal{G})_x^U(\phi_{2,U}(s)) = \phi_{2,x}(\rho(\mathcal{F})_x^U(s))$$

so it follows from the assumption that $\phi_{1,x} = \phi_{2,x}$ that

$$\rho(\mathcal{G})_x^U(\phi_{1,U}(s)) = \rho(\mathcal{G})_x^U(\phi_{2,U}(s))$$

Since $\mathcal{G}$ is a sheaf, we conclude that $\phi_{1,U}(s) = \phi_{2,U}(s)$, as required. □

Proposition 4.2.8. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on a topological space X. The following statements hold:*

(1) The map $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$ if and only if $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open U.

(2)Assume that both $\mathcal{F}$ and $\mathcal{G}$ are sheaves. Then ϕ_x are bijections for all $x \in X$ if and only if ϕ_U are bijections for all open $U \subset X$.

Proof. (1) Suppose the ϕ_x are injective for all $x \in X$. Let $s, t \in \mathcal{F}(U)$ and assume that $\phi_U(s) = \phi_U(t)$. Then for any $x \in U$, we have

$$\rho(\mathcal{G})_x^U(\phi_U(s)) = \rho(\mathcal{G})_x^U(\phi_U(t)),$$

hence

$$\phi_x(\rho_x^U(\mathcal{F})(s)) = \phi_x(\rho_x^U(\mathcal{F})(t)).$$

Since ϕ_x is injective, we conclude that $\rho(\mathcal{F})_x^U(s) = \rho(\mathcal{F})_x^U(t)$ for all $x \in U$, and therefore $s = t$ since $\mathcal{F}$ is a sheaf.

Conversely, suppose that all the ϕ_U are injective. Let $s_x, t_x \in \mathcal{F}_x$ and $\phi_x(s_x) = \phi_x(t_x)$. By definition of the stalk, we can find open neighborhood $U \ni x$ and $s_U, t_U \in \mathcal{F}(U)$ such that

$$s_x = \rho_x^U(s_U) \text{ and } t_x = \rho_x^U(t_U).$$

The fact, that $\phi_x(s_x) = \phi_x(t_x)$ means that $\phi_U(s_U) = \phi_U(t_U)$ have the same restriction to some smaller neighborhood V of x. Then

$$\phi_V(\rho_V^U(s_U)) = \rho_V^U(\phi_U(s_U)) = \rho_V^U(\phi_U(t_U)) = \phi_V(\rho_V^U(t_U)).$$

Since ϕ_V is injective, we obtain $\rho_V^U(s_U) = \rho_V^U(t_U)$. But s_U and $\rho_V^U(s_U)$, and t_U and $\rho_V^U(t_U)$ define the same elements in the stalk, so $s_x = t_x$.

(2) If all ϕ_U are bijective, then all ϕ_x are clearly bijective. For the reverse implication, the injectivity of ϕ_U follows from (1), so we only need to establish the surjectivity. Let $t \in \mathcal{G}$. For $x \in U$, since ϕ_x is surjective, we can find $s_x \in \mathcal{F}_x$ such that

$$\phi_x(s_x) = \rho(\mathcal{G})_x^U(t).$$

In turn, $s_x = \rho(\mathcal{F})_x^{U_x}(s_{U_x})$ for some open neighborhood $U_x \ni x$ and some $s_{U_x} \in \mathcal{F}(U_x)$. Then

$$\rho(\mathcal{G})_x^{U_x}(\phi_{U_x}(s_{U_x})) = \rho(\mathcal{G})_x^U(t).$$

This means that there exists an open neighborhood $U_x \supset V_x \ni x$ such that for $s_{V_x} = \rho(\mathcal{F})_{V_x}^{U_x}(s_{U_x})$, we have that

$$\phi_{V_x}(s_{V_x}) = \rho(\mathcal{G})_{V_x}^U(t). \qquad (*)$$

This proves the possibility of "local lifting". To complete the proof, we will now show that the elements $s_{V_x} \in \mathcal{F}_x$ on the open cover $U = \bigcup_{x \in U} V_x$ can be glued together to obtain $s \in U$ such that $\phi_U(s) = t$. For this, we need to check the compatibility condition in the gluing axiom, i.e. for any $x_1, x_2 \in U$ we have

$$\rho(\mathcal{F})_{V_{x_1} \cap V_{x_2}}^{V_{x_1}}(s_{V_{x_1}}) = \rho(\mathcal{F})_{V_{x_1} \cap V_{x_2}}^{V_{x_2}}(s_{V_{x_2}}).$$

Since $\phi_{V_{x_1} \cap V_{x_2}}$ is injective, it is enough to check that

$$\phi_{V_{x_1} \cap V_{x_2}}(\rho(\mathcal{F})_{V_{x_1} \cap V_{x_2}}^{V_{x_1}}(s_{V_{x_1}})) = \phi_{V_{x_1} \cap V_{x_2}}(\rho(\mathcal{F})_{V_{x_1} \cap V_{x_2}}^{V_{x_2}}(s_{x_2})),$$

which is equivalent to

$$\rho(\mathcal{G})_{V_{x_1} \cap V_{x_2}}^{V_{x_1}}(\phi_{V_{x_1}}(s_{V_{x_1}})) = \rho(\mathcal{G})_{V_{x_1} \cap V_{x_2}}^{V_{x_2}}(\phi_{V_{x_2}}(s_{V_{x_2}}))$$

But the latter holds in view of $(*)$ - both sides are equal to $\rho_{V_{x_1} \cap V_{x_2}}^U(t)$. By the gluing axiom there exists $s \in \mathcal{F}(U)$ such that $\rho_{V_x}^U(s) = s_{V_x}$ for all $x \in X$. Then $\rho(\mathcal{G})_{V_x}^U(\phi_U(s)) = \rho(\mathcal{G})_{V_x}^U(t)$ for all $x \in U$ and therefore $\phi_U(s) = t$ by the separation axiom. □

We will next consider the stalks as they relate to exact sequences. This discussion will lead us to the definition of an exact sequence of sheaves of abelian groups. We begin by recalling the well-known exactness of property of direct limits.

Let I be a filtered set, and suppose we have direct systems of abelian groups

$$A = \{A_i, \tau(A)_i^j\}, B = \{B_i, \tau(B)_i^j\} \text{ and } C = \{C_i, \tau(C)_i^j\}.$$

Furthermore, let $\phi : A \to B$ and $\psi : B \to C$ be morphisms of direct systems corresponding to the identity map of I such that the sequence

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C$$

is exact, which means that for each i we have an exact sequence

$$A_i \xrightarrow{\phi_i} B_i \xrightarrow{\psi_i} C_i,$$

and for $i \leqslant j$ the diagram

$$\begin{array}{ccccc} A_i & \xrightarrow{\phi_i} & B_i & \xrightarrow{\psi_i} & C_i \\ {\scriptstyle \tau(A)_i^j}\downarrow & & {\scriptstyle \tau(B)_i^j}\downarrow & & {\scriptstyle \tau(C)_i^j}\downarrow \\ A_j & \xrightarrow{\phi_j} & B_j & \xrightarrow{\psi_j} & C_j \end{array}$$

commutes. As we have seen before the morphisms ϕ and ψ induce morphisms between direct limits

$$\varinjlim A_i \xrightarrow{\Phi} \varinjlim B_i \text{ and } \varinjlim B_i \xrightarrow{\Psi} \varinjlim C_i.$$

Theorem 2. The sequence

$$\varinjlim A_i \xrightarrow{\Phi} \varinjlim B_i \xrightarrow{\Psi} \varinjlim C_i$$

is exact.

Proof. By our assumption, for each i we have $\psi_i \circ \phi_i = 0$, which implies that $\Psi \circ \Phi = 0$, hence $\mathrm{im}\Phi \subset \ker \Psi$. So, we only need to establish the reverse inclusion. Let $b \in \ker \Psi$. Then b is represented by some $b_i \in B_i$ such that $\psi_i(b_i)$ defined the zero element of $\varinjlim C_i$. This means that there exists $j \geqslant i$ such that

$$\tau(C)_i^j(\psi_i(b_i)) = 0.$$

Then

$$\psi_j(\tau(B)_i^j(b_i)) = 0,$$

and consequently, there exists $a_j \in A_j$ such that

$$\tau(B)_i^j(b_i) = \phi_j(a_j).$$

Let a denote the image of a_j in $\varinjlim A_i$. Since $\tau(B)_i^j(b_i)$ defines the same element of $\varinjlim B_i$ as b_i, i.e. b, we obtain that $\Phi(a) = b$ as required. □

Corollary 4.2.9. *If the sequences*

$$0 \to A_i \to B_i \to C_i \to 0$$

are exact for all i, then the sequence

$$0 \to \varinjlim A_i \to \varinjlim B_i \to \varinjlim C_i \to 0$$

is also exact.

Now, let us connect this with presheaves.

Definition 4.2.10. Let $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ be presheaves on a topological space X, and let $\phi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be morphisms of presheaves. We say that the sequence

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

is exact in the category of presheaves $\mathcal{P}sh(X)$ of abelian groups on X if the sequence

$$\mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U)$$

is exact for all open $U \subset X$.

Corollary 4.2.11. *Let* $\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ *be an exact sequence of presheaves of abelian groups. Then for each* $x \in X$, *the sequence* $\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$ *is exact. Consequently, if the sequence*

$$0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0$$

is exact in $\mathcal{P}sh(X)$, *then the sequence*

$$0 \to \mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x \to 0$$

is exact for all $x \in X$.

We use this statement to define the exactness in the category of sheaves.

Definition 4.2.12. We say that a sequence

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

of sheaves of abelian groups is exact if the sequence of stalks

$$\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

is exact for all $x \in X$.

One may wonder about how this notion relates to the exactness in the category of presheaves. It follows from the above corollary that if the sequence is exact in the category of presheaves then it is also exact in the category of sheaves. The converse is only true partially.

Theorem 3. Let $0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$ be an exact sequence of sheaves of abelian groups on a topological space X. Then for any open set $U \subset X$, the sequence

$$0 \to \mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U)$$

is exact.

Proof. We need to check the exactness at $\mathcal{F}(U)$ and $\mathcal{G}(U)$.

- Exactness at $\mathcal{F}(U)$: By our assumption, the maps $\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x$ are injective for all $x \in X$. Since $\mathcal{F}$ is a sheaf, ϕ_U is injective for every open $U \subset X$.
- Exactness at $\mathcal{G}(U)$: The kernel presheaf

$$\mathcal{K}(U) = \ker(\mathcal{G}(U) \xrightarrow{\psi_U} \mathcal{H}(U))$$

is a sheaf. By construction, we have an exact sequence in the category of presheaves

$$0 \to \mathcal{K} \to \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

hence an exact sequence of stalks

$$0 \to \mathcal{K}_x \to \mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x$$

for all $x \in X$. In particular, $\mathcal{K}_x = \ker(\mathcal{G}_x \xrightarrow{\psi_x} \mathcal{H}_x)$. Now, consider the composition $\psi \circ \phi : \mathcal{F} \to \mathcal{H}$. We have

$$(\psi \circ \phi)_x = \psi_x \circ \phi_x = 0$$

for all $x \in X$. Since $\mathcal{H}$ is a sheaf, we obtain that $\psi \circ \phi = 0$ as a morphism of sheaves. Hence $\phi_U(\mathcal{F}(U)) \subset \mathcal{K}(U)$, i.e. ϕ defines a morphism of sheaves

$$\overline{\phi} : \mathcal{F} \to \mathcal{K}.$$

According to the previous remarks, $\overline{\phi_x} : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism for any $x \in X$. Then $\overline{\phi_U} : \mathcal{F}(U) \to \mathcal{K}(U)$ is an isomorphism. Thus, $\text{im}(\phi_U) = \ker(\psi_U)$, proving the exactness.

□

An important point is that a short exact sequence

$$0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0$$

of sheaves of abelian groups may not give a short exact sequence of sections over an open set $U \subset X$, even when $U = X$. Our goal is to construct a cohomology theory that enables one to deal with the failure of exactness of the functor of global sections. More precisely, one defines abelian groups $H^i(X, \mathcal{F})$, $H^i(X, \mathcal{G})$, $H^i(X, \mathcal{H})$ for $i \geqslant 1$ that fit into a long exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to H^1(X, \mathcal{H}) \to H^2(X, \mathcal{F}) \to \cdots.$$

Example 4.2.13. *Let $X \subset \mathbb{C}$ be an open subset. For an open $U \subset X$, we let $\mathcal{O}(U)$ denote the $\mathbb{C}$-algebra of holomorphic functions on U. One easily checks that this defines a sheaf $\mathcal{O}$ on X. Let $\psi_U : \mathcal{O}(U) \to \mathcal{O}(U)$ be the operator of differentiation, i.e. $\psi_U(f) = f'$. Clearly, $\ker(\psi_U)$ is precisely the algebra $\mathcal{C}(U)$ of locally constant funtions. We then have the following exact sequence of sheaves on X:*

$$0 \to \mathcal{C} \to \mathcal{O} \xrightarrow{\psi} \mathcal{O} \to 0.$$

We only need to check the exactness at the second $\mathcal{O}$, viz. to show that ψ is surjective as a morphism of sheaves. This amounts to showing that the map on stalks $\mathcal{O}_x \xrightarrow{\psi_x} \mathcal{O}_x$ is onto for every $x \in X$. Every element of $\mathcal{O}_x$ is represented by some holomorphic function $f \in \mathcal{O}(U)$ for some open neighborhood U of x. But we can always find a smaller neighborhood $x \in U_x \subset U$ which is simply connected. In that neighborhood, $f|_{U_x}$ has an antiderivative (Morera's theorem) $g \in \mathcal{O}(U_x)$. This means that $\psi_{U_x}(g) = f|_{U_x}$, proving that the image of f in $\mathcal{O}_x$ lies in the image of ψ_x. Thus, ψ is surjective as a morphism of sheaves. On the other hand, if X is not simply connected, not every analytic function has an antiderivative, i.e. $\psi_X(\mathcal{O}(X)) \neq \mathcal{O}(X)$. Thus, the surjectivity of a morphism of sheaves on stalks does not imply its surjectivity on sections. So, this definition requires some justification. For this, we recall that a morphism $f : X \to Y$ in a category $\mathcal{C}$ is called an epimorphism if for any two morphisms

$$Y \overset{g_1}{\underset{g_2}{\rightrightarrows}} Z,$$

the fact that $g_1 \circ f = g_2 \circ f$ implies that $g_1 = g_2$, i.e. we have right cancellation for morphisms.

Example 4.2.14. . *(a) A homomorphism of abelian groups $f : X \to Y$ is an epimorphism in the category of abelian groups AbGrps if and only if it is surjective. Indeed, the surjectivity of f clearly implies that f is an epimorphism. Conversely, suppose that f is an epimorphism but $f(X) \neq Y$. Let $Z = Y \backslash f(X) = cokerf$, and take g_1 to be the canonical homomorphism, and g_2 to be the zero homomorphism. Then $g_1 \circ f = g_2 \circ f$, but $g_1 \neq g_2$, contradicting the assumption that f is an epimorphism.*

(b) Let X be a topological space. Then again a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves of abelian groups is an epimorphism in the category $\mathcal{P}sh(X, AbGrps)$ if and only if $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all U. We only need to prove that if ϕ is an epimorphism then ϕ_U is surjective. Consider the cokernel presheaf defined by

$$\mathcal{C} = \mathcal{G}(U)/\phi_U(\mathcal{F}(U)).$$

If ϕ is not surjective, then $\mathcal{C}$ is not the zero presheaf. So if, we take $g_1 : \mathcal{G} \to \mathcal{C}$ to be the canonical morphism and $g_2 : \mathcal{G} \to \mathcal{C}$ to be the zero morphism, then $g_1 \circ \phi = g_2 \circ \phi(=0)$, but $g_1 \neq g_2$ - a contradiction. Thus, ϕ must be surjective.

The argument in (b) breaks down in the category of sheaves of abelian groups, because for a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, the cokernel presheaf $\mathcal{C}$ may not be a sheaf. Instead, we need to consider the associated sheaf $\mathcal{C}^+$ together with the canonical homomorphism $\theta : \mathcal{C} \to \mathcal{C}^+$. This leads to the following statement.

Proposition 4.2.15. *Let X be a topological space, and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves of abelian groups. Then the following conditions are equivalent:*

(i) ϕ is an epimorphism in the category of sheaves of abelian groups on X;

(ii) $\mathcal{C}^+ = 0$ where $\mathcal{C}$ is the cokernel presheaf associated to ϕ;

(iii) $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjective for all $x \in X$.

Proof. First, let us prove the equivalence $(ii) \Leftrightarrow (iii)$. It follows from injectivity of the map $\prod_{x \in U} \rho_x^U : \mathcal{F}(U) \to \prod_{x \in U} \mathcal{F}_x$ that a sheaf of abelian groups is zero if and only if all the stalks are zero. On the other hand, we have the following exact sequence of presheaves:

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \to \mathcal{C} \to 0.$$

So for each $x \in X$, the sequence of stalks

$$\mathcal{F}_x \xrightarrow{\phi_x} \mathcal{G}_x \to \mathcal{C}_x \to 0.$$

is exact, i.e. $\mathcal{C}_x \simeq \mathcal{G}_x/\phi_x(\mathcal{F}_x)$. Also, as we mentioned earlier, $\mathcal{C}_x^+ = \mathcal{C}_x$. Thus,

$$\mathcal{C}^+ = 0 \Leftrightarrow \mathcal{C}_x^+ = 0 \text{ for all } x \in X \Leftrightarrow \phi_x(\mathcal{F}_x) = \mathcal{G}_x,$$

as required.

Next, let us show that $(iii) \Rightarrow (i)$. Suppose we have two morphisms

$$\mathcal{G} \underset{g_2}{\overset{g_1}{\rightrightarrows}} \mathcal{H},$$

to a sheaf $\mathcal{H}$ such that $g_1 \circ \phi = g_2 \circ \phi$. Then for any $x \in X$, we have

$$g_{1,x} \circ \phi_x = g_{2,x} \circ \phi_x.$$

Since ϕ_x is surjective, we conclude that $g_{1,x} = g_{2,x}$, hence $g_1 = g_2$ since $\mathcal{H}$ is a sheaf.

Finally, we show that $(i) \Rightarrow (iii)$. Suppose that $\phi_{x_0}(\mathcal{F}_{x_0}) \neq \mathcal{G}_{x_0}$ for some $x_0 \in X$. and consider the sequence

$$\mathcal{F} \xrightarrow{\phi} \mathcal{G} \underset{g_2}{\overset{g_1}{\rightrightarrows}} \mathcal{C} \xrightarrow{\theta} \mathcal{C}^+,$$

where g_1 is the canonical homomorphism, g_2 is the zero homomorphism and $\mathcal{C}$ is the cokernel presheaf. We obviously have $g_1 \circ \phi = g_2 \circ \phi$, hence

$$\theta \circ g_1 \circ \phi = \theta \circ g_2 \circ \phi.$$

So, invoking (i), we obtain $\theta \circ g_1 = \theta \circ g_2$. Then $\theta_{x_0} \circ g_{1,x_0} = \theta_{x_0} \circ g_{2,x_0}$. Since $\theta_{x_0} : \mathcal{C}_{x_0} \to \mathcal{C}^+_{x_0}$ is an isomorphism, we obtain $g_{1,x_0} = g_{2,x_0}$. But g_{1,x_0} is the canonical map $\mathcal{G}_{x_0} \to \mathcal{G}_{x_0}/\phi_{x_0}(\mathcal{F}_{x_0})$, and g_{2,x_0} is the zero map, a contradiction. Thus, ϕ_x must be surjective for all $x \in X$. □

We have already used the fact that $\theta_x : \mathcal{C}_x \to \mathcal{C}^+_x$ is an isomorphism several times. There is one more application.

Example 4.2.16. *Let X be a topological space, $\mathcal{C}$ be the constant presheaf on X with the value set E(i.e. $\mathcal{C}(U) = E$ for any nonempty open $U \subset X$), and $\mathcal{L}$ be the locally constant sheaf. There is the identity embedding $\iota : \mathcal{C} \to \mathcal{L}$. We have seen that $\mathcal{C}$ and $\mathcal{L}$ have the same stalks; in fact, for any $x \in X$, $\iota_x : \mathcal{C}_x \to \mathcal{L}_x$ is an isomorphism. For the sheaf $\mathcal{C}^+$ associated with $\mathcal{C}$ and the corresponding map $\theta : \mathcal{C} \to \mathcal{C}^+$, we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\iota} & \mathcal{L} \\ & {\scriptstyle\theta}\searrow \quad \nearrow{\scriptstyle\phi} & \\ & \mathcal{C}^+, & \end{array}$$

for some morphism of sheaves $\phi : \mathcal{C}^+ \to \mathcal{L}$. Since ι and θ induce isomorphism on stalks, so does ϕ. It follows that ϕ is an isomorphism of sheaves, i.e. $\mathcal{L}$ is actually the sheaf associated with the constant presheaf $\mathcal{C}$.

This argument shows that if $\iota : \mathcal{F} \to \mathcal{G}$ is a morphism of a presheaf $\mathcal{F}$ to a sheaf $\mathcal{G}$ that induces an isomorphism of stalks $\iota_x : \mathcal{F}_x \to \mathcal{G}_x$ for any $x \in X$ then $\mathcal{G}$ is the sheaf associated with the presheaf $\mathcal{F}$. What we use is the fact that if $\phi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves that induces isomorphisms on all stalks, then ϕ is an isomorphism of sheaves. It should be noted that the fact that two sheaves have isomorphic stalks does not imply in the general case that the sheaves are isomorphic. For one thing, the stalks do not reflect the restriction maps. So, one can take, for example, X to be a 2-element set $X = \{0, 1\}$ with the open subsets $X, \{1\}, \varnothing$. We will now construct two sheaves $\mathcal{F}, \mathcal{G}$ on X by assigning $\mathbb{Z}$ to X and $\{1\}$, and 0 to $\varnothing$ for both of them but taking the isomorphism $\mathbb{Z} \to \mathbb{Z}$ for $\mathcal{F}(X) \to \mathcal{F}(\{1\})$ and the zero homomorphism $\mathbb{Z} \to \mathbb{Z}$ for $\mathcal{G}(X) \to \mathcal{G}(\{1\})$. Then the stalks $\mathcal{F}_0, \mathcal{F}_1$ and $\mathcal{G}_0, \mathcal{G}_1$ are all isomorphic to $\mathbb{Z}$ but the sheaves $\mathcal{F}$ and $\mathcal{G}$ are not isomorphic.

Another, more conceptual, example can be obtained by considering nonisomorphic line bundles over a topological space or a manifold. More concretely, let $X = S^1$ be the unit circle, and let $E_1 = X \times \mathbb{R}$ be the trivial line bundle over X, and let E_2 be the Möbius strip. In either case, we let $\mathcal{F}_i$ be the sheaf of smooth sections of $E_i \to X$. It is easy to see, using the local triviality of E_i, that the stalk $\mathcal{F}_{i,x}$ at each $x \in X$ is isomorphic to the stalk of the sheaf of smooth functions on X, so all stalks are isomorphic. On the other hand, the sheaves are NOT isomorphic. Indeed, an isomorphism of sheaves would induce an isomorphism of stalks. Each stalk is a local ring with the maximal ideal consisting of functions that vanish at that point. This means that if $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ is an isomorphism, then for $\phi_X : \mathcal{F}_1(X) \to \mathcal{F}_2(X)$, we have the following: $f \in \mathcal{F}_1(X)$ vanishes at $x \in X$ if and only if $\phi_X(f) \in \mathcal{F}_2(X)$ vanishes at x. But $\mathcal{F}_1$ has a nowhere vanishing global section, while $\mathcal{F}_2$ doesn't (as otherwise the line bundle would be trivial). We will now introduce an important class of sheaves which later will be shown to be cyclic.

Definition 4.2.17. A sheaf $\mathcal{F}$ is called flasque(or flabby) if the restriction maps $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ are surjective for all open sets $V \subset U$(equivalently, the restriction maps $\rho^X_V : \mathcal{F}(X) \to \mathcal{F}(U)$ are all surjective).

It is easy to see that a skyscraper sheaf is flasque but we will see many other important examples.

Theorem 4. Let

$$0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0$$

be an exact sequence of abelian groups on X.

(i) If $\mathcal{F}$ is flasque then the sequence of global sections

$$0 \to \mathcal{F}(X) \xrightarrow{\phi_X} \mathcal{G}(X) \xrightarrow{\psi_X} \mathcal{H}(X) \to 0$$

is exact(and, in fact, we have a similar exact sequence for sections over any open $U \subset X$).

(ii) If in addition $\mathcal{G}$ is flasque then $\mathcal{H}$ is also flasque.

Proof. We only need to show that $\phi_X : \mathcal{G}(X) \to \mathcal{H}(X)$ is surjective. Let $t \in \mathcal{H}(X)$. As we already remarked above, the surjectivity of $\psi_x : \mathcal{G}_x \to \mathcal{H}_x$ means that there exists a neighborhood $U \ni x$ and a section $s_U \in \mathcal{G}(U)$ such that

$$\psi_U(s_U) = \rho^X_U(\mathcal{H})(t)$$

("local lifting"). Consider all pairs (U, s_U) satisfying this condition. This set can be partially ordered as follows

$$(U_1, s_1) \leqslant (U_2, s_2)$$

if $U_1 \subset U_2$ and $\rho^{U_2}_{U_1}(s_2) = s_1$. Suppose we have a chain

$$\{(U_\alpha, s_\alpha)\}_{\alpha \in I},$$

indexed by a set I. Set $U = \bigcup_{\alpha \in I} U_\alpha$. Then for any $\alpha, \beta \in I$ we have one of the following

$$(U_\alpha, s_\alpha) \leqslant (U_\beta, s_\beta) \text{ or } (U_\beta, s_\beta) \leqslant (U_\alpha, s_\alpha).$$

In the first case, we have

$$\rho^{U_\alpha}_{U_\alpha \cap U_\beta}(s_\alpha) = s_\alpha = \rho^{U_\beta}_{U_\alpha \cap U_\beta}(s_\beta),$$

by the definition of the order relation. Similarly, we have

$$\rho^{U_\alpha}_{U_\alpha\cap U_\beta}(s_\alpha) = \rho^{U_\beta}_{U_\alpha\cap U_\beta}(s_\beta),$$

in the second case as well. By the sheaf axioms, there exists a unique $s \in \mathcal{F}(U)$ such that $\rho^U_{U_\alpha}(s) = s_\alpha$, and then (U, s) is an upper bound for our set. By Zorn's lemma, there exists a maximal element (U, s_U). If $U = X$, we are done. Suppose there exists $x \in X\backslash U$. Then as above, there exists an open neighborhood $V \ni x$ and a section $s_V \in \mathcal{G}(V)$ such that

$$\psi_V(s_V) = \rho(\mathcal{H})^X_V(t).$$

Then, on the intersection $U \cap V$, we have

$$\psi_{U\cap V}(\rho^U_{U\cap V}(\mathcal{G})(s_U) - \rho(\mathcal{G})^V_{U\cap V}(s_V)) = \rho^X_{U\cap V}(t) - \rho^X_{U\cap V}(t) = 0,$$

which implies that

$$\rho(\mathcal{G})^U_{U\cap V}(s_U) - \rho(\mathcal{G})^V_{U\cap V}(s_V) \in \phi_U(\mathcal{F}(U\cap V)).$$

Since $\mathcal{F}$ is flasque, the restriction $\mathcal{F}(V) \to \mathcal{F}(U \cap V)$ is surjective, so there exists $r \in \mathcal{F}(V)$ such that

$$\rho(\mathcal{G})^U_{U\cap V}(s_U) - \rho(\mathcal{G})^V_{U\cap V}(s_V) = \phi_{U\cap V}(\rho^V_{U\cap V}(r)).$$

Set $s'_V = s_V + \phi_V(r)$. Then

$$\psi_V(s'_V) = \rho^X_{U\cap V}(t),$$

and

$$\rho(\mathcal{G})^U_{U\cap V}(s_U) = \rho(\mathcal{G})^V_{U\cap V}(s'_V).$$

Thus, there exists $s_{U\cup V} \in \mathcal{G}(U \cup V)$ that restricts to s_U on U and s'_V on V. Then $\psi_{U\cup V}(s_{U\cup V})$ restricts to $\rho(\mathcal{H})^X_U(t)$ on U and $\rho(\mathcal{H})^X_V(t)$ on V, so

$$\psi_{U\cup V}(s_{U\cup V}) = \rho^X_{U\cup V}(t).$$

Then $(U \cup V, s_{U\cup V})$ is strictly greater than (U, s_U) with respect to our ordering, which contradicts the maximality of U. Thus, $U = X$, which proves (i). In this argument, replacing X with an open subset $U \subset X$, we obtain the exactness of the sequence of sections over U.

(ii) It follows from (i) that we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F}(X) & \xrightarrow{\phi_X} & \mathcal{G}(X) & \xrightarrow{\psi_X} & \mathcal{H}(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho(\mathcal{F})^X_U} & & \downarrow{\scriptstyle \rho(\mathcal{G})^X_U} & & \downarrow{\scriptstyle \rho(\mathcal{H})^X_U} & & \\ 0 & \longrightarrow & \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) & \xrightarrow{\psi_U} & \mathcal{H}(U) & \longrightarrow & 0. \end{array}$$

We need to show that $\rho(\mathcal{H})^X_U$ is surjective. Let $h \in \mathcal{H}(U)$. There exists $g \in \mathcal{G}(U)$ such that $\psi_U(g) = h$. Since $\mathcal{G}$ is flasque, $\rho(\mathcal{G})^X_U$ is surjective, so there exists $g' \in \mathcal{G}(X)$ such that $\rho(\mathcal{G})^X_U(g') = g$. Let $h' = \psi_X(g')$. Then

$$\rho(\mathcal{H})^X_U(h') = \rho(\mathcal{H})^X_U(\psi_X(g')) = \psi_U(\rho(\mathcal{G})^X_U(g')) = \psi_U(g) = h,$$

as required. Thus, $\mathfrak{X}$ is flasque. □

4.3 Sheafification via étale space

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Let $\mathcal{F}$ be a presheaf on a topological space X. We will construct a topological space E together with a local homeomorphism $\pi : E \to X$ so that the sheaf of continuous sections of π defined by the sets $\Gamma(U, \pi) = \{s : U \to E \mid s \text{ continuous and } \pi \circ s = \mathrm{id}_U\}$, is precisely the sheaf $\mathcal{F}^+$ associated with the presheaf $\mathcal{F}$. It will follow from this construction that the morphism of sheafification $\theta : \mathcal{F} \to \mathcal{F}^+$ induces an isomorphism of stalks $\theta_x : \mathcal{F}_x \to \mathcal{F}^+_x$, for any $x \in X$.

Definition 4.3.1. A map $\pi : E \to X$ between two topological spaces is called a local homeomorphism if for every $e \in E$ there exist open neighborhoods $O_e \ni e$, and $U_x \ni x = \pi(e)$ such that $\pi_{O_e} : O_e \to U_x$ is a homeomorphism.

Note that π is automatically continuous since for any open set $U \subset X$ and any $e \in \pi^{-1}(U)$, one can choose open neighborhoods $O_e \ni e$, and $U_x \ni x = \pi(e)$ as in the definition so that $U_x \subset U$, and then

$$\pi^{-1}(U) = \bigcup_{e \in \pi^{-1}(U)} O_e$$

is open.

Definition 4.3.2. A pair (E, π) consisting of a topological space E and a local homeomorphism $\pi : E \to X$ is called an étale space over X. We will call X the base space, E the total space, and π the projection map. For $x \in X$, the set $E_x = \pi^{-1}(x)$ is the fiber of π over x.

We will now state (without proof) some elementary properties of local homeomorphisms.

Proposition 4.3.3. *Let $\pi : E \to X$ be a local homeomorphism. Then*

(i) π is an open map;

(ii) We have $E = \bigcup_{x \in X} E_x$, and the induced topology on each fiber is discrete;

(iii) If $s_1 : U_1 \to E$ and $s_2 : U_2 \to E$ are two sections over open subsets $U_1, U_2 \subset X$ such that $s_1(x) = s_2(x)$ for some $x \in U_1 \cap U_2$ then s_1 and s_2 coincide on some open neighborhood of x.

(iv) For any section $s : U \to E$, the image $s(U)$ is open in E and is homeomorphic to U. Moreover, $s = (\pi|_{s(U)})^{-1}$.

(v) Sets of the form $s(U)$, for $U \subset X$ open and $s \in \Gamma(U, \pi)$ form a basis of the topology on E. Consequently, the topology on E is completely determined by the topology of X and the continuous local sections of π.

Proposition 4.3.4. *Let $\pi : E \to X$ be a local homeomorphism. Then $\mathcal{F}(U) = \Gamma(U, \pi)$, with restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ for open $V \subset U$ given simply by restrictions of sections, defines a sheaf of sets on X. Furthermore, for any $x \in X$, the stalk $\mathcal{F}_x$ can be naturally identified with the fiber $E_x = \pi^{-1}(x)$.*

The verification of the fact that $\mathcal{F}$ is a sheaf is standard. So, let us just prove the assertion about the stalk. Let $x \in X$. Then for every open $U \ni x$, there is a map

$$\nu_U : \Gamma(U, \pi) \to E_x, \quad s \mapsto s(x) \in E_x$$

If two sections $s_1 : U_1 \to E$ and $s_2 : U_2 \to E$ are equivalent in the stalk $\mathcal{F}_x$, then they coincide on some open neighborhood $U_0 \ni x$ contained in $U_1 \cap U_2$; then clearly $s_1(x) = s_2(x)$. This means that we have a well-defined natural map

$$\nu_x : \mathcal{F}_x = \varinjlim_{U \ni x} \Gamma(U, \pi) \to E_x.$$

This map is surjective since, given $e \in E_x$, we can find open neighborhoods $O_e \ni e$ and $U_x \ni x$ such that

$$\pi|_{O_e} : O_e \to U_x$$

is a homeomorphism. Then $s = (\pi|_{O_e})^{-1} : U_x \to E$ is a section such that $s(x) = e$, i.e. $\nu_x(s) = e$. Finally, let us show that ν_x is injective. Suppose $s_1 \in \Gamma(U_1, \pi)$ and $s_2 \in \Gamma(U_2, \pi)$ are such that $\nu_x(s_1) = \nu_x(s_2)$. Then by part (iii) of the previous proposition, there exists a neighborhood $x \in U_0 \subset U_1 \cap U_2$ such that $s_1|_{U_0} = s_2|_{U_0}$. This means that s_1 and s_2 represent the same element of $\mathcal{F}_x$, and ν_x is injective.

We will now describe a construction of an étale space associated to a presheaf. Let $\mathcal{F}$ be a preasheaf on a topological space X. Set

$$E = \bigsqcup_{x \in X} \mathcal{F}_x \quad \text{(disjoint union of stalks)},$$

and define $\pi : E \to X$ by sending $a \in \mathcal{F}_x$ to x. Our goal is to equip E with the natural topology so that π becomes a local homeomorphism.

As we mentioned earlier, given any local homeomorphism $\pi' : E' \to X'$, the sets of the form $s(U)$ where $U \subset X'$ is open and $s : U \to E'$ is a continuous section form a basis for the topology on E'. In our situation, we topologize E by essentially reversing this process. Given an open subset U and any $s \in \mathcal{F}(U)$, we define a section $\tilde{s} : U \to E$ by $\tilde{s}(x) = \rho_x^U(s)$.

Proposition 4.3.5. *(i) There is a topology on E for which the sets $\tilde{s}(U)$ for all open $U \subset X$ and all $s \in \mathcal{F}(U)$ form a basis.*

(ii) If E is equipped with this topology then $\pi : E \to X$ is a local homeomorphism and each $\tilde{s} : U \to E$ is a continuous section.

Proof. -Omitted. ☐

We are now in a position to prove the following theorem.

Theorem 5 (**Sheafification**)**.** Let $\mathcal{F}$ be a presheaf of sets on a topological space X. Then there exists a sheaf $\mathcal{F}^+$ and a morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ such that

(i) $\theta_x : \mathcal{F}_x \to \mathcal{F}_x^+$ is a bijection

(ii) for any morphism $\phi : \mathcal{F} \to \mathcal{G}$ to a sheaf $\mathcal{G}$, there exists a unique morphism of sheaves $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\theta} & \mathcal{F}^+ \\ & \searrow^{\phi} \quad \swarrow_{\psi} & \\ & \mathcal{G}. & \end{array}$$

Proof. Let $\pi : E \to X$ be the étale space of $\mathcal{F}$, and let $\mathcal{F}^+$ be the sheaf of continuous sections of π, i.e.

$$\mathcal{F}^+(U) = \Gamma(U, \pi)$$

For each open $U \subset X$, we have a map

$$\theta_U : \mathcal{F}(U) \to \mathcal{F}^+(U), \;\; s \mapsto \tilde{s}$$

To show that these maps define a morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$, we need to check that for $V \subset U$, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\theta_U} & \mathcal{F}^+(U) \\ \rho_V^U \downarrow & & \downarrow \tilde{\rho}_V^U \\ \mathcal{F}(V) & \xrightarrow{\theta_V} & \mathcal{F}^+(V) \end{array}$$

commutes. Let $s \in \mathcal{F}(U)$. Then $\theta_U(s) = \tilde{s}$ and $\tilde{\rho}_V^U(\tilde{s}) = \tilde{s}|_V$, so for any $x \in V$, we have

$$(\tilde{s}|_V)(x) = \tilde{s}(x) = \rho_x^U(s) = \rho_x^V(\rho_V^U(s)).$$

On the other hand,

$$\theta_V(\rho_V^U(s)) = \rho_x^V(\rho_V^U(s)),$$

and the commutativity follows.

Next, let us show that $\theta_x : \mathcal{F}_x \to \mathcal{F}_x^+$ is a bijection. We already know that $\mathcal{F}_x^+ \simeq \pi^{-1}(x) = E_x = \mathcal{F}_x$ via the maps

$$\sigma_x^U : \mathcal{F}^+(U) \to \mathcal{F}_x, \;\; t \mapsto t(x).$$

Thus, we only need to show that the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\theta_U} & \mathcal{F}^+(U) \\ \rho_x^U \downarrow & & \downarrow \sigma_x^U \\ \mathcal{F}_x & \xrightarrow{=} & \mathcal{F}_x \end{array}$$

commutes. We have for $s \in \mathcal{F}(U)$,

$$\sigma_x^U(\theta_U(s)) = \sigma_x^U(\tilde{s}) = \tilde{s}(x) = \rho_x^U(s),$$

as claimed. □

Remark 3. In particular, if $\mathcal{F}$ is a sheaf, then $\theta : \mathcal{F} \to \mathcal{F}^+$ is an isomorphism of sheaves.

To check the universal property, we need the following

Lemma 4.3.6. *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sets on a topological space X. Denote by $\pi_{\mathcal{F}} : E_{\mathcal{F}} \to X$ and $\pi_{\mathcal{G}} : E_{\mathcal{G}} \to X$, the étale spaces associated to $\mathcal{F}$ and $\mathcal{G}$, respectively. Then, ϕ induces a continuous map*

$$\tilde{\phi} : E_{\mathcal{F}} \to E_{\mathcal{G}}, \;\; e \in \mathcal{F}_x \subset E_{\mathcal{F}} \mapsto \phi_x(e) \in \mathcal{G}_x \subset E_{\mathcal{G}},$$

satisfying $\pi_{\mathcal{F}} = \pi_{\mathcal{G}} \circ \tilde{\phi}$.

Proof. Since $E_{\mathcal{F}} = \bigsqcup_{x \in X} \mathcal{F}_x$ and $E_{\mathcal{G}} = \bigsqcup_{x \in X} \mathcal{G}_x$, it is clear that the diagram

$$\begin{array}{ccc} E_{\mathcal{F}} & \xrightarrow{\tilde{\phi}} & E_{\mathcal{G}} \\ {\scriptstyle \pi_{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{G}}} \\ X & \xrightarrow{=} & X \end{array}$$

commutes. So, one only needs to show that $\tilde{\phi}$ is continuous. Consider an arbitrary element $\tilde{t}(U) \subset E_{\mathcal{G}}$ of the basis of the topology on $E_{\mathcal{G}}$, where $U \subset X$ is open and $t \in \mathcal{G}(U)$. Then one easily checks that

$$\tilde{\phi}^{-1}(\tilde{t}(U)) = \bigcup \tilde{s}(V),$$

where the union is taken over all $V \subset U$ and $s \in \mathcal{F}(U)$ such that $\phi_V(s) = \rho_V^U(t)$. It follows that $\tilde{\phi}^{-1}(\tilde{t}(U))$ is open, making $\tilde{\phi}$ continuous. □

The lemma implies that given a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$, for any open $U \subset X$, we have a map

$$\phi_U^+ : \mathcal{F}^+(U) = \Gamma(U, \pi_{\mathcal{F}}) \to \mathcal{G}^+(U) = \Gamma(U, \pi_{\mathcal{G}}), \;\; s \mapsto \tilde{\phi} \circ s.$$

Clearly, the maps ϕ_U^+ define a morphism of sheaves $\phi^+ : \mathcal{F}^+ \to \mathcal{G}^+$. Moreover, the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi_U} & \mathcal{G}(U) \\ {\scriptstyle \theta_{\mathcal{F},\mathcal{U}}}\downarrow & & \downarrow{\scriptstyle \theta_{\mathcal{G},U}} \\ \mathcal{F}^+(U) & \xrightarrow{\phi_U^+} & \mathcal{G}^+(U) \end{array}$$

commutes. Indeed, let $s \in \mathcal{F}(U)$. Then

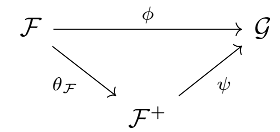
$$\begin{aligned} ((\theta_{\mathcal{G},U} \circ \phi_U)(s))(x) &= \rho(\mathcal{G})_x^U(\phi_U(s)) \\ &= \phi_x(\rho(\mathcal{F})_x^U(s)) \\ &= \tilde{\phi}(\tilde{s}(x)) \\ &= \phi_U^+(\tilde{s})(x) \\ &= (\phi_U^+ \circ \theta_{\mathcal{F},U}(s))(x). \end{aligned}$$

This means that

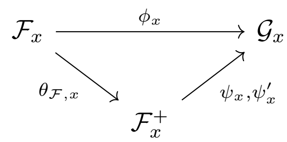
$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\ {\scriptstyle \theta_{\mathcal{F}}}\downarrow & & \downarrow{\scriptstyle \theta_{\mathcal{G}}} \\ \mathcal{F}^+ & \xrightarrow{\phi^+} & \mathcal{G}^+ \end{array}$$

is a commutative diagram of morphisms of presheaves.

Suppose now that $\mathcal{G}$ is a sheaf. Then, $\theta_{\mathcal{G}}$ is an isomorphism, and if we define $\psi = \theta_{\mathcal{G}}^{-1} \circ \phi^+ : \mathcal{F}^+ \to \mathcal{G}$, then the diagram



commutes. Besides, such ψ is unique. Indeed, suppose there are two such morphisms $\psi, \psi' : \mathcal{F}^+ \to \mathcal{G}$. For each $x \in X$, we have commutative diagrams for ψ and ψ':



Since $\theta_{\mathcal{F},x}$ is a bijection, we have

$$\psi_x = \phi_x \circ \theta_{\mathcal{F},x}^{-1} = \psi'_x.$$

Since $\mathcal{F}^+$ and $\mathcal{G}$ are sheaves, we obtain $\psi = \psi'$.

Remark 4. One can eliminate the explicit use of the étale space from the construction of sheafification. More precisely, let $\mathcal{F}$ be a presheaf on X, and $\pi : E \to X$ be the corresponding étale space. Since $E = \bigsqcup_{x \in X} \mathcal{F}_x$, a section $s : U \to E$ can be described by $(s_x) \in \bigsqcup_{x \in U} \mathcal{F}_x$, where $s_x = s(x)$. Let $\tilde{t}(V)$, where $V \subset U$ is open and $t \in \mathcal{F}(V)$, be a basis element. Then, since s is continuous, the set

$$s^{-1}(\tilde{t}(V)) = \{x \in V \mid s(x) = s_x = \rho^V_x(t)\}$$

must be open. This implies that $\Gamma(U, \pi)$ can be identified with the set of elements $(s_x) \in \prod_{x \in U} \mathcal{F}_x$ such that for each $x \in U$ there exists a neighborhood $x \in W(x) \subset U$ and $t \in \mathcal{F}(W(x))$ with the property

$$s_y = \rho^{W(x)}_y(t) \text{ for all } y \in W(x).$$

This description of sheafification is given for example in Hartshorne's book. However, the description using the étale space is much more revealing.

Example 4.3.7. *Let C be a fixed set, and let $\mathcal{F}$ be the constant presheaf on X with the value set C. Then $\mathcal{F}_x = C$ for any $x \in X$. So, the corresponding étale space E can be identified with the direct product $X \times C$, with $\pi : E \to X$ being the projection. Let $U \subset X$ be a nonempty open set, and $s \in \mathcal{F}(U) = C$. Then $\tilde{s}(U) = U \times \{s\}$. Thus, the topology on E is the product topology if C is equipped with the discrete topology. Let $\sigma : U \to E$ be a continuous section. We can write $\sigma(x) = (x, f(x))$ for some function $f : U \to C$. We have*

$$\sigma^{-1}(V \times \{s\}) = \{x \in U \cap V \mid \sigma(x) = (x, s)\} = \{x \in U \cap V \mid f(x) = s\}.$$

Taking $V = U$, we see that $f^{-1}(s) \subset U$ is open. Since this is true for all $s \in C$, the function f is locally constant. Conversely, given a locally constant function $f : U \to C$, one easily checks that $\sigma : U \to E$, $x \mapsto (x, f(x))$ defines a continuous section. It follows that $\mathcal{F}^+$ is the sheaf of locally constant functions.

Sheafification of presheaves of abelian groups. If $\mathcal{F}$ is a presheaf of abelian groups on X, then the stalks $\mathcal{F}_x, x \in X$, are all abelian groups. The existence of a group operation on each of the fibers of the corresponding étale map $\pi : E \to X$ means that we have a commutative diagram

$$\begin{array}{ccc} E \times_X E & \longrightarrow & E \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

where $E \times_X E = \{(e_1, e_2) \in E \times E \,|\, \pi(e_1) = \pi(e_2)\}$ is the fiber product over X. This motivates the following.

Definition 4.3.8. A (surjective) local homeomorphism $\pi : E \to X$ is called an étale space of abelian groups if

(1) each fiber $E_x = \pi^{-1}(x)$ is an abelian group;

(2) the maps (over X)

$$E \times_X E \xrightarrow{\phi} E, \;\; E_x \times E_x \ni (e_1, e_2) \mapsto e_1 + e_2 \in E_x$$

$$E \xrightarrow{\phi} E, \;\; E_x \ni e \mapsto -e \in E_x$$

are continuous.

It is easy to see that if $s_1, s_2 : U \to E$ are continuous sections, then

$$s_1 + s_2 = \phi \circ (s_1, s_2), \;\; (s_1, s_2)(x) = (s_1(x), s_2(x))$$

is also a continuous section, which defines an operation on $\Gamma(U, \pi)$. Using $-s = \iota \circ s$, one further establishes that $\Gamma(U, \pi)$ is an abelian group. On the other hand, we have the following.

Lemma 4.3.9. *Let $\mathcal{F}$ be a presheaf of abelian groups on X. Then the corresponding étale map $\pi : E \to X$ is an étale space of abelian groups.*

Combining these statements, we obtain that given a presheaf $\mathcal{F}$ of abelian groups, the sheaf $\mathcal{F}^+$ associated to $\mathcal{F}$ as a presheaf of sets is in fact a sheaf of abelian groups. Moreover, the canonical map $\theta : \mathcal{F} \to \mathcal{F}^+$ is a morphism of sheaves of abelian groups. This applies also to other algebraic structures.

Extension by zero and exceptional inverse image.

We have seen that for a closed embedding $i\colon Z \hookrightarrow X$ of topological spaces, and for a sheaf $\mathcal{G}$ on Z, the stalks of the sheaf $\mathcal{F} = i_*\mathcal{G}$ are described as follows.

$$\mathcal{F}_x = \begin{cases} \mathcal{G}_x, & x \in Z \\ 0, & x \notin Z. \end{cases}$$

Moreover, there exists a morphism $\eta_{\mathcal{G}}\colon \iota^{-1} i_* \mathcal{G} \to \mathcal{G}$. Since ι^{-1} does not change the stalks, one shows that $(\eta_{\mathcal{G}})_x\colon (\iota^{-1} i_*\mathcal{G})_x \to \mathcal{G}_x$ is an isomorphism for every $x \in Z$, so $\eta_{\mathcal{G}}$ is an isomorphism. Interpreting $\iota^{-1} i_* \mathcal{G} = \iota^{-1}\mathcal{F}$ as the restriction $\mathcal{F}|_Z$ of $\mathcal{F}$ to Zwe can say that $\mathcal{F}$ is obtained from $\mathcal{G}$ by extension by zero outside Z.

However, one should keep in mind that the direct image does not always provide this kind of extension.

Example. Let $j\colon U \to X$ be an open embedding with $\partial U \neq \varnothing$, and suppose that $\mathcal{G}$ is the constant sheaf on U, U_i in value group $\mathcal{S}$. Then is is easy to see that $(j_*\mathcal{G})_x = \mathcal{S}$ for all $x \in \bar{U}$. So, if $\partial U \neq \varnothing$, is is NOT an extension by zero outside U.

So, for open embeddings the construction of extension by zero needs to be modified. Let $j\colon U \to X$ be an open embedding, and let $\mathcal{G}$ be a sheaf of abelian groups on U. We define $j_!\mathcal{G}$ ("lower shriek") to be the sheaf associated to the following presheaf:

$$\mathcal{H}(V) = \begin{cases} \mathcal{G}(V), & V \subset U \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to see that for $x \in U$, the stalks $(j_!\mathcal{G})_x = \mathcal{H}_x$ can be naturally identifies with $\mathcal{G}_x$, while for $x \in X \backslash U$, for any neighborhood $V \ni x$, we have $\mathcal{H}(V) = 0$, and therefore have $j^{-1}\mathcal{H} = \mathcal{G}$, so the sheafification $\mathcal{H} \mapsto j_!\mathcal{G}$ gives rise to a morphism $\mathcal{G} \mapsto j^{-1} j_!\mathcal{G}$. Our discussion implies that this morphism induces an isomorphism on all stalks, hence is an isomorphism of sheaves. Thus, the restriction $(j_!\mathcal{G})|_U = j^{-1}j_!\mathcal{G}$ of $j_!\mathcal{G}$ to U coincides with $\mathcal{G}$, and $j_!\mathcal{G}$ is zero outside U.

For uniformity, we define $i_! = i_*$ for a closed embedding $i\colon Z \hookrightarrow X$. Then in both situations we obtain functors

$$i_!\colon \mathrm{Sh}(Z) \to \mathrm{Sh}(X) \quad \text{and} \quad j_!\colon \mathrm{Sh}(U) \to \mathrm{Sh}(X),$$

both of which are called extensions by zero. Consideration of stalks then leads to

Proposition 4.3.10. *The functors $i_!$ and $j_!$ are exact.*

We recall that a subspace X of a topological space Y is called locally closed if it is open in its closure $\bar{X}$ (equivalently, $X = U \cap V$ where $U \subset Y$ is open and $V \subset Y$ is closed, or for any $x \in X$ there exists a neighborhood $U \subset Y$ of x such that $U \cap X$ is closed in U). Since we have already define extension by zero for open and closed embeddings, we can now define it for any locally closed subspace/embedding. We will see a bit later that is it possible to construct a generalization of extension by zero functor for any continuous map of locally compact topological spaces $f\colon X \to Y$.

We will now state a very useful statement which is often used in algebraic geometry.

Proposition 4.3.11. *Let X be a topological space and $\mathcal{F}$ be a sheaf of abelian groups on X. Then for any closed subspace $Z \subset X$ and its complement $U = X \backslash Z$, there is an exact sequence*

$$0 \longrightarrow j_!(\mathcal{F}|_U) \longrightarrow \mathcal{F} \longrightarrow i_* i^{-1}\mathcal{F} = i_! i^{-1}\mathcal{F} \longrightarrow 0$$

of sheaves on X, where $i\colon Z \to X$ and $j\colon U \to X$ are the corresponding embedding.

Sketch of proof: Recall that $j_!(\mathcal{F}|_U)$ is the sheaf associated to the presheaf $\bar{\mathcal{F}}$ defined by

$$\bar{\mathcal{F}}(V) = \begin{cases} \mathcal{F}(V), & V \subset U \\ 0, & \text{otherwise.} \end{cases}$$

Next, we can define a morphism of presheaves by defining

$$\varphi_V\colon \bar{\mathcal{F}}(V) \longrightarrow \mathcal{F}(V)$$

for every open $V \subset X$ to be the identity map for $V \subset U$, and the zero map for $V \not\subset U$. By sheafification, we obtain a morphism of sheaves $j_!(\mathcal{F}|_U) \to \mathcal{F}$. On the other hand, the morphism $\mathcal{F} \to i_* i^{-1}\mathcal{F}$ is simple the unit of adjunction form the adjointness of i^{-1} and i_*. The sequence of stalks will look like

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}_x \longrightarrow 0 \longrightarrow 0$$

if $x \in U$, and

$$0 \longrightarrow 0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{F}_x \longrightarrow 0$$

if $x \notin U$ (i.e. $x \in Z$), proving the exactness.

Let us mention another application of the morphism $\varepsilon_{\mathcal{F}}\colon j_! j^{-1}\mathcal{F} \to \mathcal{F}$ be constructed in the proof of the proposition. First, for an open embedding $j\colon U \to X$ and a sheaf $\mathcal{G}$ on U, we have a natural isomorphism $\eta_{\mathcal{G}}\colon \mathcal{G} \to j^{-1} j_! \mathcal{G}$. One verifies that $\eta_{\mathcal{G}}$ and $\varepsilon_{\mathcal{F}}$ form a unit and counit of adjunction. So we set the following:

Proposition 4.3.12. *For an open embedding $j\colon U \to X$, the functor $j_!\colon Sh(U) \to Sh(X)$ is the left adjoint of $j^{-1}\colon Sh(X) \to Sh(U)$.*

Since $j_!$ is exact, we obtain the following.

Corollary 4.3.13. *For an open embedding $j\colon U \to X$, and an injective sheaf I on X, the sheaf $j^{-1}I$ (restriction of I to U) is an injective sheaf on U.*

We will now discuss an analog of extension by zero in a more general setting.

Definition 4.3.14. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X, $U \subset X$ be an open subset, and $s \in \mathcal{F}(U)$ be a section. The support of s is the set

$$\text{supp}(s) = \{x \in U \mid \rho_x^U(s) \neq 0 \text{ in } \mathcal{F}_x\}.$$

Lemma 4.3.15. *The set supp(s) is closed in U.*

Proof. Let $x \in U \backslash \mathrm{supp}(s)$, Then $\rho^U_x(s) = 0$. This means that there exists an open neighborhood $U_x \ni x$ such that $\rho^U_{U_x}(s) = 0$. Then $U_x \subset U \backslash \mathrm{supp}(s)$, showing that the latter is open. ☐

Let $j\colon U \to X$ be an open embedding, and let $\mathcal{G}$ be a sheaf on U. It immediately follows from the definitions that there is a natural morphism of sheaves $j_!\mathcal{G} \to j_*\mathcal{G}$. Moreover, for $x \in U$ we have the identification

$$(j_!\mathcal{G})_x = (j_*\mathcal{G})_x = \mathcal{G}_x,$$

and $(j_!\mathcal{G})_x = 0$ for $x \in X \backslash U = Z$. If follows that φ is an injective morphism of sheaves, so one can view $j_!\mathcal{G}$ as a subsheaf of $j_*\mathcal{G}$. In fact,

$$(j_!\mathcal{G})(W) = \{s \in (j_*\mathcal{G})(W) = \mathcal{G}(W \cap U) \mid \mathrm{supp}(s) \text{ is closed in } W\}$$

(here $\mathrm{supp}(s)$ means the suppose of s as a section of $\mathcal{G}$ over $W \cap U$). Indeed, if $s \in (j_!\mathcal{G})(W)$ then its support w.r.t. the sheaf $(j_!\mathcal{G})$ is closed in W. On the other hand, it is contained in $W \cap U$. So, this support coincides with the support of the restriction of s to $W \cap U$. But the support of the restriction is precisely $\mathrm{supp}(s)$. Conversely, suppose $\mathrm{supp}(s)$ is closed in W, and set $T = W \backslash \mathrm{supp}(s)$. Then T is open in W, and $W = (W \cap U) \cup T$. We can find $t \in (j_!\mathcal{G})(W)$ whose restriction to $W \cap U$ is $s|_{W \cap U}$ (note that $(j_!\mathcal{G})(W \cap U) = (j_*\mathcal{G})(W \cap U) = \mathcal{G}(W \cap U)$) and whose restriction to T is zero. By looking at stalks, we conclude that the image of t in $(j_*\mathcal{G})(W)$ is precisely s. Now, if Y is a locally compact space then to say that $\mathcal{S}$ is closed in Y is the same as to say that the identity map $\mathcal{S} \to Y$ is proper (recall that a continuous map (of locally compact spaces) $f\colon X \to Y$ is called proper if $f^{-1}(C)$ is compact for every compact $C \subset Y$.) It is not difficult to show that a proper map of locally compact spaces is closed.

Definition 4.3.16. Let $f\colon X \to Y$ be a continuous map of locally compact topological spaces, and let $\mathcal{F}$ be a sheaf on X. The direct image $f_!\mathcal{F}$ with proper support is the sheaf defined by

$$(f_!\mathcal{F})(V) = \{s \in \mathcal{F}(f^{-1}(V)) \mid f\colon \mathrm{supp}(s) \to V \text{ is proper}\}$$

In the algebro-geometric context, the topological notion of properness is replaces by the algebro-geometric one.

Finally, let us indicate the construction of the right adjoint $i^!$ of the function $i_! = i_*$ in the case where $i\colon Z \hookrightarrow X$ is a closed embedding. Given a sheaf of abelian groups $\mathcal{F}$ on X, let $\mathcal{F}_Z$ be the subsheaf of $\mathcal{F}$ whose sections over an open $U \subset X$ are given by

$$\mathcal{F}_Z(U) = \{s \in \mathcal{F}(U) \mid \mathrm{supp}(s) \subset Z\}.$$

We then define

$$i^!\mathcal{F} = i^{-1}\mathcal{F}_Z.$$

Since for a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ of sheaves on X, it is straightforward to check that $\varphi_U(\mathcal{F}_Z(U)) \subset \mathcal{G}_Z(U)$ for every open $U \subset X$, this yields a functor

$$i^!\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Z)$$

which is usually referred to as the exceptional inverse image functor.

Proposition 4.3.17. *For a closed embedding $i\colon Z \hookrightarrow X$, the functor $i^!\colon Sh(X) \to Sh(Z)$ is left exact.*

Sketch of proof. We need to show that if

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence in $\mathrm{Sh}(X)$, then

$$0 \longrightarrow i^!\mathcal{F} \longrightarrow i^!\mathcal{G} \longrightarrow i^!\mathcal{H}$$

is an exact sequence sequence in $\mathrm{Sh}(Z)$. Since the inverse image to an exact functor, it is enough to show that the sequence

$$0 \longrightarrow \mathcal{F}_Z \longrightarrow \mathcal{G}_Z \longrightarrow \mathcal{H}_Z \tag{$\star$}$$

is exact in $\mathrm{Sh}(X)$. But we now that for every open $U \subset X$, the sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$$

is exact, from which is follows that the sequence

$$0 \longrightarrow \mathcal{F}_Z(U) \longrightarrow \mathcal{G}_Z(U) \longrightarrow \mathcal{H}_Z(U)$$

is also exact, so $(\star)$ is exact in the category of presheaves, and therefore exact in the category of sheaves.

Proposition 4.3.18. $i_* \dashv i^!$

Summary.

- For a continuous map $f\colon X \to Y$ of topological spaces, we have the functors $f_*\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ (direct image) and $f^{-1}\colon \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ (inverse image). Furthermore, f_8 is left exact and f^{-1} is exact.
- For a closed subspace $Z \subset X$, its open complement $U = X\backslash Z$, and the inclusion maps (embeddings) $i\colon Z \hookrightarrow X$ and $j\colon U \hookrightarrow X$, we have the functors $i_! = i_*\colon \mathrm{Sh}(Z) \to \mathrm{Sh}(X)$ and $j_!\colon \mathrm{Sh}(U) \to \mathrm{Sh}(X)$ called extension by zero and $i^!\colon \mathrm{Sh}(X) \to \mathrm{Sh}(Z)$ called the exceptional inverse image. The functors $i_!$ and $j_!$ are exact, while $i^!$ is left exact. We have the following adjoint pairs

$$(j_!, j^{-1}),\ (i_!, i^!),\ (j^{-1}, j_*),\ \text{and}\ (i^{-1}, i_* = i_!).$$

Remark. We have seen that the functor $f_!$ can be constructed for more general continuous maps $f\colon X \to Y$ (for example, for arbitrary continuous maps of locally compact topological spaces), however $f^!$ has been constructed only for closed embeddings. For more general maps, a right adjoint $f^!$ of $f_!$ exists only on the level of derived categories, which plays a role in Verdier duality.

4.4 Sheaf Cohomology

We will use the general construction of right derived functors in abelian categories. So, for reference, we record the following.

Theorem 6. Let X be a topological space. Then the category $\mathrm{Sh}(X)$ of sheaves of abelian groups is an abelian category.

Proof – omitted. (One first checks that $\mathrm{Sh}(X)$ is an abelian category, and then verifies the axioms for abelian categories.)

Recall that an abelian category $\mathcal{A}$ has enough injectives if every object $A \in \mathscr{O}b(\mathcal{A})$ admits a morphism $O \to A \to I$ into an injective object I (which means that for every monomorphism $O \to A \to B$, the corresponding map $\mathrm{Hom}_{\mathcal{A}}(B, I) \to \mathrm{Hom}_{\mathcal{A}}(A, I)$ is surjective.

Proposition 4.4.1. *Sh*(X) *has enough injectives.*

Proof. Let $\mathcal{F} \in \mathscr{O}b(\mathrm{Sh}(X))$, i.e. $\mathcal{F}$ is a sheaf of abelian groups. For each point $x \in X$, the stalk $\mathcal{F}_x$ is an abelian group, so there is an injection $i_x \colon \mathcal{F}_x \to I_x$ where I_x is an injective $\mathbb{Z}$-module (i.e. a divisible abelian group). We will view i_x as a morphism of constant sheaves on $\{x\}$. Next, let $j_x \colon \{x\} \to X$ be the inclusion map. Then $j_x^{-1}\mathcal{F} = \mathcal{F}_x$ (constant sheaf on $\{x\}$). By adjunction, we obtain a morphism $\mathcal{F} \to (j_x)_* I_x$ of sheaves on X. Setting $I = \prod_{x \in X} (j_x)_* I_x$ and taking the product of these morphism over all $x \in X$, we obtain a natural morphism $i \colon \mathcal{F} \to I$. By looking at the stalks, we easily see that i is injective. each $(j_x)_* I_x$, being the direct image of an injective sheaf (explain!) I_x, is injective, and the product of injectives in injective. Thus, I is an injective sheaf, as required. □

Explicit description. For an open $U \subset X$, we have $I(U) = \prod_{x \in U} I_x$, and the homomorphism $\mathcal{F}(U) \to I(U)$ is the composition

$$\mathcal{F}(U) \xrightarrow{\prod \rho_x^U} \prod_{x \in U} \mathcal{F}_x \xrightarrow{\prod i_x} \prod_{x \in U} I_x.$$

Theorem 7. Let $\mathcal{A}$ be an abelian category that has enough injectives. Then

i. Every object $A \in \mathscr{O}b(\mathcal{A})$ admits an injective resolution, i.e. there is an exact sequence

$$I^*(A) \colon 0 \to A \to I^0 \to I^1 \to \cdots$$

where all the I^j are injective objects in $\mathcal{A}$.

ii. Let $0 \to A \to M^\bullet$ be a long exact sequence in $\mathcal{A}$, and $I^*(A')$ be an injective resolution of some $A' \in \mathscr{O}b(\mathcal{A})$. Then every morphism $A \to A'$ extends to a morphism of complexes

$$(0 \to A \to M^\bullet) \to I^*(A').$$

Any two such extensions are chain-homotopic. In particular, if $I^*(A)$ and $I^*(A')$ are two injective resolutions, then every morphism $A \to A'$ extends to a morphism of injective resolutions $I^*(A) \to I^*(A')$, and any two such extensions are chain-homotopic.

iii. Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence in $\mathcal{A}$, and let $I^*(A')$ and $I^*(A'')$ be arbitrary injective resolutions. Then there is an injective resolution $I^*(A)$ of A that fits into a short exact sequence of cochain complexes

$$0 \to I^*(A') \to I^*(A) \to I^*(A'') \to 0.$$

Definition 4.4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories.

(a) A cohomological δ-functor from $\mathcal{A}$ to $\mathcal{B}$ is a collection of functors $T^i\colon \mathcal{A} \to \mathcal{B}$, together with a morphism $\delta^i\colon T^i(A'') \to T^{i+1}(A')$ for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ and every $i \geqslant 0$ such that

- for every short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ there is a long exact sequence

$$0 \longrightarrow T^0(A') \longrightarrow T^0(A) \longrightarrow T^0(A'') \xrightarrow{\delta^0} T^1(A') \longrightarrow \cdots$$

  in $\mathcal{B}$.

- for each morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0 \end{array}$$

  in $\mathcal{A}$ there is a commutative diagram of long exact sequences

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & T^0(A') & \longrightarrow & T^0(A) & \longrightarrow & T^0(A'') & \xrightarrow{\delta^0} & T^1(A') & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T^0(B') & \longrightarrow & T^0(B) & \longrightarrow & T^0(B'') & \xrightarrow{\delta^0} & T^1(B') & \longrightarrow & \cdots \end{array}$$

  (naturality).

(b) The δ-functor $T = (T^i\colon \mathcal{A} \to \mathcal{B})_{i \geqslant 0}$ is called universal if given any other δ-functor $T' = (T'^i)$, there exists a unique sequence of natural transformations $\mathcal{F}^i\colon T^i \to T'^i$ for all $i \geqslant 0$, starting with the given $\mathcal{F}^0$, that commute with the morphism δ^i for every short exact sequence in $\mathcal{A}$.

Note that the universality implies that if $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a covariant additive functor, then there can exist at most one (up to isomorphism) universal δ-functor $T = (T^i)_{i \geqslant 0}$ with $T^0 = \mathcal{F}$.

Definition 4.4.3. An additive functor $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ between abelian categories is said to be effaceable if for each object $A \in \mathscr{O}b(\mathcal{A})$ there exists a monomorphism $u\colon A \to M$ such that $\mathcal{F}(U) = 0$.

Theorem 8 (Grothendieck)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $T = (T^i)_{i \geqslant 0}$ be a cohomological δ-functor. If T^i is effaceable for each $i > 0$, then T is universal.

Right derived functors. Let $\mathcal{A}$ be an abelian category having enough injectives, and let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a left exact additive covariant functor to another abelian category $\mathcal{B}$. For an object $A \in \mathscr{O}b(\mathcal{A})$, we pick an injective resolution

$$I^*(A)\colon 0 \to A \to I^0 \to I^1 \to \cdots$$

Applying $\mathcal{F}$ to this resolution, we obtain a complex

$$\mathcal{F}I\colon 0 \to \mathcal{F}I^0 \to \mathcal{F}I^1 \to \cdots$$

and we define $R^i\mathcal{F} = H^i(\mathcal{F}I^\bullet$ to be the ith cohomology of the complex $\mathcal{F}I$.

Theorem 9. Let $\mathcal{A}$ be an abelian category with enough injectives, and let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a left exact additive covariant functor to another abelian category $\mathcal{B}$.

i. For each object $A \in \mathscr{O}b(\mathcal{A})$ and each $i \geqslant 0$, $R^i\mathcal{F}(A)$ is independent (up to natural isomorphism) of the choice of injective resolution $I^*(A)$, and each $R^i\mathcal{F}$ is an additive functor from $\mathcal{A} \to \mathcal{B}$.

ii. There is a natural isomorphism of functors $\mathcal{F} \cong R^0\mathcal{F}$.

iii. The collection $(R^i\mathcal{F})_{i\geqslant 0}$ defined a cohomological δ-functor from $\mathcal{A}$ to $\mathcal{B}$.

iv. For every injective object I of $\mathcal{A}$, we have $R^i\mathcal{F}(I) = 0$ for all $i > 0$. Consequently, $(R^i\mathcal{F})_{i\geqslant 0}$ is a universal δ-functor.

Sketch of proof.

i. Let $I^*(A)$ and $I^*(A')$ be injective resolutions of objects A and A'. Then any morphism and any two such extensions are chain-homotopic. In particular, given two injective resolutions $I_1^*(A)$ and $I_2^*(A)$, the identity morphism $A \xrightarrow{\text{id}} A$ extends to morphisms $I_1^*(A) \xrightarrow{f_1^\bullet} I_2^*(A)$ and $I_2^*(A) \xrightarrow{f_2^\bullet} I_1^*(A)$, and the compositions $f_1^\bullet \circ f_2^\bullet$ and $f_2^\bullet \circ f_1^\bullet$ are chain-homotopic to the identity. It follows that the induced maps

$$H^i(\mathcal{F}I_1^\bullet) \to H^i(\mathcal{F}I_2^\bullet) \quad \text{and} \quad H^i(\mathcal{F}I_2^\bullet) \to H^i(\mathcal{F}I_1^\bullet)$$

are inverses of one another. Consequently, the cohomology objects $R^i\mathcal{F}(A)$ are independent of the choice of injective resolution. By the same argument, a morphism $A \xrightarrow{f} B$ gives rise to well-defined morphisms $R^i\mathcal{F}(A) \to R^i\mathcal{F}(B)$. Thus, for each I we get a functor $R^i\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$. Moreover, given two morphisms $f, g\colon A \to B$, we can extend them to morphisms of injective resolutions $f^\bullet, g^\bullet\colon I^*(A) \to I^*(B)$, and then $f^\bullet + g^\bullet\colon I^*(A) \to I^*(B)$ is an extension of $f + g\colon A \to B$. Using this particular extension, we see that each $R^i\mathcal{F}$ is additive.

ii. Since $\mathcal{F}$ is left-exact, the sequence

$$0 \to \mathcal{F}(A) \to \mathcal{F}(I^0) \to \mathcal{F}(I^1)$$

is exact, which leafs to an isomorphism of functors $R^0\mathcal{F} \cong \mathcal{F}$.

iii. Let $0 \to A' \to A \to A'' \to 0$ be a short exact sequence in $\mathcal{A}$. Then there exist injective resolutions $I^*(A')$, $I^*(A)$, and $I^*(A'')$ of $A', A,$ and A'' respectively such that we have a short exact sequence of complexes

$$0 \to I^*(A') \to I^*(A) \to I^*(A'') \to 0.$$

Since every $I^i(A)$ is injective, each of the sequences

$$0 \to I^i(A') \to I^i(A) \to I^i(A'') \to 0$$

remains exact. Thus, we have a short exact sequence of complexes

$$0 \to \mathcal{F}(I^*(A')_{\to}\mathcal{F}(I^*(A)) \to \mathcal{F}(I^*(A'')) \to 0$$

which by standard techniques gives a long exact sequence of cohomology.

iv. Since $R^i\mathcal{F}(I)$ is independent of the choice of injective resolution, we can use the injective resolution

$$0 \to I \xrightarrow{\mathrm{id}_I} I \to 0 \to \cdots$$

which clearly shows that $R^i\mathcal{F}(I) = 0$ for isomorphisms. Since A has enough injectives, Grothendieck's theorem implies that $(R^i\mathcal{F})_{i\geqslant 0}$ is a universal δ-functor.

Definition 4.4.4. Let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be as above. An object $J \in \mathscr{O}b(\mathcal{A})$ is $\mathcal{F}$-acyclic if $(R^i\mathcal{F})(I) = 0$ for all $i > 0$.

For example, every injective object is $\mathcal{F} - acyclic$.

Theorem 10. Let $A \in \mathscr{O}b(\mathcal{A})$, and let

$$0 \to A \to J^0 \to J^1 \to J^2 \to \cdots$$

be an $\mathcal{F}$-acyclic resolution of A, i.e. a long exact sequence what all the J^i are the $\mathcal{F}$-acyclic objects. Then $(R^i\mathcal{F})(A) = H^i(\mathcal{F}j^\bullet)$ for all $i \geqslant 0$.

Proof. To keep out notation simple, we will give the argument assuming that $\mathcal{A}$ and $\mathcal{B}$ are (abelian) subcategories of the category of abelian groups or more generally the category of modules over a certain ring (recall that every abelian category can be considered as a subcategory of the category of modules by the Freyd-Mitchell Embedding Theorem).

First, since $\mathcal{F}$ is left-exact, we have

$$(R^0\mathcal{F})(A) = \mathcal{F}(A) \cong H^0(\mathcal{F}J^\bullet).$$

For each $i \geqslant 0$, let

$$K^i = \ker f^i = \mathrm{im} f^{i-1}.$$

We then have the following short exact sequence:

$$0 \to K^i \xrightarrow{e^i} J^i \xrightarrow{g^i} K^{i+1} \to 0$$

where e^i is the canonical monomorphism, and g^i is the canonical epimorphism (note that by construction, $f^i = e^{i+1} \circ g^i$). Hence for each i, we have the following long exact sequence:

$$0 \to \mathcal{F}K^i \to \mathcal{F}J^i \to \mathcal{F}K^{i+1} \to (R^1\mathcal{F})(K^i) \to (R^1\mathcal{F})(J^i) \to (R^1\mathcal{F})(K^{i+1}) \to (R^2\mathcal{F})(K^i) \to \cdots$$

Since each J^i is acyclic, we have $(R^1\mathcal{F})(J^i) = 0$ for all $j > 0$, so we get natural isomorphisms

$$(R^j\mathcal{F})(K^{i+1}) \cong (R^{j+1}\mathcal{F})(K^i).$$

Consequently,

$$(R^j = 1\mathcal{F})(K^i) \cong (R^2\mathcal{F})(K^{i-1}) \cong \cdots \cong (R^{i+1}\mathcal{F})(K^0) = (R^{i+1}\mathcal{F})(A).$$

On the other hand, from the long exact sequence,

$$(R^1\mathcal{F})(K^i) = \operatorname{coker}(\mathcal{F}J^i \xrightarrow{\mathcal{F}g^i} \mathcal{F}K^{i+1}).$$

Since $\mathcal{F}$ is left-exact, the sequences

$$0 \to \mathcal{F}K^i \xrightarrow{e^i} \mathcal{F}J^i \xrightarrow{g^i} \mathcal{F}K^{i+1} \quad \text{and} \quad 0 \to \mathcal{F}K^{i+1} \xrightarrow{\mathcal{F}e^i} \mathcal{F}J^{i+1}$$

are exact, from which it follows that

$$\operatorname{coker}(\mathcal{F}J^i \xrightarrow{\mathcal{F}g^i} \mathcal{F}K^{i+1}) = \mathcal{F}K^{i+1}/\mathrm{im}\mathcal{F}g^{i-1} = \ker \mathcal{F}f^{i+1}/\mathrm{im}\mathcal{F}f^i = H^{i+1}(\mathcal{F}J^\bullet).$$

Thus, $(R^{i+1}\mathcal{F})(A) = H^{i+1}(\mathcal{F}J^\bullet)$, as required. □

Sheaf cohomology. Let X be a topological space, and $\mathrm{Sh}(X)$ be the category of sheaves of abelian groups on X. Recall that $\mathrm{Sh}(X)$ is an abelian category with enough injectives. Let

$$\Gamma(X, \bullet)\colon \mathrm{Sh}(X) \to \mathcal{A}b\mathcal{G}rps$$

be the global sections functor $\mathcal{F} \to \mathcal{F}(X)$ and $(\varphi\colon \mathcal{F} \to \mathcal{G}) \to (\varphi_X\colon \mathcal{F}(X) \to \mathcal{G}(X))$. We have seen earlier that $\Gamma(X, \bullet)$ is left-exact. So, we can consider the right derived functors.

Definition 4.4.5. Let X be a topological space. We defined the cohomology functors $H^i(X, \bullet)$ as the right derived functors of the global sections functor $\Gamma(X, \bullet)$. For a sheaf $\mathcal{F}$ of abelian groups on X, the group $H^i(X, \mathcal{F})$ is called the ith cohomology group of X with coefficients in $\mathcal{F}$.

Thus, we have $H^0(X, \mathcal{F}) = \mathcal{F}(X)$ for any $\mathcal{F} \in \mathrm{Sh}(X)$, and for any exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of sheaves of abelian groups on X, there is a long exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{G}) \to \cdots$$

Sheaf cohomology is often difficult to compute directly from the definition. One of the techniques that is frequently used for this is the comparison with Čech cohomology. We will consider this technique in detail a bit later, and will now make only one observation. Given a continuous map $f\colon X \to Y$ of topological spaces, and a sheaf of abelian groups $\mathcal{F}$ on X, there is a natural homomorphism

$$H^p(Y, f^*\mathcal{F}) \to H^p(X, \mathcal{F})$$

between Čech cohomology groups, which comes from the map between Čech complexes associated with f. We will not construct an analogous map in sheaf cohomology.

Proposition 4.4.6. *Let $f\colon X \to Y$ be a continuous map, and $\mathcal{F}$ be a sheaf of abelian groups on X. Then* *i. There is a natural homomorphism*

$$H^n(Y, f^*\mathcal{F}) \to H^n(X, \mathcal{F}).$$

ii. If f is an embedding, then $H^n(Y, f^*\mathcal{F}) \cong H^n(X, \mathcal{F})$.

For the proof of (i), we need a general fact. We recall that a morphism $\varphi^\bullet \colon M^\bullet \to N^\bullet$ of cochain complexes is a quasi-isomorphism if the induces maps on cohomology $H^n(M^\bullet) \to H^n(N^\bullet)$ are isomorphisms for all n.

Theorem 11. Let $\mathcal{A}$ be an abelian category that has enough injectives, and let $m^\bullet$ be a cochain complexes such that $M^n = 0$ for $n < 0$. There is a complex $I^\bullet$ in $\mathcal{A}$ of injective objects and a quasi-isomorphisms of complexes $\varphi^\bullet \colon M^\bullet \to I^\bullet$ such that $\varphi^n \colon M^n \to I^n$ is a monomorphism for all n.

Proof – omitted.

Proof of (i). Let $\mathcal{G}$ be a sheaf on Y, and let

$$0 \to \mathcal{G} \to J^0 \to J^1 \to c\cdots$$

be an injective resolution in $\mathrm{Sh}(Y)$. Since the inverse image functor is exact, the sequence

$$0 \to f^{-1}\mathcal{G} \to f^{-1}J^0 \to f^{-1}J^1 \to c\cdots$$

is exact in $\mathrm{Sh}(X)$, however the sheaves $f^{-1}J^i$ are not necessarily injective. By the previous theorem, there exists a complex $I^\bullet$ of injective objects together with a quasi-isomorphism $f^{-1}J^\bullet \to I^\bullet$ such that $f^{-1}J^i \to I^i$ is injective for all $i \geqslant 0$. In particular, since $f^{-1}J^i$ is exact, $I^\bullet$ is also exact. The composition $f^{-1}\mathcal{G} \to f^{-1}J^0 \to I^0$ is injective, and

$$0 \to f^{-1}\mathcal{G} \to I^0 \to I^1 \to \cdots$$

is an injective resolution of $f^{-1}\mathcal{G}$. Moreover, we have a morphism of complexes

$$\Gamma(Y, J^\bullet) \to \Gamma(X, I^\bullet)$$

obtained from the natural map

$$J^n(Y) \to (f^{-1}J^n)(X) \to I^n(X),$$

which yields a homomorphism $H^n(Y, \mathcal{G}) \to H^n(Y, f^{-1}\mathcal{G})$. Applying this to $\mathcal{G} = f_*\mathcal{F}$ for a sheaf $\mathcal{F}$ on X, we obtain maps

$$H^n(Y, f_*\mathcal{F}) \to H^n(X, f^{-1}f_*\mathcal{F})$$

for all $n \geqslant 0$. Finally, using the counit $f^{-1}f_*\mathcal{F} \to \mathcal{F}$, we obtain the required map

$$H^n(Y, f_*\mathcal{F}) \to H^n(X, f^{-1}f_*\mathcal{F}) \to H^n(X, \mathcal{F}).$$

Proof of (ii). Let

$$0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots$$

be an injective resolution. Since f_* is exact for closed embeddings and always preserves injectives, it follows that

$$0 \to f_*\mathcal{F} \to f_*I^0 \to f_*I^1 \to \cdots$$

is an injective resolution of $f_*\mathcal{F}$. Thus, the cohomology groups $H^n(Y, f_*\mathcal{F})$ can be computed using the complex

$$0 \to (f_*I^0)(Y) \to (f_*I^1)(Y) \to \cdots$$

But $f_*(I)(Y) = I(X)$, so this complex is identical to

$$0 \to f_*\mathcal{F} \to I^0(X) \to I^1(X),$$

which computes the cohomology groups $H^n(X, \mathcal{F})$. Thus, $H^n(Y, f_*\mathcal{F} \cong H^n(X, \mathcal{F})$.

Higher direct images. The second important example of right derived functors are higher direct images. Let $f: X \to Y$ be a continuous map of topological spaces and recall the direct image functor $f_*: \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$.

Definition 4.4.7. The higher direct image functors $R^if_*: \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ are defined as the right derived functors of the direct image functor f_*.

It turns out that for a sheaf $\mathcal{F}$ on X, the sheaves $R^if_*(\mathcal{F})$ admit the following concrete description:

Proposition 4.4.8. *For each $i \geqslant 0$, the sheaf $R^if_*(\mathcal{F}$ is the sheaf associated to the presheaf*

$$V \mapsto H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$$

where $V \subset Y$ open.

Proof. Let us denote the sheaf associated to the above presheaf by $\mathcal{H}^i(X, \mathcal{F})$. Since the sheafification is an exact functor $\mathrm{PSh}(X) \to \mathrm{Sh}(X)$, the functors $\mathcal{H}^i(X, \bullet)$ yield a δ-functor from $\mathrm{Sh}(X)$ to $\mathrm{Sh}(Y)$. For $i = 0$, we have

$$\mathcal{H}^0(X, \mathcal{F}) = f_*\mathcal{F} = R^0f_*\mathcal{F}.$$

Next, for an injective sheaf I on X, the sheaf f_*I is injective, so $R^if_*(I) = 0$ for $i > 0$. On the other hand, for each open $V \subset Y$, the sheaf $I|_{f^{-1}(V)}$ is injective on $f^{-1}(V)$, so $H^i(f^{-1}(V), I|_{f^{-1}(V)}) = 0$ for $i > 0$ and hence $\mathcal{H}^i(X, I) = 0$ for $i > 0$. Thus, the functors R^if_* and $\mathcal{H}^i(X, \bullet)$ are both effaceable for $i > 0$, and therefore the δ-functors $(R^if_*)_{i \geqslant 0}$ and $(\mathcal{H}^i(X, \bullet))_{i \geqslant 0}$ are universal. So, in view of the equality $R^0f_*(\mathcal{F}) = \mathcal{H}^0(X, \mathcal{F})$, we obtain an isomorphism of δ-functors $R^if_* \cong \mathcal{H}^i(X, \bullet)$, as needed. ☐

4.5 Acyclic sheaves

We have seen that sheaf cohomology can be computed using acyclic resolutions. In this section, we will describe several classes of acyclic sheaves.

Flasque sheaves. Recall that a sheaf $\mathcal{F}$ on X is flasque if the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are surjective for any open sets $V \subset U \subset X$.

Theorem 12. Let $\mathcal{F}$ be a flasque sheaf of abelian groups on X. Then $H^i * X, \mathcal{F}) = 0$ for all $i > 0$.

For the proof, we will need the following lemma which we will formulate in a somewhat more general form than we need now.

We recall that a ringed space is a pair $(X, \mathcal{O})$ consisting of a topological space X and a sheaf of commutative rings $\mathcal{O}$ on X. An $\mathcal{O}$-module is a sheaf M of abelian groups on X such that for any open set $U \subset X$ we are given a map $\mathcal{O}(U) \times M(U) \xrightarrow{m_U} M(U)$ that equips $M(U)$ with an $\mathcal{O}(U)$-module structure so that for $V \subset U$ the diagram in which the vertical arrows are restriction maps, is commutative.

Lemma 4.5.1. *Let $(X, \mathcal{O})$ be a ringed space. Then every injective $\mathcal{O}$-module I is flasque.*

Proof. For an open $U \subset X$, let $j\colon U \hookrightarrow X$ be the inclusion map, and let $\mathcal{O}_U = j_!(\mathcal{O}|_U)$ be the restriction of $\mathcal{O}$ to U extended by zero outside U. Note that $\mathcal{O}_U$ has a natural structure of an $\mathcal{O}$-module. Indeed, recall that $\mathcal{O}_U$ is a sheaf associated to the presheaf $\mathcal{F}_U$ defined by

$$\mathcal{F}_U(W) = \begin{cases} \mathcal{O}(W), & W \subset U \\ 0, & W \not\subset W. \end{cases}$$

Clearly, ever $a \in \mathcal{O}(W)$ acts by left multiplication on $\mathcal{F}_U(W)$, and this operation turns $\mathcal{F}_U(W)$ into an $\mathcal{O}(W)$-module. Using the universal property of sheafification, one transfers this structure to $\mathcal{O}_U$, making it into an $\mathcal{O}$-module (in fact, this module is generated by the identity element $1_U \in \mathcal{O}(U)$.) For open $V \subset U \subset X$, we have the inclusion $0 \to \mathcal{O}_V \to \mathcal{O}_U$ of $\mathcal{O}$-modules arising from the obvious inclusion of presheaves $0 \to \mathcal{F}_V \to \mathcal{F}_U$. Since I is an injective $\mathcal{O}$-module, we have a surjection

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_U, I) \to \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_v, I) \to 0.$$

But any morphism $\varphi\colon \mathcal{F}_U \to I$ is completely determined by $\varphi_U(1_U) \in I(U)$ (where $1_U \in \mathcal{O}(U)$ is the identity element); hence the same is true for any morphism $\mathcal{O}_U \to I$. Thus, $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_U, I) \cong I(U)$ and similarly, $\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}_V, I) \cong I(V)$. So, we obtain that the restriction $I(U) \to I(V)$ is surjective. ☐

Proof of the theorem. Let $\mathcal{F}$ be a flasque sheaf on X. Since $\mathrm{Sh}(X0$ has enough injectives, we can embed $\mathcal{F}$ into an injective sheaf I, and then consider the exact sequence

$$0 \to \mathcal{F} \to I \to \mathcal{G} \to 0$$

where $\mathcal{G}$ is the cokernel of of $\mathcal{F} \to I$. Since $\mathcal{F}$ is flasque, we have an exact sequence of global sections

$$0 \to \mathcal{F}(X) \to I(X) \to \mathcal{G}(X) \to 0.$$

On the other hand, since I is injectives, we have $H^i(X, I) = 0$ for $i > 0$. The long exact sequence then gives the exact sequence

$$0 \to \mathcal{F}(X) \to I(X) \to \mathcal{G}(X) \to H^1(X, \mathcal{F}) \to 0$$

and isomorphisms

$$H^{i-1}(X, \mathcal{G}) \cong H^i(X, \mathcal{F})$$

for $i \geqslant 2$. It follows that $H^1(X, \mathcal{F}) = 0$. Furthermore, since I is flasque, $\mathcal{G}$ is also flasque. So, using the above we obtain the result by induction on i.

Remark 5. In the theorem, we didn't specify whether $\mathcal{F}$ is a sheaf of abelian groups or a sheaf of $\mathcal{O}$-modules in case $(X, \mathcal{O})$ is a ringed space. The reason is that the notion of a flasque sheaf does not depend on whether $\mathcal{F}$ is viewed in one way of the other. For example, we can take a sheaf of $\mathcal{O}$-modules $\mathcal{F}$, take its injective resolution $\mathcal{O} \to \mathcal{F} \to I^\bullet$ in the category of $\mathcal{O}$-modules (which is abelian with enough injectives). Then this resolution is by flasque sheaves of abelian groups. It follows that the sheaf cohomology computed by viewing $\mathcal{F}$ as a sheaf of $\mathcal{O}$-modules coincides with the cohomology computed by viewing $\mathcal{F}$ as a sheaf of abelian groups.

Any sheaf of abelian groups on a topological space X has a canonical flasque resolution called the Godement resolution. To construct it, we let $\pi\colon E_{\mathcal{F}} \to X$ étale space associated to $\mathcal{F}$. Recall that $E_{\mathcal{F}} = \coprod_{x \in X} \mathcal{F}_x$, and π is defined by sending any element of the stalk $\mathcal{F}_x$ to x. We have seen that for any open $U \subset X$, we have

$$\mathcal{F}(U) = \{\text{continuous sections } s\colon U \to E_{\mathcal{F}} \text{ of } \pi\}$$

Now, for an open $U \subset X$ we let $C^0\mathcal{F}(U)$ denote the abelian of <u>all</u> (not necessarily continuous) sections

$$C^0\mathcal{F}(U)\{t\colon U \to E_{\mathcal{F}} \mid \pi \circ t = id_U\}$$

Then we have the identification $C^0\mathcal{F}(U) = \prod_{x \in U} \mathcal{F}_x$. Moreover, for open $V \subset U \subset X$, there is a natural restriction map

$$C^0\mathcal{F}(U) = \prod)x \in U \mathcal{F}_x \to \prod_{x \in V} \mathcal{F}_x = C^0\mathcal{F}(V),$$

and this data assembles to give a flasque sheaf $C^0\mathcal{F}$. It follows from the definition that $\mathcal{F}$ is a subsheaf of $C^0\mathcal{F}$, the embedding $\mathcal{F}(U) \to C^0\mathcal{F}(U)$ is given by $f \mapsto (\rho^U_x(f))_{x \in U}$. Thus, we have an exact sequence of sheaves

$$0 \to \mathcal{F} \to C^0\mathcal{F} \to Q^1 \to 0$$

where Q^1 is the cokernel of the natural inclusion. Repeating this construction, we obtain exact sequences

$$0 \to Q^1 \to C^0Q^1 \to Q^2 \to 0$$

$$0 \to Q^2 \to C^0Q^2 \to Q^3 \to 0$$

and so on. Splicing these sequences together and setting $C^k\mathcal{F} = C^0Q^k$, we obtain a long exact sequence

$$0 \to \mathcal{F} \to C^0\mathcal{F} \to C^1\mathcal{F} \to \cdots$$

Moreover, since each $C^i\mathcal{F} = C^0Q^i$ is flasque, this sequence is a flasque resolution of $\mathcal{F}$.

<u>Fine sheaves.</u> These are important for topological applications.

Definition 4.5.2. Let $\varphi\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves of abelian groups on a topological space X. For each $x \in X$, denote by $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ the corresponding morphism of stalks. We define the support of φ to be

$$\text{supp}\varphi = \{x \in X \mid \varphi_x \neq 0\}$$

Definition 4.5.3. Let X be a topological space.

1. We say that an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ is locally finite if every point $x \in X$ has a neighborhood that meets only finitely many of the U_i's.

2. We say that X is paracompact if it is Hausdorff and every open cover of X admits a locally finite refinement.

Example 4.5.4. *1. Every compact space is paracompact.*

2. Every locally compact Hausdorff second-countable space is paracompact (second-countable condition is essential – long line is not paracompact although it is locally compact).

3. Every metric space is paracompact.

Definition 4.5.5. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X, and let $\mathcal{U} = \{U_i\}_{i\in I}$ be a locally finite open cover of X. A partition of unity for $\mathcal{F}$ subordinate to $\mathcal{U}$ is a collection of morphisms $\eta_i \colon \mathcal{F} \to \mathcal{F}$ $(i \in I)$ such that

1. $\mathrm{supp}\eta_i \subset U_i$, and

2. for each $x \in X$, we have $\sum_{i\in I} \eta_{i,x} = id_{\mathcal{F}_x}$.

Note that the sum in (2) is finite because by assumption every $x \in X$ has a neighborhood that intersects only finitely many of the U_i, hence x lies in the support of only finitely many of the η_i.

Definition 4.5.6. A sheaf $\mathcal{F}$ on a topological space X is fine if for every locally finitely open cover of X there exists a partition of unity subordinate to this open cover.

Theorem 13. Let X be a topological space in which every open set is paracompact. Given a fine sheaf $\mathcal{F}$ on X, the restriction $\mathcal{F}|_U$ is acyclic for every open $U \subset X$.

The proof proceeds along the same lines as the acyclicity of flasque sheaves using the following statement:

Proposition 4.5.7. Let

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$$

be an exact sequence of sheaves of abelian groups on a paracompact space X.

i. If $\mathcal{F}$ is fine then the sequence of global sections

$$0 \to \mathcal{F}(X) \xrightarrow{\alpha_X} \mathcal{G}(X) \xrightarrow{\beta_X} \mathcal{H}(X) \to 0$$

is exact.

ii. Assume moreover that every open subset of X is paracompact. If $\mathcal{F}$ if finite and $\mathcal{G}$ is flasque then $\mathcal{H}$ is flasque.

Proof. i. We only need to prove that β_X is surjective. Let $h \in \mathcal{H}(X)$ Since $\beta_x \colon \mathcal{G}_x \to \mathcal{H}_x$ is surjective for every $x \in X$, we can find an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X such that for each $i \in I$, there exists

$g_i \in \mathcal{G}(U_i)$ with $\beta_{U_i}(g_i) = h|_{U_i}$. Since X is paracompact, we can assume that $\mathcal{U}$ is locally finite. Now, for any $i, j \in I$, we have

$$\beta_{U_{ij}}(g_i|_{U_{ij}} - g_j|_{U_{ij}}) = h|_{U_{ij}} - h|_{U_{ij}} = 0,$$

where $U_{ij} = U_i \cap U_j$ as in our discussion of Čech cohomology. Since the sequence

$$0 \to \mathcal{F}(U_{ij}) \xrightarrow{\alpha_{U_{ij}}} G(U_{ij}) \xrightarrow{\beta_{U_{ij}}} \mathcal{H}(U_{ij})$$

is exact, there exists $f_{ij} \in \mathcal{F}(U_{ij})$ such that

$$\alpha_{U_{ij}}(f_{ij}) = g_i|_{U_{ij}} - g_j|_{U_{ij}}.$$

Then for any $i, j, k \in I$, we have

$$\alpha_{U_{ijk}}(f_{ij}|_{U_{ijk}} - f_{jk}|_{U_{ijk}}) = (g_i|_{U_{ijk}} - g_j|_{U_{ijk}}) + (g_j|_{U_{ijk}} - g_k|_{U_{ijk}}) = g_i|_{U_{ijk}} - g_k|_{U_{ijk}} = \alpha_{U_{ijk}}(f_{ik}|_{U_{ijk}})$$

Since $\mathcal{F}$ is finte, there exists a partition of unity $\eta_i \colon \mathcal{F} \to \mathcal{F}$ subordinate to the cover $\mathcal{U}$. Then

$$\eta_{j,U_{ij}}(f_{ij}) \in \mathcal{F}(U_{ij}),$$

and by construction

$$S_j := \operatorname{supp}(\eta_j) \subset U_j.$$

Consider the open set $V_{ij} = U_i \backslash S_j$. Then $U_i = U_{ij} \cup V_{ij}$ and the restriction of $\eta_{j,Uij}(f_{ij})$ to $U_{ij} \cap V_{ij}$ is zero. So, there exists $\tilde{f}ij \in \mathcal{F}(U_i)$ that restricts to $\eta_{j,U_{ij}}(f_{ij})$ on U_{ij} and to zero on V_{ij}. Set

$$t_i = \sum_{k \in I} \tilde{f})ik \in \mathcal{F}(U_i)$$

This sum is understood as follows: each point in U_i has a neighborhood where all but finitely many of the $\tilde{f}_{ik}$ are zero, so the restriction of t_i to this neighborhood makes sense. These "local" sums agree on overlaps, and hence give rise to the unique element of $\mathcal{F}(U_i)$. The sums in the computation that follows are interpreted similarly. On U_{ij}, we have

$$t_i - t_J = \sum_{k \in I}(\tilde{f}_{ik} - \tilde{f}_{jk}) = \sum_{k \in I} \eta_{k,U_{ij}}(f_{ik} - f_{jk}) = \sum_{k \in I} \eta_{k,U_{ij}}(f_{ij}) = f_{ij}$$

So,

$$\alpha_{U_{ij}}(f_i - f_j) = \alpha_{U_{ij}}(f_{ij}) = g_i|_{U_{ij}} - g_j|_{U_{ij}},$$

It follows that

$$(g_i - \alpha(f_i))|_{U_{ij}} = (g_j - \alpha(f_j))|_{U_{ij}},$$

and hence there exists $g \in \mathcal{G}(X)$ such that $g|_{U_i} = g_i - \alpha(f_i)$. Then

$$\beta_X(g)|_{U_i} = \beta_{U_i}(g_i) = h|_{U_i},$$

and therefore $\beta_X(g) = h$, proving the surjectivity of β_X

ii. If every open subset $U \subset X$ is paracompact, the preceding argument gives the exactness of

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0.$$

If $\mathcal{G}$ is flasque, then one easily deduces that $\mathcal{H}$ is flasque.

□

Remark 4.5.8. The argument we just gave shows that $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ for any fine sheaf $\mathcal{F}$ and a locally finite cover $\mathcal{U}$. In fact, in this case $\check{H}^q(\mathcal{U}, \mathcal{F}) = 0$ for all $q > 0$.

Suppose now that $(X, \mathcal{O})$ is a ringed space, and assume that for every locally finite cover $\mathcal{U} = \{U_i\}_{i \in I}$, the sheaf $\mathcal{O}$ has a partition of unity subordinate to $\mathcal{U}$ in the following sense: there exist global sections $s_i \in \mathcal{O}(X)$ for all $i \in I$ such that $\operatorname{supp}(s_i) \subset U_i$ and $\sum_{i \in I} s_i = 1$, or equivalently $\sum_{i \in I} s_{i,x} = 1$ in every stalk.

Proposition 4.5.9. *Let $(X, \mathcal{O})$ be a ringed space. Assume that for every locally finite open cover $\mathcal{U} = \{U_i\}_{i \in I}$, the sheaf $\mathcal{O}$ has a partition of unity subordinate to $\mathcal{U}$. Then every sheaf $\mathcal{F}$ of $\mathcal{O}$-modules is fine.*

Proof. Let Any section $s \in \mathcal{O}(X)$ defined an endomorphism $\tilde{s} \colon \mathcal{F} \to \mathcal{F}$ such that for any open $U \subset X$, $\tilde{s}_U \colon \mathcal{F}(U) \to \mathcal{F}(U)$ is left multiplication by $\rho_U^X(s)$. Then, if $\{s_i\}_{i \in I}$ is a partition of unity subordinate to a locally finite open cover $\mathcal{U} = \{U_i\}$, for any $\mathcal{O}$-module $\mathcal{F}$, the family $\{\tilde{s}_i\}_{i \in I}$ is a partition of unity for $\mathcal{F}$. □

Example 4.5.10. *Let X be a (connected, second-countable) n-dimensional manifold. Then X is paracompact, and so is every open subset of X. Let $\mathcal{O}$ be the sheaf of rings of smooth functions. It is a classical result that for every locally finite cover $\mathcal{U} = \{U_i\}_{i \in I}$, there is a partition of unity. It follows that every sheaf of $\mathcal{O}$-modules has a partition of unity, and hence is acyclic. In particular, the sheaves $\mathcal{A}^{(k)}$ of differential k-forms are acyclic. This is crucial for the proof of de Rham's Theorem.*

Yet another important class of acyclic sheaves is soft sheaves. We recall that given an embedding of topological spaces $\iota \colon A \hookrightarrow X$, for a sheaf $\mathcal{F}$ on X, one defined the inverse image $\iota^{-1}\mathcal{F}$ as the sheafification of the following presheaf:

$$\mathcal{G}(U) = \lim_{V \supset \iota(U)} \mathcal{F}(V).$$

In particular, there are maps $\mathcal{F}(X) \to \mathcal{G}(A) \to (\iota^{-1}\mathcal{F})(A)$. We say that $\mathcal{F}$ is soft if this composite map is surjective for every closed subset $A \subset X$. One shows that if X is paracompact, the flasque and fine sheaves of abelian groups are soft. On the other hand, for any soft sheaf $\mathcal{F}$ on a paracompact space we have $H^p(X, \mathcal{F}) = 0$ for $p > 0$, i.e. $\mathcal{F}$ is acyclic.

4.6 Čech cohomology and sheaf cohomology

Let X be a topological space, $\mathcal{U} = \{U_i\}_{i \in I}$ is an open cover of X, and $\mathcal{F}$ be a sheaf of abelian groups on X. Earlier, we constructed the Čech resolution of $\mathcal{F}$:

$$0 \to \mathcal{F} \to \check{\mathscr{C}}^0(\mathcal{U}, \mathcal{F}) \to \check{\mathscr{C}}^1(\mathcal{U}, \mathcal{F}) \to \cdots$$

where $\check{\mathscr{C}}^p(\mathcal{U},\mathcal{F})$ is the sheaf on X whose sections over an open $U \subset X$ are given by

$$\check{\mathscr{C}}^p(\mathcal{U},\mathcal{F})(U) = \prod_{(i_0,\cdots,i_p)\in I^{p+1}} \mathcal{F}(U \cap U_{i_0\cdots i_p})$$

Given an injective resolution

$$0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots,$$

there is a morphism of resolutions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & \check{\mathscr{C}}^0(\mathcal{U},\mathcal{F}) & \longrightarrow & \check{\mathscr{C}}^1(\mathcal{U},\mathcal{F}) & \longrightarrow & \cdots \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F} & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \end{array}$$

Taking global sections, we obtain a commutative diagram of complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & \check{\mathscr{C}}^0(\mathcal{U},\mathcal{F})(X) & \longrightarrow & \check{\mathscr{C}}^1(\mathcal{U},\mathcal{F})(X) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I^0(X) & \longrightarrow & I^1(X) & \longrightarrow & \cdots \end{array}$$

Since the top row computes the Čech cohomology groups $\check{H}^\bullet(\mathcal{U},\mathcal{F})$, we obtain functorial (in $\mathcal{F}$) maps $\check{H}^p(\mathcal{U},\mathcal{F}) \to H^p(X,\mathcal{F})$. Our goal is to establish this map as an isomorphism. But first, we will establish the Čech acyclicity of flasque sheaves.

Proposition 4.6.1. Let $\mathcal{F}$ be a flasque sheaf on a topological space X. Then $\check{H}^p(\mathcal{U},\mathcal{F}) = 0$ for all $p > 0$ for any open cover $\mathcal{U}$ of X.

Proof. We have seen earlier that since $\mathcal{F}$ is flasque, the Čech resolution $0 \to \mathcal{F} \to \check{\mathscr{C}}^\bullet(\mathcal{U},\mathcal{F})$ is also flasque for any open cover $\mathcal{U}$, and therefore computes the sheaf cohomology. On the other hand, by design, it always computes Čech cohomology. So, we obtain that $\check{H}^p(\mathcal{U},\mathcal{F}) \cong H^p(X,\mathcal{F})$ for all $p \geqslant 0$ and any open cover $\mathcal{U}$. But $H^p(X,\mathcal{F}) = 0$ for $p > 0$ since $\mathcal{F}$ is flasque. So, $\check{H}^p(\mathcal{U},\mathcal{F}) = 0$ for $p > 0$. □

We say that an open cover $\mathcal{U} = \{U_i\}_{i\in I}$ of X is acyclic for a sheaf $\mathcal{F}$ if for any $i_0, \cdots i_n \in I$ we have

$$H^p(U_{i_0,\cdots i_n}, \mathcal{F}|_{U_{i_0,\cdots i_n}}) = 0 \quad \text{for all } p > 0.$$

Chapter 5

Grothendieck topologies

Wojciech Tralle

5.1 Introduction

One of the goals of this talk is to motivate the notions of a Grothendieck site and a sheaf on a site as generalizations of a topological space and a sheaf on a topological space. The classical definition of a sheaf begins with a topological space X. A sheaf associates information to the open sets of X. This information can be phrased abstractly by letting $Op(X)$ be the category whose objects are the open subsets U of X and whose morphisms are the inclusion maps $V \hookrightarrow U$ of open sets U and V of X. We will call such maps open immersions, just as in the context of schemes. Then a presheaf on X is a contravariant functor from $Op(X)$ to the category of sets, and a sheaf is a presheaf that satisfies the gluing axiom (here we are including the separation axiom). The gluing axiom is phrased in terms of pointwise covering, i.e. $\{U_i\}_{i\in I}$ covers U if and only if $\bigcup_{i\in I} U_i = U$. In this definition, U_i is an open subset of X. Grothendieck topologies replace each U_i with an entire family of open subsets; in this example, U_i is replaced by the family of all open immersions $V_{ij} \to U_i$. Such a collection is called a site. Pointwise covering is replaced by the notion of a covering family; in the above example, the set of all $\{V_{ij} \to U_i\}_{j\in J_i}$ as i varies is a covering family of U. Sites and covering families can be axiomatized, and once this is done open sets and pointwise covering can be replaced by other notions that describe other properties of the space X. Finally, Grothendieck topologies are the necessary machinery for étale topology used in sheaf cohomology. It turns out that the étale topology is a Grothendieck topology only but not an ordinary topology.

Definition 5.1.1. Let $\mathcal{C}$ be a category. A **family of morphisms with a fixed target** in $\mathcal{C}$ is given by an object $U \in \mathcal{C}$, a set I and for each $i \in I$ a morphism $U_i \to U$ of $\mathcal{C}$ with target U. We use the notation $\{U_i \to U\}_{i\in I}$ to indicate this. It can happen that I is empty.

Definition 5.1.2. A **Grothendieck topology** $\mathcal{T}$ on a category $\mathcal{C}$ is a collection $Cov(\mathcal{C})$ of families of morphisms $\{\phi_i : U_i \to U\}_{i\in I}$ with a fixed target, called coverings of $\mathcal{C}$, satisfying the following three properties:

(1) If $\phi : V \to U$ is an isomorphism in $\mathcal{C}$, then $\{\phi : V \to U\}$ is a covering (e.g. $\mathrm{id}_U : U \to U$ is a covering for any $U \in \mathcal{C}$).

(2) If $\{\phi_i : U_i \to U\}_{i\in I}$ is a covering and $\{\psi_{ij} : V_{ij} \to U_i\}_{j\in J_i}$ is a covering for each $i \in I$, then $\{\phi_i \circ \psi_{ij} : V_{ij} \to U\}_{i\in I, j\in J_i}$ is a covering.

(3) If $V \to U$ is a morphism in $\mathcal{C}$ and $\{U_i \to U\}_{i\in I}$ is a covering, then the fiber products $U_i \times_U V$ exist in $\mathcal{C}$ and $\{U_i \times_U V \to V\}_{i\in I}$ is a covering.

Definition 5.1.3. A category $\mathcal{C}$ together with a Grothendieck topology $\mathcal{T}$ is called a **site**. We denote it by $(\mathcal{C}, \mathcal{T})$ or simply $\mathcal{T}$ if the category $\mathcal{C}$ is clear from the context.

Example 5.1.4. *Any ordinary topology can also be viewed as a Grothendieck topology. Given a topological space X, we define $Cov(X)$ to be the families of open covers $\{U_i \to U\}_{i\in I}$ where each U_i is open, the $U_i \to U$ are the inclusion maps, and $U = \bigcup_{i\in I} U_i$. In this case, the fibered products $U_i \times_U V$ are just $U_i \cap V$.*

In one of the sections that follow we will see an example of a Grothendieck topology that does not come from an ordinary topology.

Definition 5.1.5. A **presheaf** on a site $\mathcal{T}$ is a contravariant functor $\mathcal{F}$ from the underlying category $\mathcal{C}$ to <u>*Set*</u>.

Definition 5.1.6. For each morphism $\phi : V \to U$, $\mathcal{F}$ gives a map $\mathcal{F}(\phi) : \mathcal{F}(U) \to \mathcal{F}(V)$. When ϕ is specified or unambiguous from the context we can write $\mathcal{F}(\phi)(s) = s|_V$. Given any covering $\{\phi_i : U_i \to U\}_{i\in I}$, we can consider the map

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$$

given by $s \mapsto (s|_{U_i})_{i\in I}$. We say that $\mathcal{F}$ is **separated** if this map is injective for every covering in $Cov(\mathcal{C})$.

Definition 5.1.7. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves with values in $\mathcal{C}$ is defined as a morphism of contravariant functors.

Remark 6. If we have a covering $\{\phi_i : U_i \to U\}_{i\in I}$, then we also get coverings $\{U_i \times_U U_j \to U_i\}_{j\in I}$ and $\{U_i \times_U U_j \to U_j\}_{i\in I}$. These give rise to maps $\mathcal{F}(U_i) \to \mathcal{F}(U_i \times_U U_j)$ and $\mathcal{F}(U_j) \to \mathcal{F}(U_i \times_U U_j)$. We say that a pair of elements $s_i \in \mathcal{F}(U_i)$ and $s_j \in \mathcal{F}(U_j)$ are **compatible** if

$$s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}.$$

In order for $\mathcal{F}$ to be a sheaf, we need to be able to glue compatible collections, which motivates the following definition:

Definition 5.1.8. A **sheaf** is a separated presheaf that satisfies

$$\{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i) \mid s_i|_{U_i\times_U U_j} = s_j|_{U_i\times_U U_j} \text{ for all } i,j \in I\} = \operatorname{im}(\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i))$$

for every $\{U_i \to U\}_{i\in I} \in Cov(\mathcal{C})$. In other words, a presheaf $\mathcal{F}$ is a sheaf if for every $\{U_i \to U\}_{i\in I} \in Cov(\mathcal{C})$ the following diagram is exact in $\mathcal{C}$:

$$\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j)$$

Morphisms of sheaves are defined as morphisms of underlying presheaves.

Remark 7. Note that in the special case of topological space, $U_i \times_U U_j$ becomes $U_i \cap U_j$ and we recover the gluing sheaf axiom. In fact, our new definition of sheaf agrees with the standard definition for topological spaces.

Definition 5.1.9. Let $(\mathcal{T}, \mathcal{C})$ and $(\mathcal{T}', \mathcal{C}')$ be two sites. A **morphism of Grothendieck topologies** $\mathcal{T} \to \mathcal{T}'$ is a functor $\eta : \mathcal{C} \to \mathcal{C}'$ of the underlying categories with the following two properties:

(1) If $\{\phi_i : U_i \to U\}_{i \in I} \in Cov(\mathcal{C})$ then $\{\eta(\phi_i) : \eta(U_i) \to \eta(U)\}_{i \in I} \in Cov(\mathcal{C}')$.

(2) For $\{U_i \to U\}_{i \in I} \in Cov(\mathcal{C})$ and $V \to U$ is a morphism in $\mathcal{C}$ then the canonical morphism

$$\eta(U_i \times_U V) \to \eta(U_i) \times_{\eta(U)} \eta(V)$$

is an isomorphism for all $i \in I$.

Example 5.1.10. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $(\mathcal{T}_X, Op(X))$ and $(\mathcal{T}_Y, Op(Y))$ be the corresponding sites of open sets. We have a functor $\eta : Op(Y) \to Op(X)$ via $U \mapsto f^{-1}(U)$. We claim that this gives a morphism of Grothendieck topologies.*

(1) $\{U_i \to U\}_{i \in I} \in Cov(Op(Y)) \Leftrightarrow \bigcup_{i \in I} U_i = U$. Since the inverse image of a continuous map commutes with unions, we have $\bigcup_{i \in I} f^{-1}(U_i) = f^{-1}(U) \Rightarrow \{f^{-1}(U_i) \to f^{-1}(U)\}_{i \in I} \in Cov(Op(X))$.

(2) $\{U_i \to U\}_{i \in I} \in Cov(Op(Y))$ and $V \to U$ a morphism in $Op(Y) \Leftrightarrow U = \bigcup_{i \in I} U_i$ and $V \subseteq U \subseteq Y$. Since the inverse image of a continuous map commutes with intersections, we have (by translating our notatation)

$$f^{-1}(U_i \times_U V) = f^{-1}(U_i \cap V) = f^{-1}(U_i) \cap f^{-1}(V) = f^{-1}(U_i) \times_{f^{-1}(U)} f^{-1}(V).$$

5.2 Two more examples of Grothendieck topologies

Definition 5.2.1. A morphism $U \to V$ in $\mathcal{C}$ is called an **epimorphism**, if the map $\mathrm{Hom}(V, Z) \to \mathrm{Hom}(U, Z)$ is injective for each $Z \in \mathcal{C}$. A morphism $U \to V$ is called an **effective epimorphism**, if the following diagram is exact for each $Z \in \mathcal{C}$:

$$\mathrm{Hom}(V, Z) \longrightarrow \mathrm{Hom}(U, Z) \rightrightarrows \mathrm{Hom}(U \times_V U, Z)$$

Here the two right-hand maps are induced from the projections of $U \times_V U$ onto the first and second factor.

Definition 5.2.2. A map $U \to V$ is called a **universal effective epimorphism**, if $U \times_V V' \to V'$ is an effective epimorphism for each morphism $V' \to V$ in $\mathcal{C}$.

Remark 8. These notions generalize to families of morphisms $U_i \to V$ into a fixed object V, namely a family $\{U_i \to V\}_{i \in I}$ is a family of **epimorphisms** if

$$\mathrm{Hom}(V, Z) \to \prod_{i \in I} \mathrm{Hom}(U_i, Z)$$

is injective for each $Z \in \mathcal{C}$. It is a family of **effective epimorphisms** if the diagram

$$\mathrm{Hom}(V, Z) \longrightarrow \prod_{i \in I} \mathrm{Hom}(U_i, Z) \rightrightarrows \prod_{i,j \in I} \mathrm{Hom}(U_i \times_V U_j, Z)$$

is exact for each $Z \in \mathcal{C}$. Finally, it is a family of **universal effective epimorphisms** if $\{U_i \times_V V' \to V'\}_{i \in I}$ is a family of effective epimorphisms for each morphism $V' \to V$ in $\mathcal{C}$.

Definition 5.2.3. Let $\mathcal{C}$ be a category where fiber products exist. The **canonical topology** $\mathcal{T}$ on $\mathcal{C}$ is defined by taking as the set of coverings, the collection of all families $\{U_i \to U\}_{i \in I}$ of universal effective epimorphisms in $\mathcal{C}$.

Remark 9. One can show that for the canonical topology $\mathcal{T}$ the axioms (1), (2) and (3) of Definition 5.1.2 hold so the canonical topology gives an example of a Grothendieck topology.

Remark 10. Note that it is now immediate from the definition of $\mathcal{T}$ that each representable presheaf of sets, i.e. presheaf of the form $U \mapsto \mathrm{Hom}(U, Z)$ for a fixed $Z \in \mathcal{C}$, is a sheaf.

Definition 5.2.4. Let G be a group. We define the category of (left) G-sets as the catagory whose objects are sets X with a left G-action. Its morphisms are the G-equivariant maps (recall that if X, Y are sets with left G-action then a map $f : X \to Y$ is called G-equivariant if $f(g \cdot x) = g \cdot f(x)$ for all $g \in G, x \in X$).

Remark 11. One can show that the category of G-sets has fiber products.

Example 5.2.5. *An important example of a G-set is ${}_GG$ which is the G-set whose underlying set is G and the action is given by left multiplication.*

Proposition 5.2.6. *Let G be an arbitrary group and let $\mathcal{C}$ be the category of (left) G-sets. We declare a family of morphisms $\{\phi_i : U_i \to U\}_{i \in I}$ to be a family of coverings if*

$$U = \bigcup_{i \in I} \phi_i(U_i).$$

This way, we obtain a Grothendieck topology, denoted by $\mathcal{T}_G$ on $\mathcal{C}$, i.e. $\mathcal{T}_G$ satisfies the axioms (1), (2) and (3) of Definition 5.1.2.

Proof. (1) Let $\phi : V \to U$ be an isomorphism in $\mathcal{C}$. Then $\{\phi : V \to U\}$ is a covering because $U = \phi(V)$.

(2) Let $\{\phi_i : U_i \to U\}_{i \in I}$ be a covering and $\{\psi_{ij} : V_{ij} \to U_i\}_{j \in J_i}$ be a covering for all $i \in I$. Since $\{\psi_{ij} : V_{ij} \to U_i\}_{j \in J_i}$ is a covering we have $U_i = \bigcup_{j \in J_i} \psi_{ij}(V_{ij})$ for all $i \in I$. After applying ϕ_i we obtain

$$\phi_i(U_i) = \phi_i(\bigcup_{j \in J_i} \psi_{ij}(V_{ij})) = \bigcup_{j \in J_i} \phi_i \circ \psi_{ij}(V_{ij}).$$

Taking the union over $i \in I$ we get

$$U = \bigcup_{i \in I} \phi_i(U_i) = \bigcup_{i \in I, j \in J_i} \phi_i \circ \psi_{ij}(V_{ij})$$

where the first equality follows because $\{\phi_i : U_i \to U\}_{i \in I}$ is a covering.

(3) Let $f : V \to U$ be a morphism in $\mathcal{C}$. Let $\{\phi_i : U_i \to U\}_{i \in I}$ be a covering. We know that the fiber products $U_i \times_U V$ exist in $\mathcal{C}$ so it suffices to show that $\{p_i : U_i \times_U V \to V\}_{i \in I}$ is a covering, i.e. $V = \bigcup_{i \in I} p_i(U_i \times_U V)$, which is clear. ☐

Remark 12. A more conceptual explanation why the axioms (1), (2) and (3) hold for G-sets is as follows. The fiber products of left G-sets are taken in the category of sets. In other words, the forgetful functor from the category of G-sets to the category of sets commutes with inverse limits (because it has an adjoint, the functor $S \mapsto G \times S$). Thus, taking pullbacks preserve the notion of covering, and it is easy to see the other axioms are satisfied too: if we have a cover of each of the U_i (which cover U), then collecting them gives a cover of U. Similarly, an isomorphism is a cover. This is obvious from the definition.

Remark 13. We have seen that each left G-set Z defines a sheaf on the topology $\mathcal{T}_G$ via $U \to \mathrm{Hom}_G(U, Z)$ (see the example of canonical topology). We will show that we obtain all sheaves of sets on $\mathcal{T}_G$ in this way.

Proposition 5.2.7. *The functor $Z \to Hom_G(-, Z)$ is an equivalence between the category of (left) G-sets and the category of sheaves of sets on $\mathcal{T}_G$. The functor $\mathcal{F} \to \mathcal{F}(G)$ from the category of sheaves of sets on $\mathcal{T}_G$ to the category of (left) G-sets is a quasi-inverse to $Z \to Hom_G(-, Z)$.*

Proof. Here the structure of $\mathcal{F}(G)$ as a G-set is defined as follows. For $g \in G$ and $s \in \mathcal{F}(G)$ let $g \cdot s = \mathcal{F}(\alpha_g)(s)$, where $\alpha_g : G \to G$ is the map $g' \mapsto g'g$. This is a left action because:

$$(g_1 \cdot g_2) \cdot s = \mathcal{F}(\alpha_{g_1 g_2})(s) = \mathcal{F}(\alpha_{g_2} \circ \alpha_{g_1})(s) = \mathcal{F}(\alpha_{g_1})(\mathcal{F}(\alpha_{g_2})(s)) = g_1 \cdot (g_2 \cdot s).$$

The composite of the functors $Z \to \mathrm{Hom}_G(-, Z)$ and $\mathcal{F} \to \mathcal{F}(G)$ assigns to each left G-set Z the left G-set $\mathrm{Hom}_G(G, Z)$, which can be canonically identified with Z. The composite of $\mathcal{F} \to \mathcal{F}(G)$ and $Z \to \mathrm{Hom}_G(-, Z)$ assigns to each sheaf $\mathcal{F}$ the sheaf $\mathrm{Hom}_G(-, \mathcal{F}(G))$. We have to show that there is an isomorphism

$$\mathcal{F} \xrightarrow{\cong} \mathrm{Hom}_G(-, \mathcal{F}(G))$$

which is functorial in $\mathcal{F}$. Let U be a left G-set. Then $\{\phi_u : G \to U\}_{u \in U}$ is a covering in the topology $\mathcal{T}_G$, where $\phi_u(g)$ is defined for each $u \in U$ by $\phi_u(g) = gu$. For a sheaf $\mathcal{F}$ we have the exact diagram

$$\mathcal{F}(U) \longrightarrow \prod_{u \in U} \mathcal{F}(G) \rightrightarrows \prod_{u,v \in U} \mathcal{F}(G \times_U G)$$

corresponding to this covering. It remains to show that the image of the injective map $\Phi : \mathcal{F}(U) \to \prod_{u \in U} \mathcal{F}(G) = \mathrm{Hom}(U, \mathcal{F}(G))$ is precisely the subset $\mathrm{Hom}_G(U, \mathcal{F}(G))$ of G-equivariant maps $U \to \mathcal{F}(G)$. Once we prove this, we get an isomorphism

$$\mathcal{F}(U) \xrightarrow{\cong} \mathrm{Hom}_G(U, \mathcal{F}(G))$$

which is functorial in U, hence an isomorphism of sheaves, and it is functorial in $\mathcal{F}$.

Let us prove this remaining claim, i.e. prove that $\mathrm{im}(\Phi) = \mathrm{Hom}_G(U, \mathcal{F}(G)) \subseteq \prod_{u \in U} \mathcal{F}(G)$. Denote by

$d^1 = p_1^*$ and $d^2 = p_2^*$ the maps induced by the projections $p_1 : G \times_U G \to G$ and $p_2 : G \times_U G \to G$. We have

$$\begin{array}{ccccc} \mathcal{F}(U) & \xrightarrow{\Phi} & \prod_{u\in U} \mathcal{F}(G) & = & \mathrm{Hom}(U, \mathcal{F}(G)) \\ s & \longmapsto & \Big[\Phi(s) : U & \longrightarrow & \mathcal{F}(\mathcal{G})\Big] \\ & & u & \longmapsto & \phi_u^*(s) = \mathcal{F}(\phi_u)(s) \end{array}$$

First, note that $\Phi(s)$ is G-equivariant. In fact,

$$g \cdot (\Phi(s)(u)) = \alpha_g^* \circ \phi_u^*(s) = (\phi_u \circ \alpha_g)^*(s) = \Phi(s)(gu).$$

We want to show that the equalizer of d^1 and d^2 is $\mathrm{Hom}_G(U, \mathcal{F}(G))$. If $s = (s_u)_{u\in U} \in \prod_{u\in U} \mathcal{F}(G)$ then $d^1(s)$ and $d_2(s)$ are families $d^1(s)_{u,v}$ and $d^2(s)_{u,v}$ in $\mathcal{F}(G \times_U G) = \mathcal{F}(\{(g, h) \in G \times G \,|\, gu = hv\})$ because the following diagram commutes

$$\begin{array}{ccc} G \times_U G & \xrightarrow{p_2} & G \\ {\scriptstyle p_1}\downarrow & & \downarrow \\ G & \longrightarrow & U \end{array}$$

Also $d^1(s)_{u,v} = p_1^*(s_u)$ and $d^2(s) = p_2^*(s_v)$. First, let us show that $s = (s_u)_{u\in U}$ is G-equivariant. We have $p_1^*(s_u) = p_2^*(s_v)$ for all $u, v \in U$. We want to show that $s_{gu} = \alpha_g^*(s)(u)$ for all $u \in U, g \in G$. Fix $u \in U, f \in G$ and let $v = fu$. Consider the map $\iota_g : G \to G\times_U G = \{(g, h) \in G\times G \,|\, gu = hv\}$ given by $g \mapsto (gf, g)$. Observe the following identities: $p_2 \circ \iota_f = \mathrm{id}_G$ and $p_1 \circ \iota_f = \alpha_f$. We compute $s_{fu} = s_v = \iota_f^* p_2^* s_v = \iota_f^* p_1^* s_u = \alpha_f^* s_u$, so s is G-equivariant as required. Conversely, let $s = (s_u)_{u\in U}$ be G-equivariant, i.e. $\alpha_f^* s_u = s_{fu}$. We want to show that $p_1^* s_u = p_2^* s_v$ for all $u, v \in U$ in $\mathcal{F}(\{(g, h) \in G \times G \,|\, gu = hv\})$. Let $E_{u,v} = \{g \in G \,|\, fu = v\}$ and consider the map $\Psi : G \times E_{u,v} \to G \times_U G$ given by $(g, f) \mapsto (gf, g)$. We can view $G \times E_{u,v}$ as the disjoint union $\bigsqcup_{f\in E_{u,v}} G$ so that Ψ is a map of G-sets, componentwise given by the maps ι_f. Note that Ψ is an isomorphism with inverse $(g, h) \mapsto (h, h^{-1}g)$. Since $\mathcal{F}$ is a sheaf, it is additive, in the sense that we have $\mathcal{F}(A \sqcup B) = \mathcal{F}(A) \times \mathcal{F}(B)$. Thus,

$$\mathcal{F}(G \times_U G) \cong \mathcal{F}(G \times E_{u,v}) \cong \prod_{f\in E_{u,v}} \mathcal{F}(G).$$

By construction, an element $x \in \mathcal{F}(G \times_U G)$ maps to the family $(\iota_f^* x)_{f\in E_{u,v}}$. Since this is an isomorphism, to prove $p_1^* s_u = p_2^* s_v$, it is enough to prove this equality after applying ι_f^* for all $f \in E_{u,v}$, i.e. for all f such that $fu = v$. We compute

$$\iota_f^* p_1^* s_u = \alpha_f^* s_u = s_{fu} = s_v = \iota_f^* p_2^* s_v$$

which finishes the proof. $\square$

Remark 14. Alternatively, we could have shown the required isomorphism $\mathcal{F}(U) \to \mathrm{Hom}_G(U, \mathcal{F}(G))$ by observing the following. Any left G-set U can be written as a disjoint union of orbits, $U = \bigsqcup_{i\in I} \mathcal{O}_i$. Since $\mathcal{F}$ is a sheaf, we have an isomorphism $\mathcal{F}(U) = \mathcal{F}(\bigsqcup_{i\in I} \mathcal{O}_i) \cong \prod_{i\in I} \mathcal{F}(\mathcal{O}_i)$. Similarly, $\mathrm{Hom}_G(U, \mathcal{F}(G)) \cong \prod_{i\in I} \mathrm{Hom}_G(\mathcal{O}_i, \mathcal{F}(G))$. Thus, we may assume that U is a single orbit. Consider the covering $\{\phi_u : G \to U\}_{u\in U}$ with $\phi_u(g) = gu$. Fix $u \in U$. We get the map $\Phi = \mathcal{F}(\phi_u) = \phi_u^* : \mathcal{F}(U) \to \mathcal{F}(G)$. Denote by $H = \mathrm{Stab}_G(u)$ the stabilizer of u in G. Since U is a single orbit we have $\mathrm{Hom}_G(U, \mathcal{F}(G)) = \mathcal{F}(G)^H$

the subset of H-invariant elements and one can check that $G \times_U G = \{(g, gh) \,|\, g \in G, h \in H\}$. Hence, $\mathcal{F}(G \times_U G) = \mathcal{F}(\bigsqcup_{h\in H} G) \cong \prod_{h\in H} \mathcal{F}(G)$. Thus, the sheaf property reads as follows:

$$\mathcal{F}(U) \xrightarrow{\Phi} \mathcal{F}(G) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \prod_{h\in H} F(G)$$

where p_1^* and p_2^* into a factor $\mathcal{F}(G)$ differ by multiplication by an element $h \in H$. Hence, $\mathcal{F}(U) \cong \mathrm{im}(\Phi) \cong \mathcal{F}(G)^H \cong \mathrm{Hom}_G(U, \mathcal{F}(G))$.

5.3 Sheafification and its categorical properties

We would like to have a procedure for turning an arbitrary presheaf into a sheaf. If $\mathcal{C}$ is not a topological space then it no longer makes sense to talk about the stalks of a (pre)sheaf and the familiar method of sheafification does not work. Fortunately, there is a more general construction that works for a presheaf defined on any site.

Lemma 5.3.1. *Given a pair of coverings $\{U_i \to U\}_{i\in I}$ and $\{V_j \to U\}_{j\in J}$ of a given object U of the site $\mathcal{T}$, there exists a covering which is a common refinement.*

Proof. Since $\mathcal{T}$ is site we have that for every $i \in I$ the family $\{V_j \times_U U_i \to U_i\}_{j\in J}$ is a covering. And, then another axiom implies that $\{V_j \times_U U_i \to U\}_{i,j\in I}$ is a covering of U. Clearly, this covering refines both given coverings. □

The central ingredient of this construction is the zeroth Čech cohomology group:

Definition 5.3.2. If $\mathcal{F}$ is a presheaf on $\mathcal{C}$ and $\mathcal{U} = \{U_i \to U\}_{i\in I}$ is an element of $Cov(\mathcal{C})$, then we define the zeroth Čech cohomology group by

$$\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \Big\{(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i) \,|\, s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j} \text{ for all } i, j \in I\Big\}.$$

In other words, these are collections of elements with compatible restrictions.

Remark 15. If $\mathcal{F}$ is a sheaf then $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$ is isomorphic to $\mathcal{F}(U)$.

We also need to know what a morphism of coverings is:

Definition 5.3.3. If $\mathcal{U} = \{U_i \to U\}_{i\in I}$ and $\mathcal{V} = \{V_j \to V\}_{j\in J}$ are coverings, then a **morphism of coverings** $\mathcal{V} \to \mathcal{U}$ consists of three pieces of information denoted by a triple (χ, α, χ_j). Here $\alpha : J \to I$ is a map of sets, $\chi : V \to U$ is a morphism, and $\chi_j : V_j \to U_{\alpha(j)}$ is a morphism for each $j \in J$. Finally, we require that the following diagram commutes for each $j \in J$:

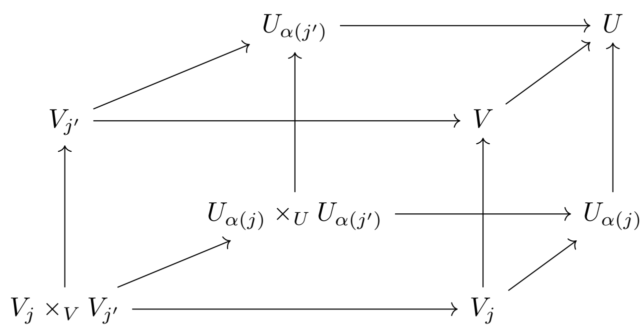
$$\begin{array}{ccc} V_j & \xrightarrow{\chi_j} & U_{\alpha(j)} \\ \downarrow & & \downarrow \\ V & \xrightarrow{\chi} & U \end{array}$$

Remark 16. Write χ_j^* to denote $\mathcal{F}(\chi_j)$. Then any morphism $(\chi, \alpha, \chi_j) : \mathcal{V} \to \mathcal{U}$ induces a map

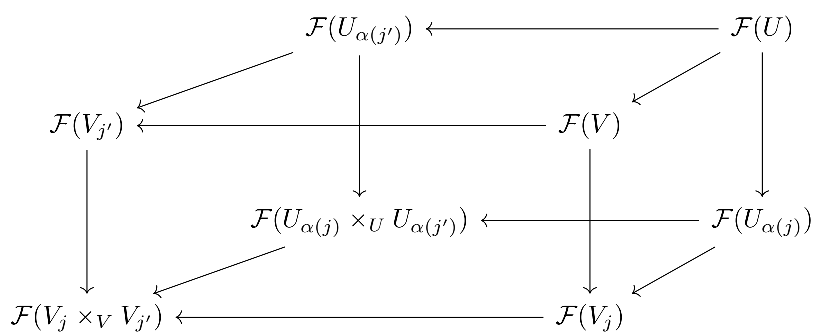
$$\Psi : \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^0(\mathcal{V}, \mathcal{F}) \text{ defined by } (s_i)_{i \in I} \mapsto (\chi_j^*(s_{\alpha(j)}))_{j \in J}.$$

Proposition 5.3.4. *This map is well-defined and depends only on* χ*.*

Proof. First we check that the map is well-defined. That is, for $(s_i)_{i \in I} \in \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$, we need to check that $(\chi_j^*(s_{\alpha(j)}))_{j \in J} \in \check{\mathrm{H}}^0(\mathcal{V}, \mathcal{F})$. For $j, j' \in J$ we need to show that $\chi_j^*(s_{\alpha(j)})$ and $\chi_{j'}^*(s_{\alpha(j')})$ have the same restriction in $\mathcal{F}(V_j \times_V V_{j'})$. The relevant objects in $\mathcal{C}$ can be put into a commutative diagram as follows



After applying $\mathcal{F}$, we get



Here $(\chi_j^*(s_{\alpha(j)}))_{j \in J}$ comes from $s_{\alpha(j)} \in \mathcal{F}(U_{\alpha(j)})$ and $\chi_{j'}^*(s_{\alpha(j')})$ comes from $s_{\alpha(j')} \in \mathcal{F}(U_{\alpha(j')})$. Since $(s_i)_{i \in I} \in \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$, we know that

$$s_{\alpha(j)}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}} = s_{\alpha(j')}|_{U_{\alpha(j)} \times_U U_{\alpha(j')}},$$

and from the commutativity of the diagram above it follows that

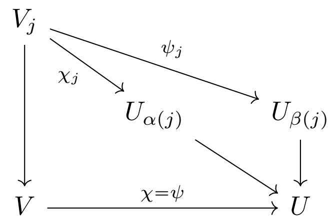
$$\chi_j^*(s_{\alpha(j)})|_{V_j \times_V V_{j'}} = \chi_{j'}^*(s_{\alpha(j')})|_{V_j \times_V V_{j'}}.$$

Thus, $(\chi_j^*(s_{\alpha(j)})) \in \check{\mathrm{H}}^0(\mathcal{V}, \mathcal{F})$ as claimed.

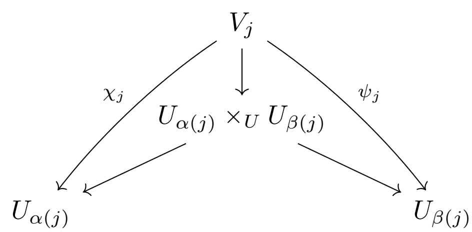
Now we want to show that the map depends only on χ. Suppose that we have two morphisms (χ, α, χ_j) and (ψ, β, ψ_j) from $\mathcal{V}$ to $\mathcal{U}$ with $\chi = \psi$. Then given an arbitrary $(s_i)_{i \in I} \in \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$, we want to show that for each j we have

$$\chi_j^*(s_{\alpha(j)}) = \psi_j^*(s_{\beta(j)}).$$

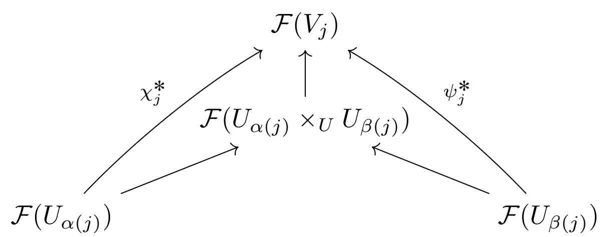
Since (χ, α, χ_j) and (ψ, β, ψ_j) are morphisms, we get the following diagram



This means that (χ_j, ψ_j) defines a map from V_j to $U_{\alpha(j)} \times_U U_{\beta(j)}$ making the following diagram commutative



After applying $\mathcal{F}$ we get



So to show that $\chi_j^*(s_{\alpha(j)}) = \psi_j^*(s_{\beta(j)})$ it is enough to show that $s_{\alpha(j)}$ and $s_{\beta(j)}$ have the same image in $\mathcal{F}(U_{\alpha(j)} \times_U U_{\beta(j)})$. But this is true by assumption since $(s_i)_{i \in I} \in \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$, so we're done. $\square$

Remark 17. If we require that $U = V$ and $\chi = \mathrm{id}_U$, then Proposition 5.3.4 tells us that we get exactly one induced map $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^0(\mathcal{V}, \mathcal{F})$.

Remark 18. Now fix an object $U \in \mathcal{C}$. For $\mathcal{U}$ and $\mathcal{V}$ covers of U, write $\mathcal{U} \leqslant \mathcal{V}$ if there exists a morphism $\mathcal{V} \to \mathcal{U}$ with $\chi = \mathrm{id}_U$. Then the covers of U form a direct system. Explicitly, if $\mathcal{U} = \{U_i \to U\}_{i \in I}$ and $\mathcal{V} = \{V_j \to U\}_{j \in J}$ are coverings, then $\mathcal{W} = \{U_i \times_U V_j \to U\}_{i \in I, j \in J}$ is a covering and $\mathcal{U}, \mathcal{V} \leqslant \mathcal{W}$. To see that $\mathcal{U} \leqslant \mathcal{W}$ it suffices to observe we have a morphism $(\mathrm{id}_U, \alpha, \chi_{ij}) : \mathcal{W} \to \mathcal{U}$ where α is given by $(i, j) \mapsto i$ and χ_{ij} is the obvious map $U_i \times_U V_j \to U_i$. The proof of $\mathcal{V} \leqslant \mathcal{W}$ is similar.

Whenever we have $\mathcal{U} \leqslant \mathcal{V}$, we get a unique map $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^0(\mathcal{V}, \mathcal{F})$. This means that we can view the groups $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$ as a direct system. This motivates the following definition:

Definition 5.3.5. We define $\mathcal{F}^+$ by taking the direct limit of this direct system:

$$\mathcal{F}^+(U) = \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}).$$

Remark 19. We make $\mathcal{F}^+$ into a presheaf by defining restriction maps. Let $\chi : V \to U$ be any morphism. Then we need to define a homomorphism $\mathcal{F}^+(U) \to \mathcal{F}^+(V)$. An element $\bar{x} \in \mathcal{F}^+(U)$ can be represented by an element $x \in \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$ for some covering $\mathcal{U} = \{U_i \to U\}_{i \in I}$. By axiom (3) of the Definition 5.1.2, we know that there is a covering $\mathcal{V} = \{U_i \times_U V \to V\}_{i\in I}$, and there is an obvious map $\mathcal{V} \to \mathcal{U}$. Let y be the image of x under the induced map $\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}^0(\mathcal{V}, \mathcal{F})$ and let $\bar{y}$ be the image of y in the direct limit that defines $\mathcal{F}^+(V)$. Then $\bar{x} \mapsto \bar{y}$ is the desired restriction map. By Proposition 5.3.4, this map is well-defined.

Lemma 5.3.6. *The map $\theta : \mathcal{F} \to \mathcal{F}^+$ has the following property: For every object U of $\mathcal{C}$ and a section $s \in \mathcal{F}^+(U)$ there exists a covering $\mathcal{U} = \{U_i \to U\}_{i\in I}$ such that $s|_{U_i}$ is in the image of $\theta_{U_i} : \mathcal{F}(U_i) \to \mathcal{F}^+(U_i)$.*

Proof. Let $\{U_i \to U\}_{i\in I}$ be a covering such that s arises from the element $(s_i)_{i\in I} \in \check{\mathrm{H}}^0(\{U_i \to U\}_{i\in I}, \mathcal{F})$. According to Proposition 5.3.4, we may consider the covering $\{U_i \to U_i\}_{i\in I}$ and the (obvious) morphism of coverings $\{U_i \to U_i\}_{i\in I} \to \{U_i \to U\}_{i\in I}$ to compute the pullback of s to an element of $\mathcal{F}^+(U_i)$. And in fact, using this covering we get exactly $\theta(s_i)$ for the restriction of s to U_i. □

Remark 20. We would like to know if $\mathcal{F}^+$ is a sheaf. The answer is not necessarily yes, but we have the following useful result:

Theorem 14. Let $\mathcal{F}$ be a presheaf of sets. The following hold:

(1) The presheaf $\mathcal{F}^+$ is separated.

(2) If $\mathcal{F}$ is separated, then $\mathcal{F}^+$ is a sheaf and the map of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ is an injective.

(3) If $\mathcal{F}$ is a sheaf, then $\theta : \mathcal{F} \to \mathcal{F}^+$ is an isomorphism.

(4) The presheaf $\mathcal{F}^{++}$ is always a sheaf.

Proof. Proof of (1). Suppose that $s, s' \in \mathcal{F}^+(U)$ and suppose that there exists some covering $\{U_i \to U\}_{i\in I}$ such that $s|_{U_i} = s'|_{U_i}$ for all i. We now have three coverings of U: the covering $\{U_i \to U\}_{i\in I}$ above, a covering $\mathcal{U}$ for s as in Lemma 5.3.6 and a similar covering $\mathcal{U}'$ for s'. By Lemma 5.3.1, we can find a common refinement, say $\{W_j \to U\}_j$. This means we have $s_j, s'_j \in \mathcal{F}(W_j)$ such that $s|_{W_j} = \theta(s_j)$, similarly for $s'|_{W_j}$, and such that $\theta(s_j) = \theta(s'_j)$. This last equality means that there exists some covering $\{W_{jk} \to W_j\}_k$ such that $s_j|_{W_{jk}} = s'_j|_{W_{jk}}$. Then since $\{W_{jk} \to U\}$ is a covering, we see that s, s' map to the same element of $\check{\mathrm{H}}^0(\{W_{jk} \to U\}, \mathcal{F})$ as desired.

Proof of (2). It is clear that $\mathcal{F} \to \mathcal{F}^+$ is injective because all the maps $\mathcal{F}(U) \to \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$ are injective. It is also clear that, if $\mathcal{U} \to \mathcal{U}'$ is a refinement, then $\check{\mathrm{H}}^0(\mathcal{U}', \mathcal{F}) \to \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$ is injective. Now, suppose that $\{U_i \to U\}_{i\in I}$ is a covering, and let $(s_i)_{i\in I}$ be a family of elements of $\mathcal{F}^+(U_i)$ satisfying the sheaf condition $s_i|_{U_i \times_U U_j} = s_j|_{U_i \times_U U_j}$ for all $i, j \in I$. Choose coverings (as in Lemma 5.3.6) $\{U_{ij} \to U_i\}$ such that $s_{i|U_{ij}}$ is the image of the (unique) element $s_{ij} \in \mathcal{F}(U_{ij})$. The sheaf condition implies that s_{ij} and $s_{i'j'}$ agree over $U_{ij} \times_U U_{i'j'}$ because it maps to $U_i \times_U U_{i'}$ and we have equality there. Hence $(s_{ij}) \in \check{\mathrm{H}}^0(\{U_{ij} \to U\}, \mathcal{F})$ gives rise to an element $s \in \mathcal{F}^+(U)$. One easily verifies that $s_{|U_i} = s_i$.

Proof of (3) is immediate from the definitions because the sheaf property says exactly that every map $\mathcal{F} \to \check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F})$ is bijective (for every covering $\mathcal{U}$ of U).

Statement (4) is now obvious. □

Definition 5.3.7. Let $\mathcal{T}$ be a Grothendieck topology on a category $\mathcal{C}$ and let $\mathcal{F}$ be a presheaf of sets on $\mathcal{C}$. The sheaf $\mathcal{F}^\# := \mathcal{F}^{++}$ together with the canonical map $\mathcal{F} \to \mathcal{F}^\#$ is called the **sheaf associated to** $\mathcal{F}$.

Remark 21. Aside from the fact that $\mathcal{F}^\#$ is always a sheaf, this construction has the categorical properties that sheafification should have. In fact, we have the following result:

Theorem 15. *The canonical map $\mathcal{F} \to \mathcal{F}^\#$ has the following universal property: For any map $\mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf of sets, there is a unique map $\mathcal{F}^\# \to \mathcal{G}$ such that $\mathcal{F} \to \mathcal{F}^\# \to \mathcal{G}$ equals the given map.*

Proof. The association $\mathcal{F} \to (\mathcal{F} \to \mathcal{F}^+)$ is a functor. In fact, if $\mathcal{F} \to \mathcal{G}$ is a map of presheaves then one easily checks that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^+. \end{array}$$

Thus, we also have the following commutative diagram

$$\begin{array}{ccccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ & \longrightarrow & \mathcal{F}^{++} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{G}^+ & \longrightarrow & \mathcal{G}^{++}. \end{array}$$

By Theorem 14, the lower horizontal maps are isomorphisms. The uniqueness follows from Lemma 5.3.6 which says that every section of $\mathcal{F}^\#$ locally comes from sections of $\mathcal{F}$. □

5.4 Direct and inverse image presheaves

Definition 5.4.1. Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We denote by

$$u^p : \mathcal{PSh}(\mathcal{D}) \to \mathcal{PSh}(\mathcal{C})$$

the functor that associates to the presheaf $\mathcal{G}$ on $\mathcal{D}$, the presheaf $u^p\mathcal{G} = \mathcal{G} \circ u$, called the **direct image presheaf** of $\mathcal{G}$.

Remark 22. For any object $V \in \mathcal{D}$, let $\mathcal{I}_V^u$ denote the category with objects,

$$Ob(\mathcal{I}_V^u) = \{(U, \phi) \,|\, U \in \mathcal{C},\, \phi : V \to u(U)\}$$

and morphisms,

$$Mor_{\mathcal{I}_V^u}((U, \phi), (U', \phi')) = \{f : U \to U' \text{ morphism in } \mathcal{C} \,|\, u(f) \circ \phi = \phi'\}.$$

We sometimes drop the subscript ^{u} from the notation and we simply write $\mathcal{I}_V$. We will use these categories to define the inverse image presheaf as a left adjoint to the functor u^p.

Lemma 5.4.2. *Let $u : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Assume*

(1) the category $\mathcal{C}$ has a final object X and $u(X)$ is a final object of $\mathcal{D}$, and

(2) the category $\mathcal{C}$ has fiber products and u commutes with them.

Then the index categories $(\mathcal{I}_V^u)^{opp}$ are filtered.

Proof. We see that $\mathcal{I}_V$ is a (possibly empty) disjoint union of directed categories. Hence it suffices to show that $\mathcal{I}_V$ is connected.

First, we show that $\mathcal{I}_V$ is nonempty. Namely, let X be the final object of $\mathcal{C}$, which exists by assumption. Let $V \to u(X)$ be the morphism coming from the fact that $u(X)$ is final in $\mathcal{D}$ by assumption. This gives an object of $\mathcal{I}_V$.

Second, we show that $\mathcal{I}_V$ is connected. Let $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$ be in $Ob(\mathcal{I}_V)$. By assumption, $U_1 \times U_2$ exists and $u(U_1 \times U_2) = u(U_1) \times u(U_2)$. Consider the morphism $\phi : V \to u(U_1 \times U_2)$ corresponding to (ϕ_1, ϕ_2) by the universal property of products. Then the object $\phi : V \to u(U_1 \times U_2)$ maps to both $\phi_1 : V \to u(U_1)$ and $\phi_2 : V \to u(U_2)$. □

Definition 5.4.3. Given $g : V' \to V$ in $\mathcal{D}$ we get a functor $\bar{g} : \mathcal{I}_V \to \mathcal{I}_{V'}$, by setting $\bar{g}(U, \phi) = (U, \phi \circ g)$ on objects. Given a presheaf $\mathcal{F}$ on $\mathcal{C}$, we obtain a functor

$$\mathcal{F}_V : \mathcal{I}_V^{opp} \to Sets, \ \ (U, \phi) \mapsto \mathcal{F}(U).$$

In other words, $\mathcal{F}_V$ is a presheaf of sets on $\mathcal{I}_V$. Note that we have $\mathcal{F}_{V'} \circ \bar{g} = \mathcal{F}_V$. We define

$$u_p\mathcal{F}(V) = \varinjlim_{\mathcal{I}_V^{opp}} \mathcal{F}_V.$$

As a direct limit, we obtain for each $(U, \phi) \in Ob(\mathcal{I}_V)$ a canonical map $\mathcal{F}(U) \xrightarrow{c(\phi)} u_p\mathcal{F}(V)$. For $g : V' \to V$ as above there is a canonical restriction map $g^* : u_p\mathcal{F}(V) \to u_p\mathcal{F}(V')$ compatible with $\mathcal{F}_{V'} \circ \bar{g} = \mathcal{F}_V$. It is the unique map that for all $(U, \phi) \in Ob(\mathcal{I}_V)$ the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{c(\phi)} & u_p\mathcal{F}(V) \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle g^*} \\ \mathcal{F}(U) & \xrightarrow{c(\phi \circ g)} & u_p\mathcal{F}(V') \end{array}$$

commutes. The uniqueness of these maps implies that we obtain a presheaf. This presheaf will be denoted $u_p\mathcal{F}$ and called the **inverse image presheaf** of $\mathcal{F}$.

Lemma 5.4.4. *There is a canonical map $\mathcal{F}(U) \to u_p\mathcal{F}(u(U))$, which is compatible with restriction maps (on $\mathcal{F}$ and on $u_p\mathcal{F}$).*

Proof. This is just the map $c(\mathrm{id}_{u(U)})$ introduced above. □

Remark 23. Note that any map of presheaves $\mathcal{F} \to \mathcal{F}'$ gives rise to compatible systems of maps between functors $\mathcal{F}_V \to \mathcal{F}'_V$ and hence to a map of presheaves $u_p\mathcal{F} \to u_p\mathcal{F}'$. In other words, we have defined a functor

$$u_p : \mathcal{PSh}(\mathcal{C}) \to \mathcal{PSh}(\mathcal{D}).$$

Theorem 16. The functor u_p is a left adjoint to the functor u^p. In other words, the formula

$$Mor_{\mathcal{PSh}(\mathcal{C})}(\mathcal{F}, u^p\mathcal{G}) = Mor_{\mathcal{PSh}(\mathcal{D})}(u_p\mathcal{F}, \mathcal{G})$$

holds bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

Proof. Let $\mathcal{G}$ be a presheaf on $\mathcal{D}$ and let $\mathcal{F}$ be a presheaf on $\mathcal{C}$. We will show that the displayed formula holds by constructing maps either way. One can show that they are mutually inverse.

Given a map $\alpha : u_p\mathcal{F} \to \mathcal{G}$, we get $u^p\alpha : u^p u_p\mathcal{F} \to u^p\mathcal{G}$. By Lemma 5.4.4 there is a map $\mathcal{F} \to u^p u_p\mathcal{F}$. The composition of the two gives the desired map. Note that by construction it is functorial in everything in sight.

Conversely, given a map $\beta : \mathcal{F} \to u^p\mathcal{G}$, we get a map $u_p\beta : u_p\mathcal{F} \to u_p u^p\mathcal{G}$. We claim that the functor $u^p\mathcal{G}_Y$ on $\mathcal{I}_Y$ has a canonical map to the constant functor with value $\mathcal{G}(Y)$. Namely, for every object (X, ϕ) of $\mathcal{I}_Y$, the value of $u^p\mathcal{G}_Y$ on this object is $\mathcal{G}(u(X))$ which maps to $\mathcal{G}(Y)$ by $\mathcal{G}(\phi) = \phi^*$. This is a transformation of functors because $\mathcal{G}$ is a functor itself. This leads to a map $u_p u^p\mathcal{G}(Y) \to \mathcal{G}(Y)$. Another trivial verification shows that this is functorial in Y leading to a map of presheaves $u_p u^p\mathcal{G} \to \mathcal{G}$. The composition $u_p\mathcal{F} \to u_p u^p\mathcal{G} = \mathcal{G}$ is the desired map. □

Chapter 6

Cohomology of sheaves for Grothendieck topologies

Valia Gazaki

6.1 Lecture 1

Let $\mathfrak{X} = (\mathcal{C}, \mathcal{T})$ be a Grothendieck site (always assume $\mathcal{C}$ has fiber products). **Goal:** For an object $U \in \mathcal{C}$ and an abelian sheaf $\mathcal{F}$ on $\mathfrak{X}$ to define sheaf cohomology $\{H^i(U, \mathcal{F})\}_{i \geqslant 0}$. Need:

1. Define the abelian categories $\mathcal{P}sh(\mathfrak{X})$, $Sh(\mathfrak{X})$ and the notion of exactness.
2. "Proceed like in the topological space situation", i.e. define $H^i(U, -)$ = ith right derived functor of

$$\Gamma(U, -) : Sh(\mathfrak{X}) \to Ab, \mathcal{F} \mapsto \mathcal{F}(U)$$

For 2, we need to show that $Sh(\mathfrak{X})$ has enough injective objects (non-trivial).

Definition 6.1.1. 1. An abelian presheaf on $\mathcal{F}$=contravariant functor $\mathcal{F} : \mathcal{C} \to Ab$, and $\mathcal{P}Sh(\mathfrak{X})$ = category of abelian presheaves on $\mathfrak{X}$ (FACT: $\mathcal{P}Sh(\mathfrak{X})$ is an abelian category). A sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $\mathcal{P}Sh(\mathfrak{X})$ is called exact if $\forall U \in Ob(\mathcal{C})$ the sequence of abelian groups $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact.

2. An abelian sheaf $\mathcal{F}$ on $\mathfrak{X}$ = abelian presheaf such that the equalizer diagram holds: If $\{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ covering, then

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is exact. (separated: if $s \in \mathcal{F}(U)$ such that $\mathcal{F}(\phi_i)(s) = 0$ for all $i \in I$, then $s = 0$. gluing: Write $\phi_{ij} : U_i \times_U U_j \to U_i$, $phi_{ji} : U_i \times_U U_j \to U_j$. Suppose $(s_i)_{i \in I} \in \prod_i \mathcal{F}(U_i)$ is such that $\mathcal{F}(\phi_{ij})(s_i) = \mathcal{F}(\phi_{ji})(s_j)$ for all i, j, then there exists $s \in \mathcal{F}(U)$ such that $\mathcal{F}(\phi_i)(s) = s_i$.)

Denote $Sh(\mathfrak{X})$ = category of abelian sheaves of $\mathfrak{X}$.

Fact: $Sh(\mathfrak{X})$ is an abelian category.

- For a morphism $\mathcal{F} \xrightarrow{\phi} \mathcal{G}$ in $Sh(\mathfrak{X})$, $\ker\phi = \ker\phi$ in $\mathcal{P}Sh(\mathfrak{X})$, i.e. $(\ker\phi)(U) = \ker(\mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U))$ is a sheaf.

- Let $a : \mathcal{P}Sh(\mathfrak{X}) \mapsto Sh(\mathfrak{X})$, $\mathcal{F} \mapsto \mathcal{F}^{\#}(= \mathcal{F}^{++} = \theta(\theta(\mathcal{F})))$ be the sheafification functor. Existence of cokernels: Let $\phi : \mathcal{F} \to \mathcal{G}$ morphism in $Sh(\mathfrak{X})$, we view ϕ as a morphism of presheaves, then $\text{coker}\phi(U) = \mathcal{G}(U)/\phi(\mathcal{F}(U))$ makes coker^p presheaf, we sheafify and have $\text{coker}\phi := (\text{coker}^p)^{\#}$.

Remark 6.1.2.
- A morphism $\phi : \mathcal{F} \to \mathcal{G}$ in $Sh(\mathfrak{X})$ is epi $\iff$ $\text{coker}\phi(U) = 0$ $\iff$ $\forall U \in \mathcal{C}$ the map $\phi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is not necessarily onto but $\forall s \in \mathcal{G}(U), \exists$ covering $\{U_i \xrightarrow{\phi_i} U\}_i$ in J such that $\mathcal{G}(\phi_i)(s)$ is in the image of $\mathcal{F}(U_i) \xrightarrow{\phi_{U_i}} G(U_i)$.

- A sequence $0 \to \mathcal{F}' \xrightarrow{\phi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \to 0$ is a short exact sequence in $Sh(\mathfrak{X})$ $\iff$ $\ker\phi = 0, \ker\psi = \text{im}\phi, \text{coker}\psi = 0$ (i.e. ψ surjective "locally on coverings").

Note: $\text{im}\phi = (\text{im}\phi^p)^{\#}$ where $\text{im}\phi^p(U) = \text{im}(\mathcal{F}'(U) \to \mathcal{F}(U))$.

Theorem 6.1.3. *1. The categories $\mathcal{P}Sh(\mathfrak{X})$, $Sh(\mathfrak{X})$ have arbitrary limits and colimits.*

2. The forgetful functor $i : Sh(\mathfrak{X}) \to \mathcal{P}Sh(\mathfrak{X})$ is left exact.

3. The sheafification functor $a : \mathcal{P}Sh(\mathfrak{X}) \to Sh(\mathfrak{X})$ is exact.

Proof. Same as topological spaces. □

Note: $a + i$ is an adjoint pair $\implies$ i is left exact and a is right exact. To show that a is left exact, it suffices to show that $\theta : \mathcal{P}Sh(\mathfrak{X}) \to \mathcal{P}Sh(\mathfrak{X}), \mathcal{F} \mapsto \mathcal{F}^+$ is left exact. Recall: For $U \in \mathcal{C}$, $\mathcal{F}^+(U) = \varinjlim \check{H}^0(\underline{U}, \mathcal{F})$. Then $\varinjlim$ exact means that it suffices to show for a fixed covering $\underline{U} = \{U_i \xrightarrow{\phi_i} U\}_i$, $\check{H}^0(\underline{U}, -)$ is exact. This follows by definition!

Remark 6.1.4. So far with this new notion of Grothendieck topology most stuff seems to generalize naturally. We don't have: stalks. For, we don't work with a fixed topological space X and points $x \in X$ but rather we use all objects $U \in \mathcal{C}$.

Recall: For X top space, $Sh(X)$ has enough injectives, this was shown using: $\mathcal{F} \in Sh(X) \implies$ for each $x \in X$ let $i_x : F_x \hookrightarrow I_x$ = injective $\mathbb{Z}$-mod. Take $\alpha_x : \{x\} \hookrightarrow I_x$ closed embedding, and then $- \to F \to (\alpha_x)_x(I_x) =$ injective! (i.e. construction of injective objects used stalks) Next: Will show $\mathcal{P}Sh(\mathfrak{X}), Sh(\mathfrak{X})$ have enough

injectives using a method of Grothendieck.

Lemma 6.1.5 (Definition)**.** *Let $\mathcal{A}$ be an abelian category. A family $(E_i)_{i\in I}$ of objects of $\mathcal{A}$ is called a family of generators if the following 2 equivalent conditions hold:*

1. *The functor*

$$\epsilon : \mathcal{A} \to AbGp,$$

given by $A \mapsto \prod_{i\in I} Hom_{\mathcal{A}}(E_i, A)$ *is faithful, i.e.* $\forall A, A' \in \mathcal{A}$ *the map:*

$$Hom_{\mathcal{A}}(A, A') \to Hom_{Ab}(\prod_{i\in I} Hom_{\mathcal{A}}(E_i, A), \prod_{i\in I} Hom_{\mathcal{A}}(E_i, A')),$$

given by $[\varphi : A \to A'] \mapsto [(f_i : E_i \to A)_i \mapsto (\phi \circ f_i)_i]$ *is injective.*

2. $\forall A \in \mathcal{A}$ *and every* $B \subsetneq A$*, there exists* $i \in I$ *and there exists a morphism* $E_i \xrightarrow{\lambda_i} A$ *such that* λ_i *does not factor through* B*.*

Proof. $a \implies b$: Let $B \subsetneq A$ in $\mathcal{A}$, then we have a short exact sequence $0 \to B \xrightarrow{\iota} A \xrightarrow{\pi} A/B \to 0$ with $A/B \neq 0$. Then $\pi \in \mathrm{Hom}_{\mathcal{A}}(A, A/B), \pi \neq 0$, so by faithfulness there exists $i \in I$ such that $\pi_* : \mathrm{Hom}(E_i, A) \to \mathrm{Hom}(E_i, A/B)$ is $\neq 0$. Hence there exists $f : E_i \to A$ such that $E_i \xrightarrow{f} A \xrightarrow{pi} A/B$ which means f does not factor through B. $b \implies a$: Let $A, A' \in \mathcal{A}$ and $f \in \mathrm{Hom}_{\mathcal{A}}(A, A')$ such that $f_{*,i} : \mathrm{Hom}_{\mathcal{A}}(E_i, A) \to$

$\mathrm{Hom}_{\mathcal{A}}(E_i, A')$ is zero $\forall i \in I$. If $f \neq 0$, then $\ker f \subsetneq A$, take $B = \ker f$ and apply (2). □

Example 6.1.6. $R =$ *unital ring,* $\mathcal{A} =$ *left* R*-mods, then* $\{E = R\}$ *is a generator for* $\mathcal{A}$*.*

Proof. $\mathcal{A} \xrightarrow{\epsilon} AbGp, A \mapsto \mathrm{Hom}_R(R, A) \simeq A \implies \epsilon =$ forgetful functor and its clearly faithful. □

Definition 6.1.7. We sat the category $\mathcal{A}$ has the property:

(AB3) : if any $\bigoplus_{i\in I} A_i$ exist in A

(AB4) : if (AB3) holds and forming direct sum is an exact functor

(AB5) : if (AB3) holds and taking $\varinjlim$ is exact.

Similarly, $(AB3^*) - (AB5^*)$ are defined dually using products/ limits.

Definition 6.1.8. An abelian category $\mathcal{A}$ is called a Grothendieck category if (AB5) holds and $\mathcal{A}$ has a family of generators.

Remark 6.1.9. If $\mathcal{A}$ is Grothendieck, then $\mathcal{A}$ has a single generator E. For, if $\{E_i\}_{i\in I}$ is a family of generators, set $E = \bigoplus_i E_i$, then $\prod_i \mathrm{Hom}_{\mathcal{A}}(E_i, A) \simeq \mathrm{Hom}_{\mathcal{A}}(E, A)$.

Theorem 6.1.10. *If* $\mathcal{A}$ *is Grothendieck, then* $\mathcal{A}$ *has enough injectives, i.e.* $\forall A$ *in*$\mathcal{A}$*,* $\exists I$ *injective object such that* $0 \to A \xrightarrow{f} I$ *in* $\mathcal{A}$*. (Recall:* I *is injective* $\iff$ $Hom_{\mathcal{A}}(-, I)$ *is exact* $\iff$ $\forall 0 \to A \to B$ *in* $\mathcal{A}$ *the map* $Hom_{\mathcal{A}}(B, I) \to Hom_{\mathcal{A}}(A, I) \to 0$ *is exact.*

Proof. Lecture 2. □

Theorem 6.1.11. *Let* $\mathfrak{X} = (\mathcal{C}, \mathcal{T})$ *be a site. Then* $\mathcal{P}Sh(\mathfrak{X})$ *and* $Sh(\mathfrak{X})$ *have enough injectives.*

Proof. STS $\mathcal{P}Sh(\mathfrak{X})$, $Sh(\mathfrak{X})$ are Grothendieck. (AB5) follows similarly to topological spaces. Need to construct a family of generators.

1. Generators for $\mathcal{PSh}(\mathfrak{X})$: Let $U \in \mathcal{C}$. Consider the presheaf $\mathbb{Z}_U^p : V \in \mathcal{C} \mapsto \bigoplus_{V \xrightarrow{f} U} \mathbb{Z} \cdot f := \mathbb{Z}_U^p(V)$. Note:

   (a) $\mathbb{Z}_U^p$ is a presheaf with restriction maps: if $V \xrightarrow{\phi} W$ in $\mathcal{C}$, then $\mathbb{Z}_U^p(W) \xrightarrow{\mathbb{Z}_U^p(\phi)_V^W} \mathbb{Z}_U^p(V), \mathbb{Z} \cdot f \mapsto \mathbb{Z}(f \circ \phi)$.

   (b) There exists a canonical map $\beta : \mathrm{Hom}_{\mathcal{PSh}(\mathfrak{X})}(\mathbb{Z}_U^p, \mathcal{F}) \to \mathcal{F}(U)$. For, let $\Phi : \mathbb{Z}_U^P \to \mathcal{F}$ morphism of presheaves, then $\Phi_I : \mathbb{Z}_U^P(U) \to \mathcal{F}(U)$, take $\Phi_U(1 \cdot id_U) \in \mathcal{F}(U)$. We claim that β is an isomorphism, i.e.e $\mathbb{Z}_U^p$ represents the functor $\mathcal{PSh}(\mathfrak{X}) \xrightarrow{\Gamma(U,-)} Ab, \mathcal{F} \mapsto \mathcal{F}(U)$.

   Proof. STS every homomorphism $\mathbb{Z}_U^p \xrightarrow{\Phi} \mathcal{F}$ is fully determined by $\Phi_U(1 \cdot id_U) \in \mathcal{F}(U)$. Let $\Phi : \mathbb{Z}_U^p \to \mathcal{F}$. Let $V \in \mathcal{C}$. If there does not exists a morphism $V \to U$, then $\mathbb{Z}_U^p(V) = 0$ and $\Phi_V = 0$, so assume $\exists V \xrightarrow{f} U$, and we want to see how $\Phi_V |_{\mathbb{Z} \cdot f}$ is defined. We have

   $$\bigoplus_{U \xrightarrow{s} U} \mathbb{Z} \cdot s = \mathbb{Z}_U^p(U) \xrightarrow{f^*} \mathbb{Z}_U^p(V) = \bigoplus_{V \xrightarrow{f'} U} \mathbb{Z} \cdot f'$$

   Observe: $\mathbb{Z} \cdot id_U \mapsto \mathbb{Z} \cdot f$ by definition. Moreover, since Φ is a presheaf homomorphism we have the following commutative diagram:

   $$\begin{array}{ccc} \mathbb{Z}_U^p(U) & \xrightarrow{\Phi_U} & \mathcal{F}(U) \\ {\scriptstyle \mathbb{Z}_U^p(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(f)} \\ \mathbb{Z}_U^p(V)) & \xrightarrow{\Phi_V} & \mathcal{F}(V) \end{array}$$

   Hence $\Phi_v(1 \cdot f) = \mathcal{F}(f)(\Phi_U(1 \cdot id_U))$ by commutativity and everything is fully determined by $\Phi_U(1 \cdot id_U)$. □

   Next we claim that the family $\{\mathbb{Z}_U^p\}_{U \in \mathcal{C}}$ are generators of $\mathcal{PSh}(\mathfrak{X})$

   Proof. The functor $\epsilon : \mathcal{PSh}(\mathfrak{X}) \to Ab, \mathcal{F} \mapsto \prod_{U \in \mathcal{C}} \mathrm{Hom}_{\mathcal{PSh}(\mathfrak{X})}(\mathbb{Z}_U^p, \mathcal{F}) = \prod_{U \in \mathcal{C}} \mathcal{F}(U)$ is faithful. For, let $\mathcal{F} \xrightarrow{\Phi} \mathcal{F}'$, $\Phi \neq 0 \implies \exists u \in \mathcal{C}$ such that $\Phi_U \neq 0$. □

2. Generators for $Sh(\mathfrak{X})$: Define $\mathbb{Z}_U = (\mathbb{Z}_U^p)^{\#}$. Using adjunction between forgeful functor and $\#$, $\mathbb{Z}_U$ represents $\Gamma(U, -)$ on $Sh(\mathfrak{X})$.

□

6.2 Lecture 2

Theorem 6.2.1. *Let $\mathcal{A}$ be an abelian category which is Grothendieck. Then $\mathcal{A}$ has enough injective objects.*

Reminders:

1. $I \in \mathcal{A}$ is injective $\iff$ $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is exact $\iff$ we have the following commutative diagram in $\mathcal{A}$

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \longrightarrow & A \\ & & {\scriptstyle f}\downarrow & \swarrow{\scriptstyle \exists \tilde{f}} & \\ & & I & & \end{array}$$

2. $\mathcal{A}$ is Grothendieck $\iff$ $\mathcal{A}$ has family of generators and satisfies AB5.

3. $\mathcal{A}$ Grothendieck $\implies$ $\mathcal{A}$ has a single generator $U \in \mathcal{A}$, i.e. the functor $\epsilon : \mathcal{A} \to AbGp, A \mapsto \mathrm{Hom}_{\mathcal{A}}(U, A)$ is faithful $\iff$ $\forall B \subsetneq A$ in $\mathcal{A} \exists f : U \to A$ such that $f(U) \nsubseteq B$.

Proposition 6.2.2. Suppose $\mathcal{A}$ is Grothendieck. Let $I \in \mathcal{A}$. Then I is injective $\iff$ we always have the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & V & \longrightarrow & U = \text{generator} \\ & & f\downarrow & \swarrow \exists \tilde{f} = \text{extension of } f & \\ & & I & & \end{array}$$

i.e. enough to check condition for $0 \to V \to U$.

Proof. Let

$$\begin{array}{ccccc} 0 & \longrightarrow & B & \longrightarrow & A \\ & & f\downarrow & & \\ & & I & & \end{array}$$

in $\mathcal{A}$. We want to show that f extends to $\tilde{f} : A \to I$. Let $\mathcal{P} = \{g : B' \to I$ such that $B \subset B' \subset A$ and $g|_B = f\}$, Then $\mathcal{P} \neq \varnothing$ since $f \in \mathcal{P}$ since $f \in \mathcal{P}$. Because $\mathcal{A}$ has colimits, we can apply Zorn's lemma and obtain that $\mathcal{P}$ has a maximal element. We may assume that $f : B \to I$ is maximal. Now we want to show that $B = A$. Suppose not, then $B \subsetneq A$. U is a generator implies that there exists $j : U \to A$ such that $j(U) \nsubseteq B$. Set $B' = B + j(U)$ so that $B \subsetneq B' \subset A$. We will sow that f extends to B' (which gives us the contradiction). Let $V = j^{-1}(B)$, then there exists a commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & V & \longrightarrow & U \\ & & j\downarrow & & j\downarrow \\ 0 & \longrightarrow & B & \longrightarrow & B' \\ & & f\downarrow & \swarrow \exists h & \\ & & I & & \end{array}$$

So we get $V \xrightarrow{\psi} U \times B \xrightarrow{\phi} B' \to 0, v \mapsto (v, -j(v))$ and $\phi(U, b) = j(U) + b$. Claim: The above sequence is exact:

- $j : U \twoheadrightarrow B'$ by definition
- $(u, b) \in \ker \phi \iff b = -j(u) \implies u = j^{-1}(B) = V$. We have

$$\begin{array}{ccccccc} V & \xrightarrow{\psi} & U \times B & \xrightarrow{\phi} & B' & \longrightarrow & 0 \\ & & \downarrow \tilde{f} & & \downarrow g(\text{want}) & & \\ & & I & & I & & \end{array}$$

So it is enough to construct $\tilde{f} : U \times B \to I$ such that $\tilde{f} \circ \psi = 0$. Define $\tilde{f} : U \times B \to I, (u, b) \mapsto h(u) + f(b)$. Note that $\tilde{f}(\psi(v)) = \tilde{f}(v, -j(v)) = h(v) - f \circ j(v) = f \circ j(v) - f \circ j(v) = 0$.

□

Sketch of proof of Theorem 6.1.10. Will construct (not additive) functor $I : \mathcal{A} \to \mathcal{A}$ with $I(A)$ injective for all $A \in \mathcal{A}$ and $0 \to A \to I(A)$. Step 1: Let $A \in \mathcal{A}$, then define $S(A) := \{g_i : V_i \to A \text{ where } 0 \to V_i \to U\}$.

Consider $S(A)$ as an index set, and consider the morphism $\epsilon_1 : \bigoplus V_i \to A \times (\bigoplus_{S(A)} U), v_i \mapsto (-g_i(v_i), v_i)$. Let $I_1(A) := \text{coker}(\epsilon_1)$. Let $f(A) : A \hookrightarrow A \times (\bigoplus_{S(A)} U) \twoheadrightarrow I_1(A), a \mapsto [a, 0]$. Easy check: $f(A)$ is injective (follows by AB4,AB5). Key Property: For every $g_i \in S(A)$ we have a diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & V_i & \longrightarrow & U \\ & & \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle \exists \tilde{g}_i} \\ 0 & \longrightarrow & A & \xrightarrow[f(A)]{} & I_1(A) \end{array}$$

Proof. $f(A) \circ g_i(V_i) = [g_i(V_i), 0]$ and since $[-g_i(V_i), V_i] = 0$ in $I_1(A)$, we have $f(A) \circ g_i(V_i) = [0, V_i]$ and so $\tilde{g}_i : U \to I_1(A), u \mapsto [0, u]$ (ith component). □

Next: Define an inductive process using transfinite induction.
Construction: For any ordinal number i define an object $I(A) \in \mathcal{A}$ and for 2 ordinal numbers $i \leqslant j$ an injective morphism $I_i(A) \hookrightarrow I_j(A)$ such that for $i < i_0$ =fixed ordinal $\{I_i(A)\}_{i<i_0}$ forms an inductive system.

- For $i = 0$, $I_0(A) = A$.
- For $i = 1$, $I_1(A)$, $I_0(A) = A \xhookrightarrow{f(A)} I_1(A)$ as in Step 1.
- If the construction has been carried out for ordinals $< i$ and $i = j + 1$, set $I_i(A) = I_1(I_j(A))$ and $I_j(A) \xhookrightarrow{f(I_j(A))} I + i(A)$

Let k be the smallest ordinal number whose cardinality is larger than the set of all subobjects of U. Take $I(A) := I_k(A)$ (If i is limit ordinal then set $I_i(A) = \varinjlim_{j<i} I_j(A)$). Claim: $I(A)$ is injective. Sketch: Previous

proposition shows that it is enough to consider the diagrams

$$\begin{array}{ccccc} 0 & \longrightarrow & V & \longrightarrow & U \\ & & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle \exists \tilde{g}} & \\ & & I_k(A) = I(A) & & \end{array}$$

Idea: Show $(*) g(V) \subset I_i(A)$ for some $i < k$. If yes, apply step 1.

$$\begin{array}{ccccccc} 0 & \longrightarrow & V & \longrightarrow & U & & \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \exists \tilde{g}_i} & & \\ 0 & \longrightarrow & I_i(A) & \xrightarrow[f(I_1(A))]{} & I(I_i(A)) & = & I_{i+1}(A) \subset I_k(A) \end{array}$$

For (*): AB5 $\implies$ $V = \varinjlim_{i<k} g^{-1}(I_i(A))$. The set of subobjects of V has cardinality $< k$ $\implies$ $\varinjlim_{i<k} g^{-1}(I_{i_0}(A))$ for some $i_0 < k$ (otherwise V would have cardinality k). □

6.2.1 Sheaf Cohomolgy

Recall: If $\phi : \mathcal{F} \to \mathcal{G}$ morphism in $Sh(\mathfrak{X})$, $\mathfrak{X} = (\mathcal{C}, \mathcal{T})$ a site, then $\ker\phi = \ker\phi^p$, i.e. $(\ker\phi)(U) = \ker\phi_U$ for all $U \in \mathcal{C}$. While $\text{coker}\phi = (\text{coker}\phi^p)^{\#}$. In general for $U \in C$, the functor $\Gamma(U, -) : Sh(\mathfrak{X}) \to AbGp, \mathcal{F} \mapsto \mathcal{F}(U)$ is left (but not right) exact. Define: $\mathrm{H}^i(U, -) := R^i\Gamma(U, -)$ =ith right derived functor of $\Gamma(U, -)$, i.e.

for $\mathcal{F} \in Sh(\mathfrak{X})$ take an injective resolution $0 \to \mathcal{F} \to I^1 \to I^2 \to \dots$, get a complex of abelian groups $0 \to I^1(U) \xrightarrow{\phi^1} I^2(U) \xrightarrow{\phi^2} \dots$, and define $\mathrm{H}^i(U, \mathcal{F}) := \frac{\ker d^i}{\mathrm{im} d^{i-1}}$. Note that $\mathrm{H}^0(U, \mathcal{F}) = \ker d^1 = \mathcal{F}(U)$. Similarly to the case of topological spaces, 2 different injective resolutions give chain homotopic complexes

and so the cohomological groups are independent of choice.

Example 6.2.3. *X topological space, and $\mathfrak{X}$ the induced Grothendieck site (where coverings=open covers in traditional sense). Then $\mathrm{H}^i(U, \mathcal{F})$ =usual sheaf cohomology groups as defined earlier for $U \in O_p(X)$.*

Direct+Inverse image sheaves

Let $\mathfrak{X} = (\mathcal{C}, \mathcal{T}), \mathfrak{X}' = (\mathcal{C}', \mathcal{T}')$. Let $f : \mathfrak{X} \to \mathfrak{X}'$ be a morphism of sites. Recall: This means $f : \mathcal{C} \to \mathcal{C}'$ is a covariant functor such that

1. $\{U_i \xrightarrow{g_i} U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C}) \implies \{f(U_i) \xrightarrow{f(g_i)} f(U)\}_i \in \mathrm{Cov}(\mathcal{C}')$.
2. For $\{U_i \xrightarrow{g_i} U\}_i \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ morphism in $\mathcal{C}$, we have $f(U_i \times_U V) \xrightarrow{\sim} f(U_i) \times_{f(U)} f(V)$.

Wojciech's Lecture: f induces morhpsim $f^p : \mathcal{P}Sh(\mathfrak{X}') \to \mathcal{P}Sh(\mathfrak{X})$ with $f^p\mathcal{F}(U) = \mathcal{F}(f(U))$ for $U \in \mathcal{C}, \mathcal{F} \in \mathcal{P}Sh(\mathfrak{X}')$.

Proposition 6.2.4. Suppose $\mathcal{F}$ is a sheaf on $\mathfrak{X}'$. Then $f^p\mathcal{F}$ is a sheaf on $\mathfrak{X}$, i.e. f induces a morhpsim $f^s : Sh(\mathfrak{X}') \to Sh(\mathfrak{X})$.

Proof. Let $\underline{U} = \{U_i \xrightarrow{g_i} U\}_i \in \mathrm{Cov}(\mathcal{C})$. We want to show that

$$\begin{array}{ccccccc} f^s\mathcal{F}(U) & \longrightarrow & \prod_i f^s\mathcal{F}(U_i) & \rightrightarrows & \prod_{i,j} f^s\mathcal{F}(U_i \times_U U_j) & & \\ \| & & \| & & \| & & \\ \mathcal{F}(f(U)) & \longrightarrow & \prod_i \mathcal{F}(f(U_i)) & \rightrightarrows & \prod_{i,j} \mathcal{F}(f(U_i \times_U U_j)) & \simeq & \prod_{i,j} \mathcal{F}(f(U_i) \times_{f(U)} f(U_j)) \end{array}$$

is exact. Then the claim follows by (1) from the previous recall since $\{f(U_i) \xrightarrow{f(g_i)} f(U)\}_i$ is a covering in $\mathcal{C}'$ and $\mathcal{F}$ is a sheaf on $\mathfrak{X}'$. $\square$

Inverse Image

Define: $f_s : Sh(\mathfrak{X}) \to Sh(\mathfrak{X}'), \mathcal{F} \mapsto (f_p\mathcal{F})^{\#}$. Recall: $(f_p\mathcal{F})(V) = \varinjlim_{I_v^{op}} \mathcal{F}_v = \varinjlim_{(U,\varphi) \in I_v^{op}} \mathcal{F}(U)$ where I_v has objects (U, φ) such that $U \in \mathcal{C}$, $\varphi : V \to f(U)$.

Proposition 6.2.5. 1. f_s is left adjoint to f^s and hence f_s is right exact, f^s is left exact.

2. f_s commutes with colimits. If f_s exact, then f^s(injective)=injective.

Proof. 1. Let $\mathcal{F} \in Sh(\mathfrak{X}), \mathcal{G} \in Sh(\mathfrak{X}')$. We want to show that

$$\begin{array}{ccc} \mathrm{Hom}_{Sh(\mathfrak{X}')}(f_s\mathcal{F},\mathcal{G}) & \simeq & \mathrm{Hom}_{Sh(\mathfrak{X})}(\mathcal{F}, f^s\mathcal{G}) \\ \wr\| & & \| \\ \mathrm{Hom}_{\mathcal{P}Sh(\mathfrak{X}')}(f_p\mathcal{F},\mathcal{G}) & \simeq & \mathrm{Hom}_{\mathcal{P}Sh(\mathfrak{X})}(\mathcal{F}, f^p\mathcal{G}) \end{array}$$

where the equality follows since $f^p\mathcal{G}$ is a sheaf and the isomorphism on the second row follows from Wojciech's results.

□

Definition 6.2.6. $R^if^s\mathcal{F}$ =ith derived functor of f^s=$\mathrm{H}^i(f^sI^\bullet)$, $I^\bullet$ =injective resolution of $\mathcal{F}$.

Remark 6.2.7. The functors f^s, f_s correspond to f_*, f^{-1} for traditional spaces. The notation confustion stems from: If $X \xrightarrow{f} Y$ continuous map of topological spaces, then we have $f : O_p(Y) \to O_p(X), U \mapsto f^{-1}(U)$ morphism of sites.

Remark 6.2.8. Additional Properties of cohomology:

1. As usual if $\mathfrak{X}$ site and have short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $Sh(\mathfrak{X})$, then we can get a long exact sequence for every $U \in \mathcal{C}$,

$$0 \to \mathrm{H}^0(U,\mathcal{F}') \to \mathrm{H}^0(U,\mathcal{F}) \to \mathrm{H}^0(U,\mathcal{F}'') \xrightarrow[\text{connecting}]{\delta} \mathrm{H}^1(U,\mathcal{F}') \to \dots$$

2. Similarly for a morphism of sites $\mathfrak{X} \xrightarrow{f} \mathfrak{X}'$ and every short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ in $Sh(\mathfrak{X}')$, we get a long exact sequence

$$\dots \to R^nf^s\mathcal{F}' \to R^nf^s\mathcal{F} \to R^nf^s\mathcal{F}'' \xrightarrow{\delta} R^{n+1}f^s\mathcal{F}' \to \dots$$

Theorem 6.2.9. *Let $\mathfrak{X}$ be a site and $\mathcal{F} \in Sh(\mathfrak{X})$. Let $g : V \to U$ be a morphism in $\mathcal{C}$. Then there exists a canonical restriction homomorphism $g^* : \mathrm{H}^i(U,\mathcal{F}) \to \mathrm{H}^i(V,\mathcal{F})$ for all $i \geqslant 0$. For $i = 0$, the map coincides with $\mathcal{F}(U) \to \mathcal{F}(V)$ the restriction map of $\mathcal{F}$. Thus we obtain an abelian presheaf $\underline{\mathrm{H}^i}(\mathcal{F}) : \mathcal{C} \to Ab, U \mapsto \mathrm{H}^i(U,\mathcal{F})$.*

Proof. Since $\mathrm{H}^i(-,\mathcal{F})$ was constructed using injective resolutions, we ahve a universal δ-functor. Thus, the result follows by universality (as for topological spaces). Since for $i = 0$ we have

$$\begin{array}{ccc} \mathrm{H}^0(U,\mathcal{F}) & \longrightarrow & \mathrm{H}^0(V,\mathcal{F}) \\ \| & & \| \\ \mathcal{F}(U) & \xrightarrow[\text{restriction}]{} & \mathcal{F}(V) \end{array}$$

g^* extends to $g^* : \mathrm{H}^i * (U,\mathcal{F}) \to \mathrm{H}^i(V,\mathcal{F})$, for all $i \geqslant 0$. □

Theorem 6.2.10. *Let $f : \mathfrak{X} \to \mathfrak{X}'$ morphism of sites. Then for all $\mathcal{F} \in Sh(\mathfrak{X}')$ and for all $i \geqslant 0$, $R^if^s\mathcal{F} =$ sheaf associated to the presheaf $U \mapsto \mathrm{H}^i(f(U),\mathcal{F})$.*

Proof. Let $0 \to \mathcal{F} \to I^0 \to I^i \to \dots$ injective resolution of $\mathcal{F}$ in $Sh(\mathfrak{X}')$. Then we get a complex in $Sh(\mathfrak{X})$:

$$0 \to f^s I^0 \to f^s I^1 \to f^s I^2 \to \dots$$

and $R^i f^s \mathcal{F} = \frac{\ker(f^s I^i \to f^s I^{i+1})}{\operatorname{im}(f^s I^{i-1} \to f^s I^i)} = \frac{\ker^p}{(\operatorname{im}^p)^\#} = \left(\frac{\ker^p}{\operatorname{im}^p}\right)^\# \dots$. □

6.3 Lecture 3

Reminders:

1. Let $\mathfrak{X} = (\mathcal{C}, \mathcal{T})$ Grothendieck site, we showed that $Sh(\mathfrak{X})$ has enough injectives, so for $U \in \mathcal{C}$, $\mathcal{F} \in Sh(\mathfrak{X})$ we defined sheaf cohomology groups $H^i(U, \mathcal{F}) := R^i\Gamma(U < \mathcal{F})$. we sometimes use notation $H^i(\mathcal{T}; U, \mathcal{F})$ to remember the site. If $\mathcal{C}$ has final object e, write $H^i(\mathcal{T}, \mathcal{F}) := H^i(\mathcal{T}; e, \mathcal{F})$.

2. Grothendieck spectral sequence (composition of functors). Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ abelian categories with $\mathcal{A}$, $\mathcal{B}$ having enough injectives. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ be composition of 2 left exact functors. Suppose F(injective) $= G$-acyclic, then we get a spectral sequence $E_2^{p,q} = (R^pG \circ R^qF)(A) \Rightarrow R^{p+1}(G \circ F)(A)$, and there exists edge homomorphisms $E_2^{h,0} \to E^n, E^n \to E_2^{0,n}$.

Example 6.3.1. *X = topological space, $\mathcal{C} = O_p(X) = \{U \text{ open} \subset X\}$. Then $\mathcal{T} = \{\text{open coverings}\} \implies X$ is a final object of $\mathcal{C}$. $\mathcal{F} \in Sh(\mathcal{C}, \mathcal{T})$ is the usual sheaf on X and $H^i(\mathcal{T}, \mathcal{F}) = H^i(X, \mathcal{F})$ as previously defined.*

Example 6.3.2 (Main example)**.** *G group, $\mathcal{T}_G$ = Grothendiect topology on left G-sets. Recall: there exist and equivalence of abelian categories:*

$$Sh(\mathcal{T}_G) \xrightarrow{\sim} \{\text{left } G\text{-modules}\}$$
$$\mathcal{F} \mapsto \mathcal{F}(G)$$
$$Hom_G(-, A) \leftarrow\!\shortmid A$$

$\mathcal{T}_G$ has a final object e = 1−element G-set, so $H^i(\mathcal{T}_G, \mathcal{F}) = H^i(\mathcal{T}; e, \mathcal{F})$. Consider the composition of functors:

$$\{\text{left } G\text{-mods}\} \xrightarrow{\Psi} Sh(\mathcal{T}_G) \xrightarrow{\Gamma(e,-)} Ab, A \mapsto Hom_G(-, A), \mathcal{F} \mapsto \Gamma(e, \mathcal{F})$$

Note: $\Gamma(e, Hom_G(-, A) = Hom_G(e, A) = A^G \implies \Gamma(e, -) \circ \Psi = (\ )^G$ G-invariants, so for A a left G-module $R^n(\Gamma(e, -) \circ \Psi)(A) = H^n(G, A)$. Grothendieck spectral sequence gives edge homomorphisms $\phi_n : E_2^{n,0} \to E^n$, and so $H^n(\mathcal{T}_G, Hom_G(-, A)) \to H^n(G, A)$. Because the functor Ψ is a n equivalence between abelian categories, it is exact, and so the ϕ_n are $\simeq$ for all n. Thus, using the new language of Grothendieck sites we recovered group cohomology!

6.3.1 The Hochschild-Serre spectral sequence

Suppose $H \lhd G$. Consider the composition of functors

$$\{G\text{-mods}\} \xrightarrow{\Phi} \{G/H\text{-mods}\} \xrightarrow{\Psi} AbGps$$

$$A \mapsto A^H \mapsto B^{G/H}$$

Note 1: $R^i\Phi(A) = H^i(H, A)$
Note 2: $\Psi \circ \Phi(A) = A^G \implies R^i(\Psi \circ \Phi) = H^i(G, A)$.
Hence we get a Grothendieck spectral sequence:

$$E_2^{pq} = H^i(G/H, H^q(H, A)) \Rightarrow H^{p+q}(G, A)$$

for $A \in G$-mods.
5-lower term exact sequence:

$$0 \to H^1(G/H, A^H) \to H^1(G, A) \to H^1(H, A)^{G/H} \to H^2(G/H, A^H) \to H^2(G, A).$$

Q: How to compute group cohomology? Use relation to Čech!

6.3.2 Čech Cohomology

$\mathfrak{X} = (\mathcal{C}, \mathcal{T})$ be a site. Let $U \in \mathcal{C}$ and $\underline{U} = \{U_i \xrightarrow{\phi_i} U\}_{i \in I}$ be a covering. Let $\mathcal{F} \in \mathcal{P}Sh(\mathfrak{X})$. Recall from previous section that $\check{\mathrm{H}}^0(\underline{U}, \mathcal{F}) := \ker(\prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$. Thus: If $\mathcal{F}$ is a sheaf, then $\check{\mathrm{H}}^0(\underline{U}, \mathcal{F}) = \mathcal{F}(U)$. Consider this as a functor:

$$\mathcal{P}Sh(\mathfrak{X}) \xrightarrow{\check{\mathrm{H}}(\underline{U}, -)} Ab$$

$$\mathcal{F} \mapsto \check{\mathrm{H}}(\underline{U}, \mathcal{F})$$

It is left exact and if we precompose with the forgetful functor $Sh(\mathfrak{X}) \xrightarrow{i} \mathcal{P}Sh(\mathfrak{X})$ we get a composition of left exact functors. Note: $\check{\mathrm{H}}^0(\underline{U}, -) \circ i = \Gamma(U, -) : Sh(\mathfrak{X}) \to Ab$.

Definition 6.3.3. Let $\mathcal{F}$ be a presheaf. Define $\check{\mathrm{H}}^q(\underline{U}, \mathcal{F}) := R^q\check{\mathrm{H}}^0(\underline{U}, -)$ = right derived functors of $\check{\mathrm{H}}^0(\underline{U}, -)$.

Description in terms of cochains: For $q \geqslant 0$ define the group of q-cochains with values in $\mathcal{F}$:

$$\check{C}^q(\underline{U}, \mathcal{F}) := \prod_{(i_0, \ldots, i_q) \in I^{q+1}} \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_q})$$

and the differential $d^q : \check{C}^q(\underline{U}, \mathcal{F}) \to \check{C}^{q+1}(\underline{U}, \mathcal{F})$ is defined as follows: For each $j \in \{0, \ldots, q\}$ let $\hat{j} : U_{i_0} \times_U \ldots \times_U U_{i_q} \to U_{i_0} \times_U \cdots \times_U \hat{U}_{i_j} \ldots \times_U U_{i_q}$ be the projection. $\mathcal{F}$ presheaf implies we get the restriction map $\mathcal{F}(\hat{j}) : \mathcal{F}(U_{i_0} \times_U \cdots \times_U \hat{U}_{i_j} \ldots \times_U U_{i_q}) \to \mathcal{F}(U_{i_0} \times_U \ldots \times_U U_{i_q})$ and $d^q(s) = \sum_{j=0}^{q+1} (-1)^j \mathcal{F}(\hat{j})(s_{i_0}, \ldots, \hat{i}_j, \ldots, i_{q+1})$. This way we get a complex:

$$0 \to \check{C}^0(\underline{U}, \mathcal{F}) \xrightarrow{d^0} \check{C}^1(\underline{U}, \mathcal{F}) \xrightarrow{d^1} \ldots$$

Theorem 6.3.4. *$\check{\mathrm{H}}^i(\underline{U}, \mathcal{F})$ is the cohomology of the above complex, i.e. for all $q \geqslant 0$,*

$$\mathrm{H}^q(\underline{U}, \mathcal{F}) = \frac{\ker d^q}{imd^{q-1}}$$

.

Sketch of proof. Set $\tilde{\mathrm{H}}^q(\underline{U}, \mathcal{F}) = \ker d^q / imd^{q-1}$. It satisfies the functoriality for morphisms of presheaves

$\theta : \mathcal{F} \to \mathcal{G}$, so it gives an additive functor $\mathcal{P}Sh(\mathfrak{X}) \to Ab, \mathcal{F} \mapsto \tilde{\mathrm{H}}^q(\underline{U}, \mathcal{F})$. Claims:

1. The sequence $\{\tilde{\mathrm{H}}^q(\underline{U}, \mathcal{F})\}_{q \geqslant 0}$ forms a cohomological δ-functor (i.e. gives long exact sequence in Ab for each short exact sequence in $\mathcal{P}Sh(\mathfrak{X})$.
2. $\tilde{H}, \check{H}$ agree on $q = 0$ (clear from definition).
3. Both $\{\tilde{\mathrm{H}}^q(\underline{U}, \mathcal{F})\}_{q \geqslant 0}, \{\check{\mathrm{H}}^q(\underline{U}, \mathcal{F})\}_{q \geqslant 0}$ are universal δ-functors. Hence, since they agree on H^0 they agree everywhere.

Pf of claim 3. $\{\check{\mathrm{H}}^q\}_{q \geqslant 0}$ universal is clear since $\check{\mathrm{H}}^q$ is the right derived functor of the left exact functor $\mathcal{P}Sh(\mathfrak{X}) \to Ab$. For $\{\tilde{\mathrm{H}}^q\}_{q \geqslant 0}$, need to show the functor is effaceable, i.e. for $q \geqslant 0$ there exists a monomorphism $0 \to \mathcal{F} \to \mathcal{F}''$ in $\mathcal{P}Sh(\mathfrak{X})$ such that $\tilde{\mathrm{H}}^q(\underline{U}, \mathcal{F}'') = 0$. Suffices to show that $\tilde{\mathrm{H}}^q(\underline{U}, I) = 0$ for all I injective presheaves. Recall from Lecture 1 (Valia): The presheaf $\mathbb{Z}_U^p$ represents the functor $\mathcal{P}Sh(\mathfrak{X}) \to Ab, \mathcal{F} \mapsto \mathcal{F}(U)$ where $\mathbb{Z}_U^p(V) = \bigoplus_{V \xrightarrow{f} U} \mathbb{Z} \cdot f$, i.e. $\mathcal{F}(U) \simeq \mathrm{Hom}_{\mathcal{P}Sh(\mathfrak{X})}(\mathbb{Z}_U^p, \mathcal{F})$. Suppose I is injective, the the complex for I becomes:

$$\cdots \to I(U_{i_0} \times_I \cdots \times_U U_{i_q}) \xrightarrow{d^q} I(U_{i_0} \times_U \cdots \times_U U_{i_{q+1}}) \to \ldots$$

$$\cdots \to \mathrm{Hom}_{\mathcal{P}Sh}(\mathbb{Z}^p_{U_{i_0} \times_I \cdots \times_U U_{i_q}}, I) \xrightarrow{d^q} \mathrm{Hom}_{\mathcal{P}Sh}(\mathbb{Z}^p_{U_{i_0} \times_U \cdots \times_U U_{i_{q+1}}}, I) \to \ldots$$

I injective implies that the above complex is exact is $\cdots \leftarrow \mathbb{Z}^p_{U_{i_0} \times_I \cdots \times_U U_{i_q}} \leftarrow \mathbb{Z}^p_{U_{i_0} \times_U \cdots \times_U U_{i_{q+1}}} \leftarrow \ldots$ is exact in $\mathcal{P}Sh(\mathfrak{X})$, which holds if and only if for all $V \in \mathcal{C}$, the complex

$$\cdots \leftarrow \mathbb{Z}^p_{U_{i_0} \times_I \cdots \times_U U_{i_q}}(V) \leftarrow \mathbb{Z}^p_{U_{i_0} \times_U \cdots \times_U U_{i_{q+1}}}(V) \leftarrow \ldots$$

is exact in Ab (skip the proof of this exactness). □

□

6.3.3 From Čech to Sheaf Cohomology

Have composition of functors $Sh(\mathfrak{X}) \xrightarrow{i} \mathcal{P}Sh(\mathfrak{X}) \xrightarrow{\check{\mathrm{H}}^q(\underline{U}, -)} Ab$. Write $R^q i = \mathcal{H}^q(\ )$ so for each $\mathcal{F}$ sheaf obtain a preheaf $\mathcal{H}^q(\mathcal{F})$ with $\mathcal{H}^0(\mathcal{F}) = i(\mathcal{F}) = \mathcal{F}$ as a presheaf. Note: Let I be an injective sheaf. It follows from the proof of Thm 6.3.4 that $\check{\mathrm{H}}^q(\underline{U}, I) = 0$ for all $q > 0$ i.e. $i(I)$ is $\check{\mathrm{H}}^0(\underline{U}, -)$-acyclic, and so we obtain the Grothendieck spectral sequence

$$E_2^{p,q} = \check{\mathrm{H}}^q(\underline{U}, \mathcal{H}^q(\mathcal{F}) \Rightarrow E^{p+q} = \mathrm{H}^{p+q}(U, \mathcal{F})$$

which is functorial in $\mathcal{F}$ and the edge homomorphisms $\mathrm{H}^p(\underline{U}, \mathcal{F}) \to \mathrm{H}^p(U, \mathcal{F})$.

Corollary 6.3.5. Let $\underline{U} = \{U_i \to U\}_{i \in I}$ be a covering in $\mathcal{T}$ and $\mathcal{F} \in Sh(\mathfrak{X})$ such that $\mathrm{H}^q(U_{i_0} \times_U \cdots \times_U U_{i_r}, \mathcal{F}) = 0$ for all $q > 0$ and $(i_0, \ldots, i_r) \in I^{r+1}$. Then the edge homomorphisms $\mathrm{H}^p(\underline{U}, \mathcal{F}) \to \mathrm{H}^p(U, \mathcal{F})$ are isomorphisms for all p.

Example 6.3.6. *$\mathcal{T}_G$, e = 1-element G-set. Consider the covering $\{G \twoheadrightarrow e\}$ in $\mathcal{T}_G$. We get the edge homomorphisms for A = left G-module*

$$\mathrm{H}^p(\{G \to e\}, Hom_G(-, A)) \to \mathrm{H}^p(G, A)$$

. We want to show $\simeq$. By the corollary, it suffices to show that $\mathrm{H}^q(G \times_e \cdots \times_e G, Hom_G(-, A)) = 0$. $\mathrm{H}^q(G, Hom_G(-, A))$: As before this is the qth right derived functor of the composition:

$$\{\textit{left } G\textit{-mods}\} \to Sh(\mathcal{T}_G) \to Ab$$

$$A \mapsto Hom_G(G, A) \simeq A \textit{ as abelian groups}$$

This is just the forgetful functor which is exact, and so $\mathrm{H}^q(G, Hom_G(-, A)) = 0$ for all $q > 0$. Similar argument holds for the other cases.

We can use this to compute group cohomology via cochains: $\mathrm{H}^p(\{G \to e\}, \mathrm{Hom}_G(-, A))$ computed via the Čech complex:

$$0 \to \mathrm{Hom}_G(G, A) \xrightarrow{d^0} \mathrm{Hom}_G(G \times_e G, A) \ldots$$

$$d^0(f)(\sigma, \tau) = \tau a - \sigma a), d^1 f(\rho, \sigma, \tau) = f(\tau, \rho) - f(\sigma, \rho) + f(\sigma, \tau), \text{ etc...}$$

Remark 6.3.7. This Čech complex recovers computation obtained by projective resolution $P_\bullet : \cdots \to P_1 \to \mathbb{Z}^G \to \mathbb{Z} \to 0$, P_r =free $\mathbb{Z}-$mod with basis the $r+1$-tuples $(g_0, \ldots, g_r) \in G^{r+1}$ with diagonal action. We showed the above complex is a projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]-$module in previous years. Then $\mathrm{H}^i(G, A) = \mathrm{H}^i(\mathrm{Hom}_G(P_\bullet, A))$ for all A G-module.