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Contents

Chapter 1

Flatness and Geometry

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The goal of this chapter is to discuss flatness with an eye towards applications in algebraic geometry. The discussion is motivated by the following geometric theorem which we shall prove in detail.

Theorem 1.0.1. *Let $f : X \to Y$ be a flat morphism of finite type between Noetherian schemes. Then, f is open.*

We start by studying flat modules.

1.1 Flat Modules

We start by studying the effect of tensoring on the exactness of a sequence. To this end, we have the following proposition.

Proposition 1.1.1. *If R is a ring, M, M', M'' and N are R-modules, and*

$$M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

is an exact sequence, then

$$M' \otimes N \xrightarrow{\varphi \otimes 1} M \otimes N \xrightarrow{\psi \otimes 1} M'' \otimes N \longrightarrow 0$$

is exact.

Proof. The exactness of the first sequence is equivalent to the existence of an isomorphism $\overline{\psi} : M \to M''$ such that the following diagram

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M'' \\ {\scriptstyle \pi}\downarrow & \nearrow_{\overline{\psi}} & \\ M/\operatorname{im}\varphi & & \end{array} \tag{1.1}$$

commutes. Now, consider the map $M'' \times N \to (M \otimes N)/\mathrm{im}(\varphi \otimes 1_N)$, which maps (m'', n) to $[m \otimes n]$, with m chosen so that $\psi(m) = m''$.

First, note that such an m always exists since ψ is bijective. By the exactness of the first sequence, this map is independent of the choice of m, and therefore it is well-defined.

Second, it is clear that this map is R-bilinear, and therefore lifts to the tensor product.

Furthermore, this is clearly an inverse of the map $(M \otimes N)/\mathrm{im}(\varphi \otimes 1_N) \to M'' \otimes N$ induced by the map $\psi \otimes 1$. Therefore, we have an analogous diagram to (1.1) with $M \otimes N, M'' \otimes N$ and $(M \otimes N)/\mathrm{im}(\varphi \otimes 1_N)$ in place of M, M'' and $M/\operatorname{im}\varphi$. Therefore, the second sequence is exact, and we are done. □

In contrast to Proposition 1.1.1, we will give two examples to illustrate that the tensor product does not preserve injectivity in general.

Example 1.1.2. Consider the inclusion map $\iota : 2\mathbb{Z} \to \mathbb{Z}$. This is clearly an embedding of $\mathbb{Z}$ modules. On the other hand, tensoring with $\mathbb{Z}/2\mathbb{Z}$, it is clear that

$$(\iota \otimes 1)(2m \otimes n) = 2m \otimes n = m \otimes 2n = 0,$$

for any $2m \in 2\mathbb{Z}, n \in \mathbb{Z}/2\mathbb{Z}$.

Example 1.1.3. Let $R = \mathbb{C}[x, y], I = (x, y)$, and consider the embedding $\iota : I \to R$. It is easy to see that $R \otimes_R I \cong R$ through the map $f \otimes g \to fg$. Therefore, to show that $\iota \otimes 1_I$ is not injective, it is equivalent to show that

$$\begin{aligned} \varphi : I \otimes_R I &\to R \\ f \otimes g &\mapsto fg \end{aligned}$$

is not injective. First, note that $\varphi(x \otimes y - y \otimes x) = xy - yx = 0$. We will now show that $x \otimes y - y \otimes x$ is nonzero in the tensor product $I \otimes_R I$. It is easy to show that the map $\psi : I \times I \to R$ given by $\psi(f, g) = \frac{\partial f}{\partial x}(0, 0) \cdot \frac{\partial g}{\partial y}(0, 0)$ is R-bilinear, and therefore defines a map on $I \otimes_R I$, which we also denote by ψ. It is also clear that $\psi(x \otimes y - y \otimes x) = 1 \neq 0$, and therefore, $x \otimes y - y \otimes x$.

These examples motivate the following definition.

Definition 1.1.4. Let R be a ring and N an R-module. We say that N is flat if for every injective morphism $\varphi : M' \to M$, $\varphi \otimes 1_N : M' \otimes N \to M \otimes N$ is also injective.

We now give examples of flat modules.

Example 1.1.5. R is a flat R-module. This is clear because the map $\varphi \otimes 1_R : M' \otimes R \to M \otimes R$ can be identified with $\varphi : M' \to M$ as in Example 1.1.3.

To give more examples of flat modules, we first study the relation between flatness and direct sums.

Proposition 1.1.6. *Let $(N_i)_{i \in I}$ be a collection of R-modules. Then, $N = \bigoplus_{i \in I} N_i$ is flat if and only if each N_i is flat.*

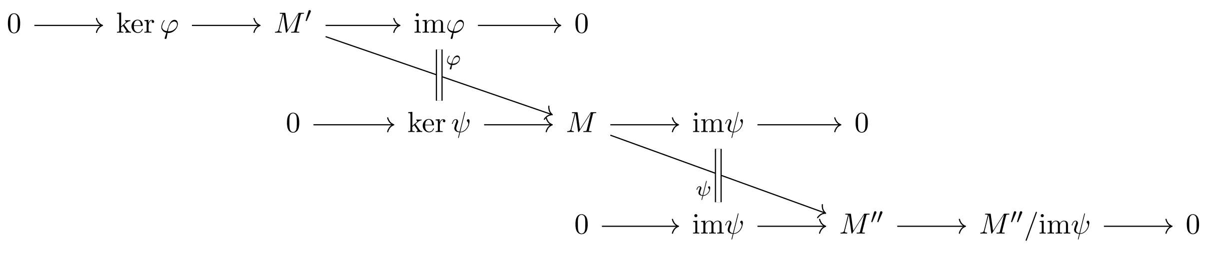
Proof. This follows clearly from the relation $M \otimes N = \bigoplus_{i \in I} M \otimes N_i$ for any R-module M. □

Example 1.1.7. Example 1.1.5 and Proposition 1.1.6 show that all free modules are flat. In fact, all projective modules are flat. To see this, let Q be a projective module. Then, there exists an R-module P such that $P \otimes Q$ is free, and thus flat. Applying Proposition 1.1.6 again, we have that Q is itself flat.

We have the following equivalent characterizations of flat modules.

Proposition 1.1.8. *Let N be an R-module. Then, the following are equivalent.*
(1) N is flat.
(2) If the sequence $0 \to M' \to M \to M'' \to 0$ of R-modules is exact, then $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact.
(3) If $M' \to M \to M''$ is exact, then $M' \otimes N \to M \otimes N \to M'' \otimes N$ is exact.

Proof. That (1) and (2) are equivalent is clear. Also, (3) clearly implies (2). Now, it remains to prove that (2) implies (3). To this end, note that the rows of the following diagram



are exact sequences and that it is commutative. Tensoring with N, by (2), we get exact rows as well. In particular, we have that $M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N$ is exact, and we are done. □

Corollary 1.1.9. *Let $\varphi : M' \to M$ be any morphism of R-modules, and let N be a flat R-module. Then,* $\ker(\varphi \otimes 1_N) = \ker(\varphi) \otimes N$, $\mathrm{im}(\varphi \otimes 1_N) = \mathrm{im}(\varphi) \otimes N$ *and* $\mathrm{coker}(\varphi) \otimes N$.

Proof. This is easy. □

An interesting fact is that the flatness of an R-module N is determined by how it tensors with ideals of R. More precisely, the following proposition holds.

Proposition 1.1.10. *Let N be an R-module. Then, the following are equivalent.*
(1) N is flat. (2) For all ideals I of R, $I \otimes_R N \to R \otimes_R N \cong N$ is injective, where this map is $\iota \otimes 1_N$ with ι the inclusion morphism of I into R.

The fact that (2) follows from (1) is trivial. To prove the reverse implication, we introduce the notion of M-flatness.

Definition 1.1.11. Let M and N be R-modules. We say that N is M-flat if for all injective morphisms $\varphi : M' \to M$, the map $\varphi \otimes 1_N : M' \otimes_R N \to M \otimes N$ is also injective.

From this definition, it is clear that N is a flat R-module if and only if N is M-flat for all modules M. Therefore, to prove Proposition 1.1.10, we therefore prove that if N satisfies (2), then N is M-flat for all modules M.

The idea of the proof is as follows. Every R-module M is a quotient of a free R-module, and (2) says that N is R-flat. Therefore, it is natural to consider how M-flatness behaves when we take direct sums and quotients of M.

For quotients, we have the following lemma.

Lemma 1.1.12. *Let M_2 be a quotient module of M. If N is M-flat, then N is M_2-flat.*

Proof. First, note that if M_2 is a quotient module of M, then, by definition of quotient modules, there exists a submodule $\mathfrak{a}$ of M such that $0 \to \mathfrak{a} \to M \to M_2 \to 0$ is exact.

If M_2' is an R-module that embeds into M_2, we can assume without loss of generality that M_2' is a submodule of M_2. This implies that there exists a submodule M_1 of M such that $M_2' = M_1/\mathfrak{a}$.

This is all summarized in the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & M_1 & \longrightarrow & M_2' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{a} & \longrightarrow & M & \longrightarrow & M_2 & \longrightarrow & 0 \end{array}$$

Tensoring with N, we have the following commutative diagram.

$$\begin{array}{ccccccc} \mathfrak{a} \otimes N & \longrightarrow & M_1 \otimes N & \longrightarrow & M_2' \otimes N & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \\ \mathfrak{a} \otimes N & \longrightarrow & M \otimes N & \longrightarrow & M_2 \otimes N & \longrightarrow & 0 \end{array}$$

The map of the first column is an isomorphism. The second map is injective because N is M-flat. Therefore, the map $M_2' \otimes N \to M_2 \otimes N$ is also injective by the snake lemma.

□

Before proving the analogous lemma for direct sums, we first show that we can reduce to the case of finitely generated submodules, and that therefore, we can consider only direct sums.

Lemma 1.1.13. *For an R-module N to be M-flat, it is necessary and sufficient that for every finitely generated submodule M' of M, the canonical homomorphism*

$$\iota \otimes 1_N : M' \otimes_R N \to M \otimes N$$

is injective.

Proof. If N is M-flat, the conclusion follows from the definition.

Now, assume the condition holds, and let $M'' \to M$ be an injective map. We can assume without loss of generality that M'' is a submodule of M. Suppose $z = \sum x_i \otimes y_i \in M'' \otimes N$ is 0 as an element of $M \otimes N$. Then, we are to show that z is 0 in $M'' \otimes N$.

But then, consider M' to be the R-submodule of M'' generated by x_i. Then, M' is finitely generated, and the map $M' \otimes N \to M \otimes N$, factors through $M'' \otimes N$, and therefore, takes $z \to 0$. Therefore, by hypothesis, $z = 0$ in $M' \otimes N$ and thus in $M'' \otimes N$. □

Before we prove the analogue for direct sums, we will consider how $M' \hookrightarrow M$ responds to tensoring when M' is a direct factor of M. This is given by the following lemma.

Lemma 1.1.14. *Let N be* **any** *R-module, and assume $M' \subset M$ is a direct factor of M. Then, $M' \otimes N$ embeds into and is a direct factor of $M \otimes N$.*

Proof. This is easy. Let $M = M' \bigoplus M''$. Then $M \otimes N = (M' \otimes N) \bigoplus (M'' \otimes N)$. □

We are now able to prove the following lemma.

Lemma 1.1.15. *Let $(M_i)_{i \in I}$ be a collection of R-modules. If N is M_i-flat for all $i \in I$, then N is M-flat where $M = \bigoplus\limits_{i \in I} M_i$.*

Proof. We first consider the case $I = \{1, 2\}$, that is $M = M_1 \oplus M_2$. Consider $M' \hookrightarrow M$, and let $M'_1 = M_1 \cap M', M'_2 = M_2 \cap M'$. We are to show that $M' \otimes N$ embeds into $M \otimes N$.

To this end, consider the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'_1 & \xrightarrow{\iota'} & M' & \xrightarrow{p'} & M'_2 & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & M_1 & \xrightarrow{\iota} & M & \xrightarrow{p} & M_2 & \longrightarrow & 0
\end{array}$$

where ι, ι' are the inclusion embeddings, and p, p' are the projection maps. By the definition of direct sums, it is clear that the rows are exact. Furthermore, the first column and third column maps are clearly injective as they are inclusions.

Therefore, taking tensor products with N, we get

$$\begin{array}{ccccc}
M'_1 \otimes N & \xrightarrow{\iota' \otimes 1} & M' \otimes N & \xrightarrow{p' \otimes 1} & M'_2 \otimes N \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
M_1 \otimes N & \xrightarrow{\iota \otimes 1} & M \otimes N & \xrightarrow{p \otimes 1} & M_2 \otimes N
\end{array}$$

The first and third columns are injective maps because N is M_1-flat and M_2-flat. Furthermore, the rows are exact by Proposition 1.1.1. Finally, the map $M_1 \otimes N \to M \otimes N$ is also injective because M_1 is a direct factor of M. Therefore, by the Snake Lemma, $M' \otimes N \to M \otimes N$ is injective, and thus, we conclude that N is M-flat.

By induction, we have that the lemma holds for any finite direct sum of modules M_i. By the previous lemma, it suffices to show that $\iota \otimes 1_N : M' \otimes_R N \to M \otimes_R N$ is injective for any finitely generated M' submodule of M.

Let M' be such a submodule. Then, since M' is finitely-generated $M' \subset \bigoplus_{i \in J} M_i$ for some finite index set J. Since N is M_J-flat, $M' \otimes_R N \to M_J \otimes_R N$ is injective. Furthermore, M_J is a direct factor of M, so $M_J \otimes_R N \to M \otimes_R N$ is injective, and therefore, the composite map, $M' \otimes_R N \to M \otimes_R N$ is injective, and we are done.

□

We are now ready to prove Proposition 1.1.10.

Proof of Proposition 1.1.10. By (2), N is R-flat, and therefore, N is M-flat for any free R-module M. Therefore, N is M'-flat for any quotient M' of a free R-module, and therefore, $N$is M-flat for any R-module, that is, N is a flat R-module. □

1.2 Varieties and Schemes

We recall notions about varieties and use them to motivate schemes. Most of the material here is given without details/proof, and this section simply serves as a motivation and recollection of important geometric constructions.

The main goal of algebraic geometry is to study the geometry of zero sets of polynomials. This motivates the following definition.

Definition 1.2.1. Let k be an algebraically closed field, and let $\mathbb{A}_k^n = \{(a_1, \ldots, a_n) \in k^n\}$. Let $\mathfrak{a}$ be an ideal of $k[\mathbb{A}_k^n] := k[x_1, \ldots, x_n]$. We define

$$Z(\mathfrak{a}) = \{P \in \mathbb{A}_k^n, f(P) = 0 \text{ for all } f \in \mathfrak{a}\}$$

The sets $Z(\mathfrak{a})$ allow us to topologize the space $\mathbb{A}_k^n$ due to the following lemma.

Lemma 1.2.2. *The sets $\{Z(\mathfrak{a}), \mathfrak{a} \lhd k[\mathbb{A}^n]\}$ satisfy the axioms for closed sets of a topology. This topology is called the Zariski topology on $\mathbb{A}_k^n$*

In other words, for every ideal $\mathfrak{a}$ of $k[\mathbb{A}_k^n]$, we have a closed subset $Z(\mathfrak{a})$ of $\mathbb{A}_k^n$.

If we take $n = 1$, $\mathfrak{a} = (x), \mathfrak{b} = (x^2)$, it is clear that $\mathfrak{a} \neq \mathfrak{b}$ and yet $Z(\mathfrak{a}) = Z(\mathfrak{b}) = \{0\}$. This indicates that the correspondence is not bijective.

However, if we define for a set $S \subset \mathbb{A}_k^n$, $I(S) := \{f \in k[\mathbb{A}^n], f(P) = 0 \text{ for all } P \in S\}$, a simple argument shows that $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, where $\sqrt{\mathfrak{a}}$ is the radical ideal of $\mathfrak{a}$. Therefore, we have a bijective correspondence between radical ideals of $k[\mathbb{A}^n]$ and A_k^n, where the maps are give by Z and I respectively.

We now study this correspondence when we restrict our scope of sets. For example, we consider the following important definition.

Definition 1.2.3. Let S be a subset of a topological space X. S is said to be irreducible if it cannot be written as $S = S_1 \cup S_2$ where S_1 and S_2 are proper subsets of S which are also closed in the subspace topology of S.

Directly from the definition, one sees that we may think of irreducible subsets as the building blocks of sets in a topological space.

We will state without proof that the irreducible closed subsets of $\mathbb{A}_k^n$ in the Zariski topology correspond to the prime ideals of $k[\mathbb{A}^n]$ under the above correspondence.

To conclude our discussion around this correspondence, let us note what sets correspond to the maximal ideals. To this end, let $P \in A_k^n$ and consider the evaluation map

$$\begin{aligned} \varepsilon_P : k[\mathbb{A}^n] &\to k \\ f &\to f(P). \end{aligned}$$

It is clear that $Z(\ker \varepsilon_P)$ is precisely $\{P\}$. Furthermore, since k is a field and ε_P is clearly nonzero - take the constant maps -, then by the first isomorphism theorem, $k[\mathbb{A}^n]/\ker \varepsilon_P \cong k$, and therefore, $\ker \varepsilon_P$. It is a fact that all maximal ideals arise such way, and therefore maximal ideals correspond to singletons.

Definition 1.2.4. If X is a closed irreducible subset of $\mathbb{A}_k^n$ in the Zariski topology, we say that X is an affine variety.

In general, we are interested in studying these affine varieties rather than $\mathbb{A}_k^n$ itself, which is uninteresting. Whereas two different polynomials over an infinite field, and therefore for any algebraically closed field, must evaluate to different values for some $P \in \mathbb{A}_k^n$, this is not guaranteed for X.

To illustrate this, consider $\mathbb{A}_{\mathbb{C}}^2$ and consider the variety $X = \{(x, y) \in \mathbb{A}_{\mathbb{C}}^2, y = x^2\}$. Then, the polynomials $f(x, y) = x \in k[\mathbb{A}^2]$ and $g(x, y) = x + y - x^2 \in k[\mathbb{A}^2]$ agree as polynomial functions on X. To rectify this, we give the following definition.

Definition 1.2.5. Let X be an affine variety, say X is an irreducible closed subset of $\mathbb{A}_k^n$. Then, we write $k[X] := k[\mathbb{A}_k^n]/I(X)$, where $I(X)$ is the collection of polynomials in $k[\mathbb{A}_k^n]$ which vanish on X. We call $k[X]$ the coordinate ring of X.

As is a recurring theme in mathematics, to understand objects, we aim to define morphisms between them. Since the polynomials are the building block for these algebraic varieties, the following definition is natural. We note that this is insufficient motivation, but it is enough for our purposes here.

Definition 1.2.6. A map $\phi : X \to \mathbb{A}_k^n$ is regular if $\phi = (f_1, \ldots, f_n)$ where $f_1, \ldots, f_n \in k[X]$. If $\phi(X) \subset Y$, where Y is some affine variety, we say that $\phi : X \to Y$ is a regular map.

As we associated algebraic objects (coordinate rings) for geometric objects (affine varieties), we aim to associate a corresponding algebraic quantity to this geometric morphism (regular map).

To this end, note that if $\phi : X \to Y$ is a regular map, and $f \in k[Y]$, then $\phi^*(f) := f \circ \phi$ is in $k[X]$. Therefore, the regular map $\phi : X \to Y$ gives a k-algebra homormophism $\phi^* : k[Y] \to k[X]$. It is easy to see that the reverse holds as well.

In fact, for two affine varieties X and Y, we know that X and Y are isomorphic as varieties, that is, there are regular maps $\phi : X \to Y$ and $\psi : Y \to X$ with $\phi \circ \psi = 1_Y$ and $\psi \circ \phi = 1_X$ if and only if $k[Y]$ and $k[X]$ are isomorphic as k-algebras. This indicates that studying affine varieties is equivalent to studying finitely generated k-algebras.

To obtain a natural consequence of this, let us fix some notation first. Let X and Y be varieties. For a ring R, we denote by $\operatorname{Specm} R$ the collection of maximal ideals of R, in other words, the maximal spectrum. For $P \in X, Y$, we write $\mathfrak{m}_P$ to denote the kernel of the evaluation map at P.

For $P \in X, Y$, we write $r_P : X, Y$ to the maximal ideal $\mathfrak{m}_P$. Also, it is easy to see that $(\phi^*)^{-1}$ takes maximal ideals in $k[X]$ to maximal ideals $k[Y]$. We then have the following commutative diagram.

$$k[X] \xleftarrow[\phi^*]{} k[Y]$$

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle r_P}\downarrow & & \downarrow{\scriptstyle r_{\phi(P)}} \\ \operatorname{Specm} k[X] & \xrightarrow{(\phi^*)^{-1}} & \operatorname{Specm} k[Y] \end{array}$$

Since studying X and Y is equivalent to studying $k[X]$ and $k[Y]$, and since the diagram commutes above, a natural generalization is to consider rings A and B in place of $k[X]$ and $k[Y]$, and where we can consider their geometry by taking $\operatorname{Specm} A$ and $\operatorname{Specm} B$.

However, an immediate problem arises. If $\phi : A \to B$ is a ring homomorphism, it is not true in general that the preimage of a maximal ideal is maximal. To amend this, we consider the next best thing. Namely, the preimage of a prime ideal is prime, and we consider the collection of prime ideals of A and B, denoted by $\operatorname{Spec} A$ and $\operatorname{Spec} B$ respectively.

Therefore, with each ring homomorphism $\phi : A \to B$, we have an induced map $\hat{\phi} : \operatorname{Spec} B \to \operatorname{Spec} A$, which maps $\mathfrak{p} \in \operatorname{Spec} B$ to $\phi^{-1}(\mathfrak{p})$.

Now, to give a topology on the sets $\operatorname{Spec} A$ and $\operatorname{Spec} B$, and study those maps with respect to that topology, we do an analogue of the definition for varieties. Namely, recall that $P \in X$, where X is some affine variety in $\mathbb{A}_k^n$ is equivalent to $\mathfrak{m}_P \supset I(X)$. Since $X = Z(\mathfrak{a})$ for some prime ideal $\mathfrak{a}$, we then have that this is equivalent to $\mathfrak{m}_P \supset \mathfrak{a}$. Therefore, the closed set $I(X)$, corresponds to the collection of maximal ideals $\mathfrak{m}_P$ that contain $Z(I(X)) = \mathfrak{a}$. Since we are dealing with prime ideals instead of maximal ideals, we put forward the following definition/lemma.

Definition 1.2.7. Let $\mathfrak{a}$ be an ideal of a ring A. We write $V(\mathfrak{a}) = \{\mathfrak{p} \in \operatorname{Spec} A, \mathfrak{p} \supset \mathfrak{a}\}$. The sets $V(\mathfrak{a})$ satisfy the axioms for closed sets of a topology, and the induced topology is called the Zariski topology on $\operatorname{Spec} A$.

In the Zariski topology, we have the following lemma.

Lemma 1.2.8. *If $\phi : A \to B$ is a ring homomorphism, then $\hat{\phi} : \operatorname{Spec} B \to \operatorname{Spec} A$ is continuous.*

Proof. The proof is trivial by following definitions. □

We are generally interested in Noetherian rings, that is, rings where any ascending chain of ideals stabilizes. If A is such a ring, $\operatorname{Spec} A$ is a Noetherian topological space, that is, any descending chain of closed subsets stabilizes.

Now that we have defined varieties and $\operatorname{Spec} A$ for rings A, we introduce the sheaf of regular map on varieties and extrapolate to a definition on $\operatorname{Spec} A$.

Note that for a variety X, we considered that it was natural to consider $f \in k[X]$. A question that arises naturally is why we didn't consider rational functions on $k[X]$ as regular on X instead. This is due to the following lemma which we give without proof (see Hartshorne).

Lemma 1.2.9. *Let $f : X \to k$ be a continuous function such that, for every $x \in X$, there exists a neighborhood of x on which f is a rational function. Then, $f \in k[X]$.*

However, since this is not necessarily true if we restrict f to U, we have the following definition.

Definition 1.2.10. If U is an open set, we define

$$\begin{aligned}\mathcal{O}_X(U) := \Big\{ & f : U \to k \text{ continuous such that for every } P \in U, \\ & \text{there exists } U_P \text{ an open neighborhood of } P \text{ with} \\ & f|_{U_P} = \frac{g|_{U_P}}{h|_{U_P}} \text{ for some } g, h \in k[X], h(P) \neq 0. \Big\}\end{aligned}$$

We call this the collection of regular functions on U.

In fact, $\mathcal{O}_X(U)$ is more than just a set. To be more precise, we have the following proposition.

Proposition 1.2.11. *The following are true.*
(1) For each U open in X, the set $\mathcal{O}_X(U)$ is a ring.
(2) If $V \subset U$ is an open set, then the restriction map $\rho_V^U : f \to f|_V$ maps $\mathcal{O}_X(U)$ into $\mathcal{O}_X(V)$. Furthermore, (1) and (2) give a presheaf structure on U. The presheaf $\mathcal{O}_X$ is a sheaf.

Proof. The proof is an easy conclusion of the fact that $\mathcal{O}_X(U)$ is defined locally. □

Even though the definition of $\mathcal{O}_X(U)$ looks abstract, the ring $\mathcal{O}_X(U)$ is easy to express on certain sets called the principal open sets.

Definition 1.2.12. Let X be a variety, $f \in k[X]$. We write $D(f) := \{P \in X, f(P) \neq 0\}$. Such sets $D(f)$ are called principal open sets.

The sets $D(f)$ are interesting to us in the following sense.

Lemma 1.2.13. *The following are true.*
(1) The sets $D(f)$ form a basis of X in the Zariski topology.
(2) $\mathcal{O}_X(D(f)) = k[X]_f$, where $k[X]_f$ denotes the localization of $k[X]$ by the multiplicative set $S = \{1, f, f^2, f^3, \ldots\}$.

We now use this lemma to generalize the sheaf of regular functions to $\operatorname{Spec} A$ where A is an arbitrary ring.

First, note that by our rationale before, the natural analogue of $D(f)$ is the set $D(a) = \{\mathfrak{p} \in \operatorname{Spec} A, a \notin \mathfrak{p}\}$, where $a \in A$. It is easy to prove that these sets form a basis for the Zariski topology on $\operatorname{Spec} A$.

To obtain an analogous sheaf, we define $\mathcal{O}(D(a)) = A_a$, where A_a is the localization of A with respect to the set $\{1, a, a^2, a^3, \ldots\}$. Since the sets $D(a)$ form a basis, there is a unique sheaf $\mathcal{O}$ on $\operatorname{Spec} A$ such that $\mathcal{O}(D(a)) = A_a$. It is given by the projective limit $\mathcal{O}(U) = \varprojlim_{D(a) \subset U} A_a$, where the ordering is given by $U \leq V$ whenever $U \subset V$. This is called the structure sheaf on $\operatorname{Spec} A$.

Finally, let us explain how to study the local behavior of a regular map on a variety, and give an analogous definition for sheaves. For $P \in X$, where X is a variety, define the direct system $I := \{U \text{ open in } X, P \in U\}$ ordered by reverse inclusion, that is, intuitively, ordered by "how local it is at P." From here, it is clear that the direct limit $\mathcal{O}_P = \varinjlim_{P \in U} \mathcal{O}(U)$ encodes the local behavior of regular functions at P, or so to speak, the germs of regular functions at P.

In this notation, we then have the following lemma.

Lemma 1.2.14. *The following are true.*
(1) If X is a variety, $P \in X$ and $\mathcal{O}$ is the sheaf of regular functions on X, then $\mathcal{O}_P = k[X]_{\mathfrak{m}_P}$, where $k[X]_{\mathfrak{m}_P}$ is the localization of $k[X]$ away from $\mathfrak{m}_P$, in other words, with respect to the set $S = \{f \in k[X] \setminus \mathfrak{m}_P\}$.
(2) If A is a ring, $X = \operatorname{Spec} A$, $\mathfrak{p} \in X$, and $\mathcal{O}$ is the structure sheaf on X, then $\mathcal{O}_\mathfrak{p} = A_S$, where $S = \{a \in A \setminus \mathfrak{p}\}$.

To encapsulate the notion of $\operatorname{Spec} A$ with its topology and sheaf in a more abstract manner, we give the following definitions.

Definition 1.2.15. We call $(X, \mathcal{O}_X)$ a ringed space if X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

Definition 1.2.16. We say that the ring space $(X, \mathcal{O}_X)$ is a locally ringed space if $\mathcal{O}_P$ is a local ring for every $P \in X$.

Definition 1.2.17. Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be ringed spaces. A map of ringed spaces is a pair

$$(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y),$$

where $f : X \to Y$ is a continuous map and for every $V \subset Y$ an open subset, we have a map

$$f_V^\# : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)),$$

such that for all open subsets U, V of Y with $U \subset V$, the following diagram

$$\begin{array}{ccc} \mathcal{O}_Y(V) & \xrightarrow{f_V^\#} & \mathcal{O}_X(f^{-1}(V)) \\ \rho(Y)_U^V \big\downarrow & & \big\downarrow \rho(X)_{f^{-1}(U)}^{f^{-1}(V)} \\ \mathcal{O}_Y(U) & \xrightarrow[f_U^\#]{} & \mathcal{O}_X(f^{-1}(U)) \end{array}$$

commutes. A map of locally ringed spaces is a map of ringed spaces such that the corresponding maps at a point, that is the map induced on the stalk $\mathcal{O}(Y)_f(P) \to \mathcal{O}(X)_P$ for some $P \in X$ are local maps.

To show that this is an abstraction of the situation for spectra of rings, let us give a few details that shows that spectra are an example.

Example 1.2.18. Let A and B be rings, $\phi : A \to B$ be a ring homomorphism. The ring morphism induces a continuous map $f : \operatorname{Spec} B \to \operatorname{Spec} A$. For ease of notation, we denote $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$.

We know that $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, where $\mathcal{O}_X$ and $\mathcal{O}_Y$ are the structure sheaves, are locally ringed spaces. Therefore, we now aim to construct $f^\#$ as above.

Since the structure sheaves were constructed in terms of the principal open sets, let us first determine $f^{-1}(D_Y(g))$, where $g \in A$.

We clearly have that

$$\begin{aligned} f^{-1}(D_Y(g)) &= \{\mathfrak{p} \in \operatorname{Spec} B, f(\mathfrak{p}) \not\ni g\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} B, \phi^{-1}(\mathfrak{p}) \not\ni g\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} B, \mathfrak{p} \not\ni g\} \\ &= D_X(\phi(g)). \end{aligned}$$

Therefore, we want to first define maps $f^{\#}_{D_Y(g)} : \mathcal{O}_Y(D_Y(g)) \to \mathcal{O}_X(D_X(\phi(g)))$ in a natural way. However, note that $\mathcal{O}_Y(D_Y(g)) = A_g$ and $\mathcal{O}_X(D_X(\phi(g))) = B_{\phi(g)}$.

Thus, the natural definition is that $f^{\#}_{D_Y(g)}$ is the map $\phi_g : A_g \to B_{\phi(g)}$ induced on the localizations from ϕ. It is easy to check that this system is compatible (we need some sheaf theory for this), and we can use these morphisms on basis to construct $f^{\#}_V$ for all open subsets V of Y.

The example above warrants the following definition.

Definition 1.2.19. A locally ringed space $(X, \mathcal{O}_X)$ is called an affine scheme if it is isomorphic to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$, where A is some ring, as locally ringed spaces, that is, there are morphisms of locally ringed spaces which are inverses of each other.

In the general case, one considers some gluing of affine varieties, and also of schemes. We will not expand upon this, but we will simply give the definition.

Definition 1.2.20. $(X, \mathcal{O}_X)$ is called a scheme if for all $P \in X$, there exists an open neighborhood U of P such that $(U, \mathcal{O}_X|_U)$ is an affine scheme.

1.3 Proof of Main Result

We will now prove our main result. For clarity of exposition, we first recall it.

Theorem 1.3.1. *If $f : X \to Y$ is a flat morphism of finite type of Noetherian schemes, then f is open.*

Before we explain what finite type morphisms are, and before we prove the lemmas required, we first illustrate how badly this fails through an example.

Example 1.3.2. Let $X = Y = \mathbb{A}^2_k$, and consider the regular map

$$\begin{aligned} f :& X \to Y \\ &(x, y) \to (x, xy). \end{aligned}$$

An easy computation shows that $f(X) = (\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1) \cup \{(0,0)\}$. Therefore, $f(X)$ is not open. However $(\mathbb{A}^1 \setminus \{0\} \times \mathbb{A}^1)$ is open and $\{(0,0)\}$ is closed. We now introduce an appropriate topological notion of which this example is an illustration.

Definition 1.3.3. A subset of a topological space X is said to be locally closed if it is the intersection of a closed and open subset of X. This includes both open and closed sets.

The image $f(X)$ in the example above was a union of two locally closed subsets. This warrants the following definition.

Definition 1.3.4. A subset of a topological space X is said to be constructible if it is the finite union of locally closed sets.

Before we state any theorems regarding the constructibility of images of varieties/schemes, let us first write down some natural properties of the collection of constructible sets.

Lemma 1.3.5. *If X_1 and X_2 are constructible, then so are $X_1 - X_2, X_1 \cup X_2$ and $X_1 \cap X_2$.*

Proof. We will only prove the case $X_1 \cap X_2$. The remaining statements are analogous.

Since X_1 and X_2 are constructible, then, by definition, there exists open sets $U_1, \ldots, U_n, V_1, \ldots, V_m$ and closed sets $F_1, \ldots, F_n, G_1, \ldots, G_m$ such that

$$X_1 = (U_1 \cap F_1) \cup \ldots \cup (U_n \cap F_n),$$

and

$$X_2 = (V_1 \cap G_1) \cup \ldots \cup (V_m \cap G_m).$$

Therefore, it is easy to see that

$$X_1 \cap X_2 = \bigcup_{\substack{1 \le i \le n \\ 1 \le j \le m}} ((U_i \cap V_j) \cap (F_i \cap G_j)),$$

which is clearly constructible. □

We now give an alternative more intuitive definition of constructibility in the case of Noetherian spaces.

Theorem 1.3.6. *Let X be a Noetherian topological space. A subset Y of X is constructible if and only if the following implication holds.*

If F is a closed irreducible subset of X and $Y \cap F$ is dense in F, then there exists an open subset U of X such that $\emptyset \neq U \cap F \subset Y \cap F$.

Proof. First, assume that Y is constructible, and let F be a closed irreducible subset of X such that $Y \cap F$ is dense in F. Since Y is constructible, there exists open sets $U_1, \ldots, U_n$ of X, and closed sets $F_1, \ldots, F_n$ of X such that $Y = (U_1 \cap F_1) \cup \ldots \cup (U_n \cap F_n)$.

We then have that $Y \cap F = (U_1 \cap (F_1 \cap F)) \cup \ldots \cup (U_n \cap (F_n \cap F))$. If $\overline{Y \cap F} = F$, where $\overline{Y \cap F}$ is the closure of $Y \cap F$, then we have

$$F = \overline{U_1 \cap F_1 \cap F} \cup \ldots \cup \overline{U_n \cap F_n \cap F}.$$

Since F is assumed to be irreducible, then $F = \overline{U_m \cap F_m \cap F}$ for some $m \in \{1, \ldots, n\}$. But then, we have that

$$F \subset \overline{U_m \cap F_m \cap F} \subset \overline{F_m \cap F} = F_m \cap F \subset F,$$

and therefore, $F = F \cap F_m$. This implies that

$$U_m \cap F = U_m \cap (F_m \cap F) = (U_m \cap F_m) \cap F \subset Y \cap F,$$

and we proved our claim.

Now, assume that the implication holds and Y is not constructible. We will use the principle of Noetherian induction to obtain a contradiction. To this end, define

$$\mathcal{P} := \{\emptyset \neq Z \subset X, Z \text{ is closed and } Y \cap Z \text{ is not constructible.}\}.$$

Since Y is not constructible, $\mathcal{P} \neq \emptyset$. Since X is Noetherian, $\mathcal{P}$ contains a minimal element, say Z_0.

First, we claim that Z_0 is irreducible. Assume that $Z_0 = Z_1 \cup Z_2$ where Z_1 and Z_2 are proper closed subsets of Z_0. Then, $Y \cap Z_0 = (Y \cap Z_1) \cup (Y \cap Z_2)$. By minimality o Z_0, we have that $Y \cap Z_1$ and $Y \cap Z_2$ are constructible, and therefore, $Y \cap Z_0$ is constructible, a contradiction. Therefore, Z_0 is irreducible.

Second, we claim that $Y \cap Z_0$ is dense in Z_0. Assume not. Then, $\overline{Y \cap Z_0}$ is a proper closed subset of Z_0 and by minimality of Z_0 in $\mathcal{P}$, we have that $Y \cap \overline{Y \cap Z_0}$ is constructible. However, it is clear that

$$Y \cap Z_0 \subset Y \cap \overline{Y \cap Z_0} \subset Y \cap \overline{Z_0} = Y \cap Z_0,$$

and therefore, $Y \cap Z_0$ is constructible, a contradiction. Therefore, $Y \cap Z_0$ is dense in Z_0.

Finally, using the implication, we have that there exists an open subset U of X such that $U \cap Z_0 \subset Y \cap Z_0$. Since U is open and Z_0 is closed, $U \cap Z_0$ is locally closed, and thus constructible. In addition, we can write

$$Y \cap Z_0 = (U \cap Z_0) \cup (Y \cap (Z_0 \setminus U)).$$

Since $U \cap Z_0 \neq \emptyset$, we have that $Z_0 \setminus U$ is a proper closed subset of Z_0, and thus, by minimality of Z_0, we have that $Y \cap (Z_0 \setminus U)$ is constructible, and therefore, $Y \cap Z_0$, being a union of two constructible sets, is constructible, a contradiction.

Therefore, Y is constructible, and we are done. □

Now, we give the definition of morphisms of schemes of flat type.

Definition 1.3.7. Let $f : X \to Y$ be a morphism of schemes.
(i) We say that f is locally of finite type at $x \in X$ if there exists $U = \operatorname{Spec} B, V = \operatorname{Spec} A$ neighborhoods of x and $f(x)$ respectively such that the ring morphism $\phi : A \to B$ which induces $f|_U : U \to V$ is of finite type, that is, B is finitely generated as an A-algebra through ϕ.
(ii) We say that f is of finite type if it is locally of finite type at all $x \in X$.

We now have the exact theorem which illustrates what happened in Example 1.3.2.

Theorem 1.3.8 (Chevalley's Theorem)**.** *Let $f : X \to Y$ be a morphism of finite type of Noetherian schemes. Then, $f(X)$ is constructible.*

An immediate corollary is the following.

Corollary 1.3.9. *If $f : X \to Y$ is a morphism as in Chevalley's theorem, and Z is a constructible subset of X, then $f(Z)$ is constructible.*

Proof. This follows by the fact that the restriction $f|_Z : Z \to Y$ satisfies the hypothesis of Chevalley's theorem when Z is constructible. □

We now prove Theorem 1.3.8.

Proof of Chevalley's Theorem. Since Y is a Noetherian scheme, it can be covered by finitely many affine open sets V_i and since X is a Noetherian scheme, $f^{-1}(V_i)$ can be covered by finitely many open affine subsets of X. Therefore, it suffices to prove this theorem when $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$ and $f : X \to Y$ is induced from a ring homomorphism $\phi : A \to B$.

Since Y is Noetherian, to show that $f(X)$ is constructible in Y, by Theorem 1.3.6, we are required to show the following.

If $F \subset Y$ is an irreducible closed set with $f(X) \cap F$ dense in F, then $f(X) \cap F$ must contain a nonempty subset $U \cap F$ with U open in Y.

First, note that the irreducible closed subsets of Y are given by $V(\mathfrak{p}) := \{\mathfrak{p}' \in \operatorname{Spec} A, \mathfrak{p}' \supset \mathfrak{p}\}$ for $\mathfrak{p} \in \operatorname{Spec} A$. It is a well-known fact of algebra that the prime ideals contained in $V(\mathfrak{p})$ correspond to the prime ideals of $A/\mathfrak{p}$. It can be shown that this holds in a scheme-theoretic sense, in the sense that $V(\mathfrak{p})$ is isomorphic to $\operatorname{Spec} A/\mathfrak{p}$.

Under this identification, the image $f(X) \cap V(\mathfrak{p}) = \phi^{-1}(\operatorname{Spec} B) \cap V(\mathfrak{p})$ is identified with the image of the map $\operatorname{Spec} B/\mathfrak{p}B \to \operatorname{Spec} A/\mathfrak{p}$ induced from the natural map $\tilde{\phi} : A/\mathfrak{p} \to B/\mathfrak{p}B$. This is called base-change.

Since we are interested in the case when $f(X) \cap F$ is dense in F, we compute the closure of the image under these identifications.

Let $\tilde{\phi} : A/\mathfrak{p} \to B/\mathfrak{p}B$. Then,

$$\begin{aligned}
\overline{\tilde{\phi}^{-1}(\operatorname{Spec} B/\mathfrak{p}B)} &= \overline{\{\tilde{\phi}^{-1}(\mathfrak{q}), \mathfrak{q} \in \operatorname{Spec} B/\mathfrak{p}B\}} \\
&= V(\cap \tilde{\phi}^{-1}(\mathfrak{q})), \text{ where } \mathfrak{q} \text{ ranges over } \operatorname{Spec} B/\mathfrak{p}B. \\
&= V(\tilde{\phi}^{-1}(\cap \mathfrak{q})) \\
&= V(\tilde{\phi}^{-1}(0)) \\
&= \{\mathfrak{p}' \in \operatorname{Spec} A/\mathfrak{p}, \mathfrak{p}' \supset \ker \tilde{\phi}\}.
\end{aligned}$$

Therefore, $f(X) \cap F$ is dense in F is equivalent, after identification, to $\ker \tilde{\phi} = 0$, that is, to $\tilde{\phi} : A/\mathfrak{p} \to B/\mathfrak{p}B$.

To recapitulate, we want to prove that if $\tilde{\phi} : A/\mathfrak{p} \to B/\mathfrak{p}B$ is injective, then, $\tilde{\phi}^{-1}(\operatorname{Spec} B/\mathfrak{p}B)$ contains an open set. Therefore, since $\mathfrak{p}$ is a prime ideal of A, that is $A/\mathfrak{p}$ is an integral domain, and $B/\mathfrak{p}B$, by our hypothesis, is a finitely generated $A/\mathfrak{p}$-algebra, then it suffices to prove the following algebraic fact.

If $\psi : A \to B$ is an injective homomorphism where B is a finitely generated A-algebra and A is an integral domain, then, there exists a nonzero element $a \in A$ such that every prime ideal of A lying in the set $D(a) = \{\mathfrak{p} \in \operatorname{Spec} A, a \notin \mathfrak{p}\}$ is the inverse image of a prime ideal of B.

To prove this, write $B = A[x_1, \dots, x_n]$, where $x_1, \dots, x_r$ are algebraically independent over A and $x_{r+1}, \dots, x_n$ are algebraic over $A[x_1, \dots, x_r]$. In other words, for each $j, r+1 \le j \le n$, we have an equation

$$p_{j0}x_j^{d_j} + p_{j1}x_j^{d_j-1} + \ldots + p_{jd_j} = 0,$$

where $p_{ji} \in A[x_1, \dots, x_n]$ and $p_{j0} \neq 0$. Then $\prod_{j=r+1}^{n} p_{j0}$ is a nonzero polynomial. We take a to be any nonzero coefficient of this polynomial.

Now, if $\mathfrak{p}$ is a prime ideal of A that does not contain a, then clearly, $\prod_{j=r+1}^{n} p_{j0} \notin \mathfrak{p}[x_1, \dots, x_r]$, and $\mathfrak{p}[x_1, \dots, x_r]$ is a prime ideal of $A[x_1, \dots, x_r]$, which we denote by $\mathfrak{p}'$.

Since $\prod_{j=r+1}^{n} p_{j0} \notin \mathfrak{p}'$, we then have that $p_{j0} \notin \mathfrak{p}'$ for all j, and therefore, $B_{\mathfrak{p}'}$ is integral over $A[x_1, \dots, x_r]_{\mathfrak{p}'}$. By the going-up theorem, there exists a prime ideal $\mathfrak{q}$ of $B_{\mathfrak{p}'}$ such that its inverse image (recall that the map is injective, so we can use the going-up theorem) is $\mathfrak{p}'$, and therefore, the inverse image of $\mathfrak{q}$ in A is $\mathfrak{p}$. □

Now, we have proved that for morphisms of finite type of Noetherian schemes, the image of a constructible set is constructible. Therefore, to prove that the image is open under a certain condition, it is enlightening to find a condition for which a constructible set is open. To answer this, we first give the following definitions.

Definition 1.3.10. Let X be a topological space, $x, y \in X$. We say that y is a generalization of x if $x \in \overline{\{y\}}$.

Definition 1.3.11. Let X be a topological space, $x \in X$. We say that x is a generic point of X if $\overline{\{a\}} = X$.

We now give the following lemma without proof.

Lemma 1.3.12. *If X is a Noetherian scheme, then every irreducible closed subset of X contains a generic point.*

Now, we characterize open constructible sets in Noetherian schemes, with an even more generality than we actually need.

Theorem 1.3.13. *Let X be a Noetherian topological space such that every irreducible closed subset of X has a generic point. Let U be a constructible subset of X and $x \in U$. Then, U contains an open neighborhood of x (in X) if and only if U contains every generalization of x.*

Proof. If U contains an open neighborhood of x, the implication is trivial.

Now, assume U contains every generalization of x. To obtain a contradiction, assume that U contains no open neighborhood of x. We again argue by Noetherian induction.

To this end, let $\mathcal{P} := \{Z, x \in Z \subset X$ is closed and $U \cap Z$ contains no open neighbohood of x in $Z\}$. Since $X \in \mathcal{P}$, we have that $\mathcal{P} \neq \emptyset$ and therefore, it must contain a minimal element, say Z_0.

First, we claim that Z_0 is irreducible. Assume not, then $Z_0 = Z_1 \cup Z_2$, where Z_1 and Z_2 are proper closed subsets of Z_0. As a first case, assume that Z_1 and Z_2 both contain x. By the minimality of Z_0, there exists V_1, V_2 open neighborhoods of x in X such that $V_1 \subset Z_1 \cap U \cap Z_1$ and $V_2 \subset Z_2 \cap U \cap Z_2$. Therefore, we have that $(V_1 \cup V_2) \cap U \subset Z_0 \cap U$, a contradiction with our choice of Z_0. As a second case, assume Z_1 contains x and Z_2 doesn't. Then, simply taking V_1 as above, we have that $V_1 \setminus Z_2$ contains x, and is therefore nonempty. Then, we have that $(V_1 \setminus Z_2) \cap Z = V_1 \cap Z_1 \subset U \cap Z_1 \subset U \cap Z$, again, a contradiction. The case where Z_1 doesn't contain x and Z_2 does is handled in exactly the same manner. Therefore, Z_0 is irreducible.

Second, we claim that $U \cap Z_0$ is dense in Z_0. Let y be a generic point of Z_0. Then, y is a generalization of x. Therefore, by assumption, U contains y. Therefore, $\overline{U \cap Z_0} \supset \overline{y} = Z_0.s$

Now, since U is constructible, Z_0 is irreducible closed subset of X and $U \cap Z_0$ is dense in Z_0, we have that there exists an open subset V of X such that $\emptyset \neq V \cap Z_0 \subset U \cap Z_0$.

We now claim that V contains x. Assume not. Then, $Z_0 \setminus (V \cap Z_0)$ is a proper closed subset of Z_0, and since this proper subset contains x, we have that there exists W an open neighborhood of x such that $W \cap (Z_0 \setminus (V \cap Z_0)) \subset U \cap (Z_0 \setminus (V \cap Z_0))$. But then, $W \cap Z_0 = W \cap (Z_0 \setminus (V \cap Z_0)) \cup W \cap V \cap Z_0 \subset U \cap (Z_0 \setminus (V \cap Z_0)) \cup V \cap Z_0 \subset U \cap Z_0$, a contradiction with our choice of Z_0.

But then, $V \cap Z_0$ is an open neighborhood (in Z_0) of x contained in $U \cap Z_0$, again, a contradiction with our choice of Z_0. Therefore, we conclude that U contains an open neighborhood of x.

□

To recapitulate, let $f : X \to Y$ be a flat morphism of finite type of Noetherian schemes. We have that $f(X)$ is constructible. Therefore, to prove that $f(X)$ is open, we have to show that $f(X)$ is closed under generalization.

To do this, we first derive an algebraic condition for $f(X)$ to be closed under generalization. This is encapsulated in the following proposition.

Proposition 1.3.14. *Let $f : X \to Y$ be a morphism of finite type of Noetherian schemes. Let $x \in X, y = f(x)$. Then, the following are equivalent.*
(i) f maps neighborhoods of x to neighborhoods of y.
(ii) For every generalization y' of y, there exists a generalization x' of x such that $f(x') = y'$.
(iii) $(f_x^{\#})^{-1} : \operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{Y,y}$ is surjective.

Before we prove this proposition, a remark is in order regarding the map $(f_x^{\#})^{-1}$. First, we have that $f_U^{\#} : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ for every open subset U of Y. By taking direct limits, passing to the stalk, we then have an induced map $f_x^{\#} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. Note that this is a ring morphism, and therefore, defines a map on the spectra, namely $(f_x^{\#})^{-1} : \operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{Y,y}$.

Proof of Proposition 1.3.14. We first prove that (ii) and (iii) is equivalent

To this end, by taking affine neighborhoods of x and y, we lose no generality if we assume $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$ and $f : X \to Y$ is the map induced by the ring morphism $\phi : A \to B$.

Now, if $y \in Y = \operatorname{Spec} A$ and y' is a generalization of y, that is, $y \in \overline{\{y'\}} = V(y')$, that is, y is a prime ideal containing y'. In other words, the generalizations of y can be identified with the prime ideals of A_y and the generalizations of x can be identified with the prime ideals of x.

Since $\mathcal{O}_{X,x} = B_x$ and $\mathcal{O}_{Y,y} = A_y$, and by the remark above, $(f_x^{\#})^{-1}$ is clearly the map on the spectra induced by $\phi_x : A_x \to B_y$, the equivalence between (ii) and (iii) follows immediately.

Now, we prove that (i) implies (ii). Suppose that y' is a generalization of y. Let F be the union of the irreducible components of $f^{-1}(\overline{\{y'\}})$ not containing x. Then, $X - F$ is a neighborhood of x, and $f(X - F)$ is an open neighborhood of y. Therefore, $f(X - F)$ contains every generalization of y, and thus $y' \in f(X - F)$, say, $y' = f(x_1)$. Then, x_1 lies in an irreducible closed component C of $\overline{X - F}$. Let x' be the generic point of C. Then, $x_1 \in \overline{x'}$, and thus, since f is continuous, $y' = f(x_1) \in \overline{f(x')}$. On the other hand, $x' \in C \subset f^{-1}(\overline{\{y'\}})$, and thus, $f(x') \in \overline{\{y'\}}$. Therefore, $f(x') = y'$.

Finally, let us prove that (ii) implies (i). Let U be an open neighborhood of x. Then U is constructible, and therefore, $f(U)$ is constructible. Therefore, Furthermore, by (ii), $f(U)$ contains every generalization of y, and therefore, $f(U)$ contains an open neighborhood of y. □

We are now ready to define flat morphisms, where flatness is an algebraic condition, and proceed to prove our main theorem.

Definition 1.3.15. A morphism $f : X \to Y$ of schemes is called flat if $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is flat as a map of modules for every $x \in X, y = f(x)$, that is, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module under f.

By the previous proposition and its proof, we want to prove the following. Let $\phi : R \to A$ be a ring morphism of finite type, $\mathfrak{q} \in \operatorname{Spec}(A)$, with $\phi^{-1}(\mathfrak{q}) = \mathfrak{p}$, and consider the induced map $\tilde{\phi} : R_{\mathfrak{p}} \to A_{\mathfrak{q}}$. If $\tilde{\phi}$ is flat, then $(\tilde{\phi})^{-1} : \operatorname{Spec} A_{\mathfrak{q}} \to \operatorname{Spec} R_{\mathfrak{p}}$ is surjective.

Since $A_{\mathfrak{q}}$ and $R_{\mathfrak{p}}$ are both local rings and $\tilde{\phi}$ is a local map, we are done if we prove the following algebraic fact.

Lemma 1.3.16. *Let $\phi : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a local morphism of local rings such that B is finitely generated as an A-algebra, and ϕ is flat. Then, the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective.*

Proof. Let $\mathfrak{p}$ be a prime ideal in A, and consider the ring $B \otimes_A Q(A/\mathfrak{p})$, where $Q(A/\mathfrak{p})$ is the field of fractions of $A/\mathfrak{p}$, which is well-defined since $A/\mathfrak{p}$ is an integral domain.

First, let us prove that $B \otimes_A Q(A/\mathfrak{p}) \neq 0$. Since B is a flat A-module, we have that $B \otimes_A A/\mathfrak{p}$ embeds into $B \otimes_A Q(A/\mathfrak{p})$. Furthermore, since $A/\mathfrak{p}$ embeds into $A/\mathfrak{m}$ and since B is a flat A-module, we have that $B \otimes_A A/\mathfrak{p}$ embeds into $B \otimes_A A/\mathfrak{m}$. Since ϕ is local, we thus have that $B \otimes_A A/\mathfrak{m} \cong B/\mathfrak{n}$, and therefore, is nonzero. Thus, $B \otimes_A Q(A/\mathfrak{p}) \neq 0$.

But then, the ring $B \otimes_A Q(A/\mathfrak{p})$ contains some prime ideal $\mathfrak{r}$. Now, $\mathfrak{r} \cap Q(A/\mathfrak{p})$ is a prime ideal of $Q(A/\mathfrak{p})$, and thus, the zero ideal. Therefore, $\mathfrak{r} \cap A = \mathfrak{p}$, and we are done. □

Chapter 2

Faithfully Flat Descent

Wojciech Tralle

Introduction

The theory of faithfully flat descent starts with the theory of flat and faithfully flat modules.

Definition 2.0.1. An A-module M is called **flat** if for every injective homomorphism of A-modules $N \xrightarrow{f} N'$ the map

$$N \otimes_A M \xrightarrow{f\otimes 1} N' \otimes_A M$$

is injective.

Definition 2.0.2. Let M be a flat A-module. We say that M is **faithfully flat** over A if for any A-module N, if $M \otimes_A N = 0$, then $N = 0$.

Let B be a commutative A-algebra. In this talk we are going to address the following descent problems.

(1) Let M and N be a pair of A-modules, and $g \in \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$. When does g have the form $g = f \otimes 1$ for some $f \in \mathrm{Hom}_A(M, N)$?

(2) Let M be a B-module. When do we have

$$M \cong N \otimes_A B$$

for some A-module N?

Note that without strong assumptions on B these questions would be meaningless. However, if one imposes the condition of faithful flatness on B, the situation changes. In particular, we obtain the following results.

Theorem 2.0.3. *(Descent of Homomorphisms) Let B be a faithfully flat A-algebra over a commutative ring A, and M, N a pair of A-modules. One defines certain maps $\mathfrak{F}_0, \mathfrak{F}_1$ so that the following sequence is exact*

$$0 \to \mathrm{Hom}_A(M, N) \xrightarrow{\mathfrak{F}} \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B) \xrightarrow{\mathfrak{F}_0 - \mathfrak{F}_1} \mathrm{Hom}_{B\otimes_A B}(M \otimes_A B \otimes_A B, N \otimes_A B \otimes_A B),$$

where $\mathfrak{F}(f) = f \otimes 1$. As a consequence, any homomorphism $g \in \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$ is of the form $g = f \otimes 1$ for some $f \in \mathrm{Hom}_A(M, N)$ if and only if $g \in \ker(\mathfrak{F}_0 - \mathfrak{F}_1)$.

This theorem gives necessary and sufficient conditions for Question (1) to have a positive answer. The key assumption is the faithful flatness of B. We will prove the exactness of the sequence above in Proposition 2.2.4.

Answering Question (2) is more complicated: one needs to formulate the answer in terms of the *descent datum*, that is, in terms of a special homomorphism

$$g : B \otimes_A M \to M \otimes_A B$$

and three homomorphisms g_1, g_2 and g_3 on $B \otimes_A B \otimes M$ and on $B \otimes_A M \otimes_A B$, respectively. If these homomorphisms satisfy some additional assumptions, we say that g is a *descent datum*. The Main Theorem we want to prove, provides sufficient conditions to obtain a positive answer to Question (2). It is usually referred to as the Theorem of Faithfully Flat Descent for Modules.

Theorem 2.0.4. *(Descent of Modules) Let B be a commutative faithfully flat A-algebra. Let M be a B-module and let $g : B \otimes_A M \to M \otimes_A B$ be a descent datum for M over B. Then there exists an A-module N and an isomorphism $\nu : N \otimes_A B \to M$ of B-modules such that the diagram of $B \otimes_A B$-modules*

$$\begin{array}{ccc} B \otimes_A N \otimes_A B & \xrightarrow{1\otimes\nu} & B \otimes_A M \\ \tau\downarrow & & \downarrow g \\ N \otimes_A B \otimes_A B & \xrightarrow{\nu\otimes 1} & M \otimes_A B \end{array} \tag{2.1}$$

commutes, where $\tau(a \otimes b \otimes c) = b \otimes a \otimes c$. Up to isomorphism, these properties uniquely determine the module N and the isomorphism ν.

It turns out that we can extend Theorem 2.0.4 to algebras. The formulation for algebras is as follows.

Theorem 2.0.5. *(Descent of Algebras) Let B be a commutative faithfully flat A-algebra. Let M be a B-algebra and $g : B \otimes_A M \to M \otimes_A B$ be a descent datum for M over B such that g is an isomorphism of $B \otimes_A B$-algebras. Then there exists an A-algebra N and an isomorphism $\nu : N \otimes_A B \to M$ of B-algebras such that the diagram of $B \otimes_A B$-algebras*

$$\begin{array}{ccc} B \otimes_A N \otimes_A B & \xrightarrow{1\otimes\nu} & B \otimes_A M \\ \tau\downarrow & & \downarrow g \\ N \otimes_A B \otimes_A B & \xrightarrow{\nu\otimes 1} & M \otimes_A B \end{array} \tag{2.2}$$

commutes, where $\tau(a \otimes b \otimes c) = b \otimes a \otimes c$. Up to isomorphism, these properties uniquely determine the module N and the isomorphism ν.

We can formulate our problem in terms of category theory as follows. There is a functor $- \otimes_A B$ from the category of A-modules to the category of B-modules, which sends every A-module N, to its extension of scalars, namely $N \otimes_A B$, which is naturally endowed with the structure of a B-module. We are addressing the following question.
Question: When can we go the other way? In other words, when is a B-module M of the form $N \otimes_A B$ for some A-module N?

The technique of faithfully flat descent was developed by A. Grothendieck in a series of Bourbaki seminars over the period 1959-1962.

2.1 Flatness and faithful flatness

2.1.1 Definitions and examples

Definition 2.1.1. An A-module M is called **flat** if for every injective homomorphism of A-modules $N \xrightarrow{f} N'$ the map

$$N \otimes_A M \xrightarrow{f\otimes 1} N' \otimes_A M$$

is injective.

Remark 2.1.2. *The tensor product is known to be right-exact, so flat modules are precisely those modules that send short exact sequences to short exact sequences.*

Example 2.1.3. All free modules (and more generally all projective modules) are flat.

Example 2.1.4. $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. In fact, let $L \overset{\psi}{\hookrightarrow} M$ be an inclusion of $\mathbb{Z}$-modules. We want to show that $L \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\psi\otimes 1} M \otimes_{\mathbb{Z}} \mathbb{Q}$ is injective. First, note that every element of $L \otimes_{\mathbb{Z}} \mathbb{Q}$ is of the form $l \otimes \frac{1}{d}$ for $l \in L, d \in \mathbb{Z} \setminus \{0\}$. In fact, let $\sum_i l_i \otimes \frac{a_i}{b_i} \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ with $l_i \in L$ and $a_i, b_i \in \mathbb{Z}$ with $b_i \neq 0$. Denote $d_i = \prod_{j\neq i} b_j$, $d = \prod_j b_j$ and $l = \sum_i d_i a_i l_i$. We have

$$\begin{aligned}\sum_i l_i \otimes \frac{a_i}{b_i} = \sum_i a_i l_i \otimes \frac{1}{b_i} &= \sum_i a_i l_i \otimes \frac{\prod_{j\neq i} b_j}{\prod_j b_j} \\ &= \sum_i a_i l_i \otimes \frac{d_i}{d} \\ &= \sum_i d_i a_i l_i \otimes \frac{1}{d} \\ &= l \otimes \frac{1}{d}.\end{aligned}$$

If $l \otimes \frac{1}{d} \in \ker(\psi \otimes 1)$ then $\psi(l) \otimes \frac{1}{d} = 0$ in $M \otimes_{\mathbb{Z}} \mathbb{Q} \cong S^{-1}M$ with $S = \mathbb{Z} \setminus \{0\}$ (recall the $S^{-1}A$-module isomorphism $M \otimes_A S^{-1}A \cong S^{-1}M, m \otimes \frac{a}{s} \mapsto \frac{am}{s}$ for any multiplicative set S). Thus, $c \cdot \psi(l) = \psi(cl) = 0$ for some $c \in \mathbb{Z} \setminus \{0\}$, so $cl = 0$ because ψ is injective. Hence, $l \otimes \frac{1}{d} = l \otimes \frac{c}{cd} = cl \otimes \frac{1}{cd} = 0$ in $L \otimes_{\mathbb{Z}} \mathbb{Q}$. Consequently, $\psi \otimes 1$ is injective as desired.

Note that this example can be easily generalized as follows. Replace $\mathbb{Z}$ with any domain A and $\mathbb{Q}$ with the field of fractions $\mathrm{Frac}(A)$ of A. We will see that even more is true, namely localizations are flat.

Example 2.1.5. For any $n \geq 2$, $\mathbb{Z}/n\mathbb{Z}$ is not flat over $\mathbb{Z}$. In fact, after tensoring the canonical injection

$$n\mathbb{Z} \hookrightarrow \mathbb{Z}$$

with $\mathbb{Z}/n\mathbb{Z}$, the image of $n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$ is equal to $n \cdot \mathbb{Z}/n\mathbb{Z} = 0$ but

$$n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z} \neq 0$$

which shows that the map

$$n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$$

is not injective.

Example 2.1.6. $k[x]$ is flat over $k[x^2]$. In fact, $k[x] = (k[x^2])[x]$ is a polynomial ring over $k[x^2]$ so it is free and therefore flat.

Example 2.1.7. The $A = k[x, xy]$-module $M = k[x, y]$ is not flat. We will apply the criterion for flatness with ideals (see Theorem 2.1.9). Take the ideal $I = (x, y)$ generated by x and y. Then $x \otimes y - xy \otimes 1 \neq 0$ in $I \otimes_A M$ but it goes to 0 under the canonical map $I \otimes_A M \to IM$. In fact, it is enough to find a bilinear map $I \times M \to IM$ that sends $(x, y) - (xy, 1)$ to a nonzero element of IM. For $f = \sum_{i,j} c_{ij} x^i y^j \in I$ consider the bilinear map $s : I \times M \to IM,\ (f, m) \mapsto c_{10} \cdot m$. Then $s((x, y)) - s((xy, 1)) = 1 \cdot y - 0 \cdot 1 = y \neq 0$, so $x \otimes y - xy \otimes 1 \neq 0$.

2.1.2 Properties of flatness

Proposition 2.1.8. *Let A be a commutative ring. The following statements hold.*

(1) Every free module is flat

(2) (Product) The tensor product of modules that are flat over A is flat over A

(3) (Base Change) Let B be an A-algebra. If M is flat over A, then $M \otimes_A B$ is flat over B

(4) (Transitivity) Let B be a flat A-module. Then every B-module that is flat over B is flat over A.

Proof. We only prove (1) and (3). The other two properties are proved similarly.

(1) If M is free then $M = \bigoplus_{i \in I} A$. Let $N' \overset{\iota}{\hookrightarrow} N$ be an inclusion of A-modules. Note that $N' \otimes_A M \cong N' \otimes_A \bigoplus_{i \in I} A \cong \bigoplus_{i \in I} N' \otimes_A A \cong \bigoplus_{i \in I} N'$. Similarly, $N \otimes_A M \cong \bigoplus_{i \in I} N$, so the map $N' \otimes_A M \to N \otimes_A M$ is the map $\bigoplus_{i \in I} N' \xrightarrow{\bigoplus_{i \in I} \iota} \bigoplus_{i \in I} N$ which is injective because ι is injective. Hence M is flat.

(3) Let $N' \hookrightarrow N$ be an inclusion of B-modules. Then $N' \otimes_B (M \otimes_A B) \cong N' \otimes_B (B \otimes_A M) \cong N' \otimes_A M$ and similarly, $N \otimes_B (M \otimes_A B) \cong N \otimes_A M$ so the map $N' \otimes_B (M \otimes_A B) \to N \otimes_B (M \otimes_A B)$ is injective because M is flat over A. Hence $M \otimes_A B$ is flat over B. □

Theorem 2.1.9. *(Criterion for flatness using ideals) Let M be an A-module. Then M is flat if and only if for any ideal I of A the canonical homomorphism $I \otimes_A M \to IM$ is an isomorphism.*

Proof. Suppose that M is flat and consider the canonical inclusion $I \overset{\iota}{\hookrightarrow} A$. Then $I \otimes_A M \overset{\iota \otimes 1}{\hookrightarrow} A \otimes_A M \cong M$ and $\mathrm{im}(\iota \otimes 1) = IM$ so $I \otimes_A M \cong IM$.

Conversely, suppose that we have this isomorphism for all ideals I of A. Let $N' \hookrightarrow N$ be an inclusion of A-modules. We argue by considering more and more general cases of N.

Case 1: N is free of finite rank n, so $N \cong A^n$.

If $n = 1$ then $N \cong A$ so N' can be viewed as an ideal of N, and $N' \otimes_A M \hookrightarrow N \otimes_A M$ by assumption. Suppose that $n \geq 2$ and suppose that the result holds for all free modules of rank $< n$. Write $N = N_1 \oplus N_2$ for two free submodules N_1, N_2. Let $N_1' = N_1 \cap N'$ and N_2' be the image of N' in $N_2 \cong N/N_1$. We get the following commutative diagram with exact horizontal arrows:

$$\begin{array}{ccccc}
N_1' & \hookrightarrow & N' & \twoheadrightarrow & N_2' \\
\downarrow & & \downarrow & & \downarrow \\
N_1 & \hookrightarrow & N & \twoheadrightarrow & N_2
\end{array}$$

Tensoring with M gives

$$\begin{array}{ccccc}
N_1' \otimes_A M & \longrightarrow & N' \otimes_A M & \longrightarrow & N_2' \otimes_A M \\
\beta \downarrow & & \downarrow & & \downarrow \gamma \\
N_1 \otimes_A M & \xrightarrow{\alpha} & N \otimes_A M & \longrightarrow & N_2 \otimes_A M
\end{array}$$

with horizontal arrows still exact because $- \otimes_A M$ is right exact. Now

- α is injective (by distributivity of the tensor product over direct sum)
- β and γ are injective (by induction hypothesis applied to N_i)

so $N' \otimes_A M \to N \otimes_A M$ is injective. In fact, it follows from exactness of the rows:

$$\begin{array}{ccccc}
0 & \longrightarrow & \ker(\phi) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
N_1' \otimes_A M & \longrightarrow & N' \otimes_A M & \longrightarrow & N_2' \otimes_A M \\
\downarrow & & \downarrow \phi & & \downarrow \\
N_1 \otimes_A M & \longrightarrow & N \otimes_A M & \longrightarrow & N_2 \otimes_A M
\end{array}$$

Case 2: N is free of arbitrary rank.

If N_0 is a direct factor of N of finite rank then $(N' \cap N_0) \otimes_A M \hookrightarrow N_0 \otimes_A M$ so $(N' \cap N_0) \otimes_A M \hookrightarrow N \otimes_A M$ because N_0 is a direct factor of N. Since every element of the tensor product $N' \otimes_A M$ is a finite sum of simple tensors for any $x \in N' \otimes_A M$, there is N_0 such that x is contained in the image of $(N' \cap N_0) \otimes_A M \hookrightarrow N \otimes_A M$, so $N' \otimes_A M \hookrightarrow N \otimes_A M$.

Case 3: N is an arbitrary A-module.

There is a free A-module L and a surjective homomorphism $L \twoheadrightarrow N$ (take $L = \bigoplus_{n \in N} A$ and $p : L \to N,\ (0, \dots, 0, 1, 0, \dots) \mapsto n$). Let $L' = p^{-1}(N')$. We have the following commutative diagram

$$\begin{array}{ccccccc} \ker(p) & \hookrightarrow & L' & \twoheadrightarrow & N' & \longrightarrow & 0 \\ \| & & \downarrow & & \downarrow & & \\ \ker(p) & \hookrightarrow & L & \twoheadrightarrow & N & \longrightarrow & 0 \end{array}$$

with exact horizontal arrows so after tensoring we get

$$\begin{array}{ccccccc} \ker(p) \otimes_A M & \longrightarrow & L' \otimes_A M & \longrightarrow & N' \otimes_A M & \longrightarrow & 0 \\ \| & & \downarrow f & & \downarrow g & & \\ \ker(p) \otimes_A M & \longrightarrow & L \otimes_A M & \longrightarrow & N \otimes_A M & \longrightarrow & 0 \end{array}$$

with the rows still exact. Since f is injective, g is also injective. □

Corollary 2.1.10. *Let A be a PID. An A-module M is flat if and only if it is torsion-free.*

Proof. For $a \in A$, $a \neq 0$, let t_a and u_a be left multiplication maps by a in A and M, respectively. For the ideal $I := (a)$, $t_a : A \to I$, $x \mapsto ax$ is an isomorphism. We have the following commutative diagram

$$\begin{array}{ccccc} A \otimes_A M & \cong & M & \xrightarrow{t_a \otimes 1} & I \otimes_A M \\ & & {\scriptstyle u_a}\downarrow & \swarrow {\scriptstyle f} & \\ & & IM & & \end{array}$$

If M is flat then $t_a \otimes 1$ is an isomorphism, so u_a is an isomorphism. Hence M is torsion-free.

Conversely, suppose that M is torsion-free. It suffices to show that f is injective. Let $x = \sum_i ax_i \otimes m_i \in I \otimes_A M$, where $I = (a), a \neq 0$ and A is a PID. If $x \in \ker(f)$ then

$$f(x) = \sum_i ax_i m_i = a \cdot \sum_i x_i m_i = 0,$$

so $\sum_i x_i m_i = 0$ because M is torsion-free. Thus, $x = \sum_i ax_i \otimes m_i = a \otimes \sum_i x_i m_i = 0$, which proves that f is injective. □

Lemma 2.1.11. *For any multiplicative set S of A the canonical homomorphism $A \to S^{-1}A$ is flat, i.e. $S^{-1}A$ is flat over A for its A-module structure.*

Proof. If $N \hookrightarrow M$ is an inclusion of A-modules then after tensoring we get $S^{-1}N \cong N \otimes_A S^{-1}A \to M \otimes_A S^{-1}A \cong S^{-1}M$ which is injective because localizations are exact. □

Lemma 2.1.12. *Let M be an A-module. Then $M = 0$ if and only if $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \in \operatorname{Spec\,max}(A)$.*

Proof. If $M = 0$ then clearly $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of A.

Conversely, suppose that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of A. Let $x \in M$ and let $I = \{a \in A \,|\, ax = 0\}$. We claim that $I = A$. In fact, if I is a proper ideal of A then it must be contained in some maximal ideal $\mathfrak{m}$ of A. By assumption, $M_{\mathfrak{m}} = 0$ so there exists $s \in S = A \setminus \mathfrak{m}$ such that $sx = 0$, so $s \in I$. Thus, $I \not\subset \mathfrak{m}$, which is a contradiction. Thus, $I = A$ and $1 \in I$ so $x = 1 \cdot x = 0$ which proves that $M = 0$. □

Proposition 2.1.13. *Let M be an A-module. The following are equivalent:*

(1) M is flat over A

(2) $M_{\mathfrak{p}}$ is flat over $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$

(3) $M_{\mathfrak{m}}$ is flat over $A_{\mathfrak{m}}$ for all $\mathfrak{m} \in \operatorname{Spec}\max(A)$.

Proof. (1) $\Rightarrow$ (2) Suppose that M is flat over A. Let $\mathfrak{p} \in \operatorname{Spec}(A)$, $S = A \setminus \mathfrak{p}$. Then $M_{\mathfrak{p}} = S^{-1}M \cong M \otimes_A S^{-1}A$ is flat over $S^{-1}A = A_{\mathfrak{p}}$ (follows from base change property, see Proposition 2.1.8).

(2) $\Rightarrow$ (3) clear.

(3) $\Rightarrow$ (1) Let $N' \hookrightarrow N$ be an inclusion of A-modules. Let $L = \ker(N' \otimes_A M \to N \otimes_A M)$. By Lemma 2.1.11, $A \to S^{-1}A$ is flat for any S, so for any maximal ideal $\mathfrak{m}$ of A we get an exact sequence

$$0 \to L \otimes_A A_{\mathfrak{m}} \to (N' \otimes_A M) \otimes_A A_{\mathfrak{m}} \to (N \otimes_A M) \otimes_A A_{\mathfrak{m}}.$$

which can be rewritten as

$$0 \to L \otimes_A A_{\mathfrak{m}} \to N'_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}} \to N_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}}$$

In fact, we have the following isomorphism

$$N'_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}} \cong S^{-1}N' \otimes_{S^{-1}A} S^{-1}M \cong (N' \otimes_A S^{-1}A) \otimes_{S^{-1}A} (M \otimes_A S^{-1}A) \cong N' \otimes_A M \otimes_A S^{-1}A \cong (N' \otimes_A M) \otimes_A A_{\mathfrak{m}}.$$

Since $N'_{\mathfrak{m}} \hookrightarrow N_{\mathfrak{m}}$ and $M_{\mathfrak{m}}$ is flat over $A_{\mathfrak{m}}$, we get $N'_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}} \hookrightarrow N_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} M_{\mathfrak{m}}$. Thus, $L \otimes_A A_{\mathfrak{m}} = L_{\mathfrak{m}} = 0$, so $L = 0$ by Lemma 2.1.12. □

2.1.3 Relation between projective and flat modules

Definition 2.1.14. A module is **projective** if it is a direct summand of a free module.

Definition 2.1.15. An A-module M is **finitely presented** if there is an exact sequence

$$F_1 \to F_0 \to M \to 0$$

where F_0, F_1 are free with finite bases.

The goal of this section is to prove the following theorem:

Theorem 2.1.16. *A finitely presented module is projective if and only of it is flat.*

Proposition 2.1.17. *Projective modules are flat.*

Proof. By definition, projective modules are direct summands of free modules so it is enough to show that for any A-module M of the form $M = \bigoplus_{i \in I} M_i$,

$$M \text{ is flat if and only if each } M_i \text{ is flat .}$$

Let $N \overset{\iota}{\hookrightarrow} L$ be an inclusion of A-modules. We have the following commutative diagram (with vertical maps the isomorphisms expressing the bilinearity of $\otimes$ over $\oplus$):

$$\begin{array}{ccc} N \otimes_A (\bigoplus_{i \in I} M_i) & \xrightarrow{\iota \otimes 1} & L \otimes_A (\bigoplus_{i \in I} M_i) \\ \downarrow \cong & & \cong \downarrow \\ \bigoplus_{i \in I} (N \otimes_A M_i) & \longrightarrow & \bigoplus_{i \in I} (L \otimes_A M_i) \end{array}$$

Since the top map is injective if and only if the bottom map is injective, the result follows. □

Definition 2.1.18. If M is an A-module then its character module is defined by

$$M^* = \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}).$$

It is an A-module by defining $a \cdot f(m) = f(am) = f(ma)$.

Lemma 2.1.19. *A sequence of A-modules*

$$M_1 \xrightarrow{\alpha} M \xrightarrow{\beta} M_2 \ (\star)$$

is exact if and only if the sequence

$$M_2^* \xrightarrow{\beta^*} M^* \xrightarrow{\alpha^*} M_1^* \ (\star\star)$$

is exact.

Proof. If $(\star)$ is exact then $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ carries it into an exact sequence.

Let us prove the converse. We want to show that

$$\ker(\alpha^*) = \mathrm{im}(\beta^*) \Rightarrow \ker(\beta) = \mathrm{im}(\alpha).$$

Suppose that $m_1 \in M_1$ and $\alpha(m_1) \notin \ker(\beta)$, so $\beta \circ \alpha(m_1) \neq 0$. Since $\mathbb{Q}/\mathbb{Z}$ is a cogenerator, there is $f : M_2 \to \mathbb{Q}/\mathbb{Z}$ with $f \circ \beta \circ \alpha(m_1) \neq 0$, $f \in M_2^*$, so $f \circ \beta \circ \alpha \neq 0$. Thus, $\alpha^* \circ \beta^*(f) \neq 0$, a contradiction. This proves that $\mathrm{im}(\alpha) \subset \ker(\beta)$. Now suppose that there is $m \in \ker(\beta)$, $m \notin \mathrm{im}(\alpha)$. Then $m + \mathrm{im}(\alpha) \neq \bar{0}$ in $M/\mathrm{im}(\alpha)$. Thus, there exists $g : M/\mathrm{im}(\alpha) \to \mathbb{Q}/\mathbb{Z}$ such that $g(m + \mathrm{im}(\alpha)) \neq 0$. Consider $f := g \circ \pi : M \to \mathbb{Q}/\mathbb{Z}$ where $\pi : M \to M/\mathrm{im}(\alpha)$ is the natural projection. Then $f(m) \neq 0$ and $f(\mathrm{im}(\alpha)) = 0$, so $0 = f \circ \alpha = \alpha^*(f)$ and $f \in \ker(\alpha^*) = \mathrm{im}(\beta^*)$. Hence, $f = \beta^*(h) = h \circ \beta$ for some $h \in M_2^*$, which implies $f(m) = h \circ \beta(m) = 0$ but $f(m) \neq 0$, a contradiction. □

Lemma 2.1.20. *Let A, B be rings. Let M be a finitely presented A-module and N an (A, B)-bimodule (i.e. a left A-module, a right B-module and $(am)b = a(mb)$ holds for all $a \in A, b \in B, m \in N$). Then $\sigma : N^* \otimes_A M \to \mathrm{Hom}_A(M, N)^*$ given by $\phi \otimes m \mapsto [\psi \mapsto \phi(\psi(m))]$ for $\phi \in N^*, m \in M, \psi \in \mathrm{Hom}_A(M, N)$ is an isomorphism.*

Proof. Since M is finitely presented, there exist $m, n \in \mathbb{N}$ such that

$$A^m \to A^n \to M \ (\star)$$

is an exact sequence of A-modules. We claim that $N^* \otimes_A A^m \cong \mathrm{Hom}_A(A^m, N)^*$. Tt follows from properties of Hom functor and the tensor product. In fact,

$$\begin{aligned} N^* \otimes_A A^m &\cong N^* \otimes_A \bigoplus_{i=1}^m A \\ &\cong \bigoplus_{i=1}^m (N^* \otimes_A A) \\ &\cong \bigoplus_{i=1}^m \mathrm{Hom}_A(A, N)^* \\ &\cong \bigoplus_{i=1}^m \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_A(A, N), \mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\bigoplus_{i=1}^m \mathrm{Hom}_A(A, N), \mathbb{Q}/\mathbb{Z}) \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_A(A^m, N), \mathbb{Q}/\mathbb{Z}) \\ &= \mathrm{Hom}_A(A^m, N)^* \end{aligned}$$

After applying $N^* \otimes_A -$ to $(\star)$ we get:

$$\begin{array}{ccccccc} N^* \otimes_A A^m & \longrightarrow & N^* \otimes_A A^n & \longrightarrow & N^* \otimes_A M & \longrightarrow & 0 \\ \downarrow \cong & & \downarrow \cong & & \downarrow \sigma & & \\ \mathrm{Hom}_A(A^m, N)^* & \longrightarrow & \mathrm{Hom}_A(A^n, N)^* & \longrightarrow & \mathrm{Hom}_A(M, N)^* & \longrightarrow & 0 \end{array}$$

By the Three Lemma, σ is an isomorphism. □

Fact: P is a projective A-module if an only if $\mathrm{Hom}_A(P, -)$ is right exact.

Proof. (of the Theorem 2.1.16): Let $N \xrightarrow{\phi} N_0$ be an exact sequence of A-modules. By the fact above it is enough to show that

$$\mathrm{Hom}_A(M, N) \xrightarrow{\phi^*} \mathrm{Hom}_A(M, N_0) \to 0$$

is exact. By Lemma 2.1.19, $0 \to N_0^* \to N^*$ is exact. After applying $- \otimes_A M$, we get the following commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & N_0^* \otimes_A M & \longrightarrow & N^* \otimes_A M \\ & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \mathrm{Hom}_A(M, N_0)^* & \longrightarrow & \mathrm{Hom}_A(M, N)^* \end{array}$$

Since M is flat the top row is exact and the vertical maps are isomorphisms so the bottom row is exact. By Lemma 2.1.19 again, we get that

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N_0) \to 0$$

is exact. Hence, M is projective. □

2.1.4 Faithfully flat modules

Proposition 2.1.21. *Let M be a flat A-module. Then the following are equivalent:*

(1) $M \neq \mathfrak{m}M$ for every maximal ideal $\mathfrak{m}$ of A.

(2) Let N be an A-module. If $M \otimes_A N = 0$, then $N = 0$.

(3) Let $f : N_1 \to N_2$ be a homomorphism of A-modules. If $f \otimes 1_M : N_1 \otimes_A M \to N_2 \otimes_A M$ is an isomorphism, then so is f.

Proof. See See [[2], Corollary 2.20., pp. 12] □

Definition 2.1.22. Let M be a flat module over a ring A. We say that M is **faithfully flat** over A if it verifies one of the properties of the proposition above. Let $f : A \to B$ be a ring homomorphism. We say that B is faithfully flat over A if it is faithfully flat as an A-module. We will also say that f is faithfully flat.

Remark 2.1.23. *One can verify that Proposition 2.1.8 remains true if we replace "flat" by "faithfully flat" and take only non-zero modules.*

Corollary 2.1.24. *Let $f : A \to B$ be a flat ring homomorphism. The following properties are equivalent:*

(1) f is faithfully flat.

(2) For every prime ideal $\mathfrak{p}$ of A, there exists a prime ideal $\mathfrak{q}$ of B such that $f^{-1}(\mathfrak{q}) = \mathfrak{p}$.

(3) For every maximal ideal $\mathfrak{m}$ of A, there exists a maximal ideal $\mathfrak{n}$ of B such that $f^{-1}(\mathfrak{n}) = \mathfrak{m}$.

Proof. See [[2], Corollary 2.20., pp. 13] □

2.2 Faithfully flat descent

2.2.1 Amitsur complex

Definition 2.2.1. Let $\theta : A \to B$ be a homomorphism of commutative rings. By $B^{\otimes r}$ we denote $B \otimes_A \cdots \otimes_A B$, the tensor product of r copies of B. For any $0 \leq j \leq r$, define the A-module homomorphisms

$$\begin{aligned} e_j : B^{\otimes(r+1)} &\to B^{\otimes(r+2)} \\ b_0 \otimes \cdots \otimes b_r &\mapsto b_0 \otimes \cdots b_{j-1} \otimes 1 \otimes b_j \otimes \cdots \otimes b_r \end{aligned}$$

The **Amitsur complex** for B over A is

$$0 \to A \xrightarrow{\theta} B \xrightarrow{d^0} B^{\otimes 2} \xrightarrow{d^1} B^{\otimes 3} \xrightarrow{d^2} \cdots$$

where the coboundary map $d^r : B^{\otimes(r+1)} \to B^{\otimes(r+2)}$ is defined by

$$d^r = \sum_{i=0}^{r+1} (-1)^i e_i.$$

We denote the Amitsur complex by $\mathcal{C}^\bullet(B/A)$.

Remark 2.2.2. One can check that $e_j e_i = e_{i+1} e_j$ for $j \leq i$, and that this is in fact a complex of A-modules. In fact, we have

$$\begin{aligned} e_{i+1} \circ e_j(x_0 \otimes \cdots \otimes x_n) &= e_{i+1}(x_0 \otimes \cdots x_{j-1} \otimes 1 \otimes x_j \otimes \cdots \otimes x_n) \text{ where } x_j \text{ appears in the } (j+1)\text{st slot} \\ &= x_0 \otimes \cdots \otimes x_{j-1} \otimes 1 \otimes x_j \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_n \end{aligned}$$

where in the last line x_{i-1} is in the ith slot. On the other hand

$$\begin{aligned} e_j \circ e_i(x_0 \otimes \cdots \otimes x_n) &= e_j(x_0 \otimes \cdots x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_n) \text{ where } x_i \text{ appears in the } (i+1)\text{st slot} \\ &= x_0 \otimes \cdots \otimes x_{j-1} \otimes 1 \otimes x_j \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_n \end{aligned}$$

where in the last line x_{i-1} is in the ith slot, so $e_{i+1} e_j = e_j e_i$.

Now we check that Amitsur complex is in fact a complex. Let $b_0 \otimes \cdots \otimes b_r \in B^{\otimes(r+1)}$. Then

$$\begin{aligned} d^{r+1} \circ d^r(b_0 \otimes \cdots \otimes b_r) &= d^{r+1}\Big(\sum_{i=0}^{r+1} (-1)^i e_i(b_0 \otimes \cdots \otimes b_r)\Big) \\ &= \sum_{i=0}^{r+1} (-1)^i d^{r+1} e_i(b_0 \otimes \cdots \otimes b_r) \\ &= \sum_{i=0}^{r+1} (-1)^i \sum_{j=0}^{r+2} (-1)^j e_j e_i(b_0 \otimes \cdots \otimes b_r) \\ &= \sum_{i=0}^{r+1} \sum_{j=0}^{r+2} (-1)^{i+j} e_j e_i(b_0 \otimes \cdots \otimes b_r) \\ &= \sum_{i=0}^{r+1} \sum_{j \leq i} (-1)^{i+j} e_j e_i(b_0 \otimes \cdots \otimes b_r) + \sum_{i=0}^{r+1} \sum_{j \geq i+1} (-1)^{i+j} e_j e_i(b_0 \otimes \cdots \otimes b_r) \\ &= \sum_{i=0}^{r+1} \sum_{j \leq i} (-1)^{i+j} e_{i+1} e_j(b_0 \otimes \cdots \otimes b_r) + \sum_{i=0}^{r+1} \sum_{j \geq i} (-1)^{i+j+1} e_{j+1} e_i(b_0 \otimes \cdots \otimes b_r) \\ &= \sum_{i=0}^{r+1} \sum_{j \leq i} (-1)^{i+j} e_{i+1} e_j(b_0 \otimes \cdots \otimes b_r) - \sum_{i=0}^{r+1} \sum_{j \geq i} (-1)^{i+j} e_{j+1} e_i(b_0 \otimes \cdots \otimes b_r) \\ &= 0 \end{aligned}$$

Proposition 2.2.3. *Let B be a commutative faithfully flat A-algebra.*

(1) The Amitsur complex $\mathcal{C}^\bullet(B/A)$ is an exact sequence.

(2) If M is any A-module, then the complex $M \otimes_A \mathcal{C}^\bullet(B/A)$:

$$0 \to M \xrightarrow{1\otimes\theta} M \otimes_A B \xrightarrow{1\otimes d^0} M \otimes_A B^{\otimes 2} \xrightarrow{1\otimes d^1} M \otimes_A B^{\otimes 3} \xrightarrow{1\otimes d^2} \cdots$$

is an exact sequence.

Proof. (1) Step 1: We show that $\mathcal{C}^\bullet(B/A)$ is exact if there exists an A-module homomorphism $\sigma : B \to A$ which is a left inverse for the structure homomorphism $\theta : A \to B$. Define a homotopy operator $k^r : B^{\otimes(r+2)} \to B^{\otimes(r+1)}$ by $k^r(x_0 \otimes \cdots \otimes x_{r+1}) = \sigma(x_0)x_1 \otimes \cdots \otimes x_{r+1}$. It follows from

$$\begin{aligned}
k^r d^r(x_0 \otimes \cdots \otimes x_r) &= k^r \sum_{i=0}^{r+1} (-1)^i e_i(x_0 \otimes \cdots \otimes x_r) \\
&= k^r(1 \otimes x_0 \otimes \cdots \otimes x_r) - k^r(x_0 \otimes 1 \otimes x_1 \otimes \cdots \otimes x_r) + \sum_{i=2}^{r+1}(-1)^i k^r e_i(x_0 \otimes \cdots \otimes x_r) \\
&= \sigma(1) \cdot x_0 \otimes x_1 \otimes \cdots \otimes x_r - \sigma(x_0) \cdot 1 \otimes x_1 \otimes \cdots \otimes x_r + \sum_{i=2}^{r+1}(-1)^i k^r e_i(x_0 \otimes \cdots \otimes x_r) \\
&= x_0 \otimes x_1 \otimes \cdots \otimes x_r - \sigma(x_0) \otimes x_1 \otimes \cdots \otimes x_r + \sum_{i=2}^{r+1}(-1)^i k^r e_i(x_0 \otimes \cdots \otimes x_r)
\end{aligned}$$

and from

$$\begin{aligned}
d^{r-1}k^{r-1}(x_0 \otimes \ldots \otimes x_r) &= d^{r-1}(\sigma(x_0)x_1 \otimes \cdots \otimes x_r) \\
&= \sum_{i=0}^{r} (-1)^i e_i(\sigma(x_0)x_1 \otimes \cdots \otimes x_r) \\
&= 1 \otimes \sigma(x_0)x_1 \otimes x_2 \otimes \cdots \otimes x_r - \sigma(x_0)x_1 \otimes 1 \otimes x_2 \otimes \cdots \otimes x_r \\
&\quad + \sum_{i=2}^{r} (-1)^i e_i(\sigma(x_0)x_1 \otimes \cdots \otimes x_r) \\
&= \sigma(x_0) \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_r - \sigma(x_0)x_1 \otimes 1 \otimes x_2 \otimes \cdots \otimes x_r \\
&\quad + \sum_{i=2}^{r} (-1)^i e_i(\sigma(x_0)x_1 \otimes \cdots \otimes x_r)
\end{aligned}$$

that $k^r d^r + d^{r-1}k^{r-1}$ is the identity map on $B^{\otimes(r+1)}$. This proves that k is a contracting homotopy (meaning identity and zero map are homotopic), so the complex is an exact sequence.

Step 2: If C is another commutative A-algebra, then $\mathcal{C}^\bullet(B \otimes_A C/C)$, the Amitsur complex for $B \otimes_A C$ over C, is obtained by applying the functor $- \otimes_A C$ to the complex $\mathcal{C}^\bullet(B/A)$. This is because

$$B^{\otimes r} \otimes_A C \cong B^{\otimes r} \otimes_A (C \otimes_C C \otimes_C \cdots \otimes_C C) \cong (B \otimes_A C) \otimes_C (B \otimes_A C) \otimes_C \cdots \otimes_C (B \otimes_A C) \cong (B \otimes_A C)^{\otimes r}.$$

Step 3: Consider $\rho : B \to B \otimes_A B$, defined by $b \mapsto 1 \otimes b$. Define $\mu : B \otimes_A B \to B$, by $\mu(b \otimes b') = bb'$. Then μ is a left inverse for ρ. In fact, for any $b \in B$, we have $\mu \circ \rho(b) = \mu(1 \otimes b) = 1 \cdot b = b$. By Step 1, the Amitsur complex $\mathcal{C}^\bullet(B \otimes_A B/B)$ for $\rho : B \to B \otimes_A B$ is exact. This complex looks as follows:

$$0 \longrightarrow A \otimes_A B \xrightarrow{\rho=\theta\otimes 1_B} B \otimes_A B \xrightarrow{d^0\otimes 1_B} B^{\otimes 2} \otimes_A B \xrightarrow{d^1\otimes 1_B} B^{\otimes 3} \otimes_A B \xrightarrow{d^2\otimes 1_B} \cdots$$

where we identify $A \otimes_A B \cong B$. Since $\mathcal{C}^\bullet(B \otimes_A B/B)$ is exact and B is faithfully flat, by Step 2 applied to B, it follows that $\mathcal{C}^\bullet(B/A)$ is exact. In fact, as noted above the complex $\mathcal{C}^\bullet(B \otimes_A B/B)$ is obtained from $\mathcal{C}^\bullet(B/A)$ by tensoring it with $- \otimes_A B$.

(2) One argues as in (1) by assuming that if there is a left inverse to $\theta : A \to B$ then we can construct a contracting homotopy. In fact, the exact same reasoning works for the maps $\widetilde{k^r} = 1_M \otimes k^r$ and $\widetilde{d^r} = 1_M \otimes d^r$, where $\widetilde{k^r}$ plays the role of a new homotopy. The computation as in (1) shows that $\widetilde{k^r}\widetilde{d^r} + \widetilde{d^{r-1}}\widetilde{k^{r-1}}$ is the identity map on $M \otimes_A B^{\otimes(r+1)}$, so $1_{M\otimes_A B^{\otimes(r+1)}}$ determines the zero map on cohomology, i.e. the complex is exact. We consider the maps $\widetilde{\rho} = 1_M \otimes \rho$ and $\widetilde{\mu} = 1_M \otimes \mu$ with ρ and μ as in Step 3 in (1). One checks that $\widetilde{\mu}$ is a left inverse for $\widetilde{\rho}$, so the complex $M \otimes_A \mathcal{C}^\bullet(B \otimes_A B/B)$ is exact. This complex looks as follows.

$$
\begin{array}{ccccccc}
0 \longrightarrow M \otimes_A B & \xrightarrow{\widetilde{\rho}=1_M\otimes\rho} & (M \otimes_A B) \otimes_B (B \otimes_A B) & \xrightarrow{\widetilde{d^0}\otimes 1_B} & (M \otimes_A B) \otimes_B (B \otimes_A B)^{\otimes 2} & \xrightarrow{\widetilde{d^1}\otimes 1_B} & \cdots \\
 & & \Vert & & \Vert & & \\
 & & M \otimes_A B^{\otimes 2} & & M \otimes_A B^{\otimes 3} & &
\end{array}
$$

Since B is faithfully flat, we conclude that the required complex $M \otimes \mathcal{C}^\bullet(B/A)$ is also exact. This complex looks as follows.

$$
0 \longrightarrow M \otimes_A A \xrightarrow{\widetilde{\theta}} M \otimes_A B \xrightarrow{\widetilde{d^0}} M \otimes_A B^{\otimes 2} \xrightarrow{\widetilde{d^1}} M \otimes_A B^{\otimes 3} \xrightarrow{\widetilde{d^2}} \cdots
$$

where $\widetilde{\theta} = 1_M \otimes \theta$. □

2.2.2 The Descent of Homomorphisms

Let B be a commutative A-algebra and M and N a pair of A-modules. Note that one can consider $M \otimes_A B$ and $N \otimes_A B$ as B-modules. The goal is to find sufficient conditions on a homomorphism $g \in \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$ so that $g = f \otimes 1$ for some $f \in \mathrm{Hom}_A(M, N)$. For $i = 0, 1$, we can define the maps

$$
e_i : M \otimes_A B \to M \otimes_A B \otimes_A B
$$

by $e_0(x \otimes b) = x \otimes 1 \otimes b$ and $e_1(x \otimes b) = x \otimes b \otimes 1$. Define homomorphisms $\mathfrak{F}_i$ by demanding the following diagram to be commutative

$$
\begin{array}{ccc}
M \otimes_A B & \xrightarrow{e_i} & M \otimes_A B \otimes_A B \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathfrak{F}_i(g)} \\
N \otimes_A B & \xrightarrow{e_i} & N \otimes_A B \otimes_A B
\end{array}
$$

for $i = 0, 1$. Let us check that it is possible. If $g \in \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$, then we can write $g(x \otimes b) = \sum_i x_i \otimes b_i$ where $b, b_i \in B$ and $x \in M, x_i \in N$. Thus, $e_0 \circ g(x \otimes b) = \sum_i x_i \otimes 1 \otimes b_i$, so we define $\mathfrak{F}_0(g)(x \otimes 1 \otimes b) = \sum_i x_i \otimes 1 \otimes b_i$ and extend it by bilinearity, meaning

$$
\mathfrak{F}_0(g)(x \otimes b \otimes b') = (b \otimes 1) \cdot \mathfrak{F}_0(g)(x \otimes 1 \otimes b') = (b \otimes 1) \cdot \sum_i x_i \otimes 1 \otimes b'_i = \sum_i x_i \otimes b \otimes b'_i,
$$

where $g(x \otimes b') = \sum_i x_i \otimes b'_i$. Similarly, one defines $\mathfrak{F}_1$ by $\mathfrak{F}_1(g)(x \otimes b \otimes 1) = \sum_i x_i \otimes b_i \otimes 1$, where $g(x \otimes b) = \sum_i x_i \otimes b_i$.

Proposition 2.2.4. *Let A be a commutative ring, B a faithfully flat commutative A-algebra, and M and N a pair of A-modules. The sequence*

$$
0 \to \mathrm{Hom}_A(M, N) \xrightarrow{\mathfrak{F}} \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B) \xrightarrow{\mathfrak{F}_0 - \mathfrak{F}_1} \mathrm{Hom}_{B\otimes_A B}(M \otimes_A B \otimes_A B, N \otimes_A B \otimes_A B)
$$

is exact, where $\mathfrak{F}(f) = f \otimes 1$ and $\mathfrak{F}_0, \mathfrak{F}_1$ are defined as above.

Proof. Since each $\mathfrak{F}_i$ is an additive functor, $\mathfrak{F}_0 - \mathfrak{F}_1$ is a $\mathbb{Z}$-module homomorphism. If $f \in \mathrm{Hom}_A(M, N)$, then the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & M \otimes_A B & \xrightarrow{d^0} & M \otimes_A B \otimes_A B \\ & & \downarrow f & & \downarrow f\otimes 1 = \mathfrak{F}(f) & & \downarrow f \otimes 1 \otimes 1 \\ 0 & \longrightarrow & N & \longrightarrow & N \otimes_A B & \xrightarrow{d^0} & N \otimes_A B \otimes_A B \end{array}$$

commutes. Also the rows are exact because these are Amitsur complexes. We claim that $\mathfrak{F}$ is injective. In fact, if $\mathfrak{F}(f) = 0$, then $\mathfrak{F}(f)(x \otimes 1) = (f \otimes 1)(x \otimes 1) = f(x) \otimes 1 = 0$ for all $x \in M$, so $f(x) = 0$ for all $x \in M$, proving that $f = 0$. We claim that $\mathrm{im}(\mathfrak{F}) \subset \ker(\mathfrak{F}_0 - \mathfrak{F}_1)$. In fact, let $g = \mathfrak{F}(f) = f \otimes 1 \in \mathrm{im}(\mathfrak{F})$ for some A-module homomorphism $f : M \to N$. Then

$$\begin{aligned} \mathfrak{F}_0(f \otimes 1)(x \otimes b \otimes b') &= (b \otimes 1) \cdot \mathfrak{F}_0(f \otimes 1)(x \otimes 1 \otimes b') \\ &= (b \otimes 1) \cdot (f(x) \otimes 1 \otimes b') \\ &= f(x) \otimes b \otimes b' \\ &= (1 \otimes b') \cdot (f(x) \otimes b \otimes 1) \\ &= \mathfrak{F}_1(f \otimes 1)(x \otimes b \otimes b') \end{aligned}$$

Thus, $g \in \ker(\mathfrak{F}_0 - \mathfrak{F}_1)$ as desired. To complete the proof, we show that $\ker(\mathfrak{F}_0 - \mathfrak{F}_1) \subset \mathrm{im}(\mathfrak{F})$. Let $g \in \mathrm{Hom}_B(M \otimes_A B, N \otimes_A B)$ and assume $\mathfrak{F}_0(g) = \mathfrak{F}_1(g)$. Given $x \in M$ we have $e_0(x \otimes 1) = e_1(x \otimes 1)$, so

$$e_0 \circ g(x \otimes 1) = \mathfrak{F}_0(g) \circ e_0(x \otimes 1) = \mathfrak{F}_0(g) \circ e_1(x \otimes 1) = \mathfrak{F}_1(g) \circ e_1(x \otimes 1) = e_1 \circ g(x \otimes 1).$$

By exactness of the bottom row (Amitsur complex), we have that

$$g(x \otimes 1) \in \ker(e_0 - e_1) = \ker(d^0) = \mathrm{im}(N \to N \otimes_A B) = N \otimes_A 1 \cong N.$$

Define $f : M \to N$ by $f(x) = g(x \otimes 1)$. Then $g = \mathfrak{F}(f)$. In fact, identifying $N \otimes_A 1 \cong N$, we see that

$$\mathfrak{F}(f)(x \otimes b) = (f \otimes 1)(x \otimes b) = f(x) \otimes b = b \cdot g(x \otimes 1) = g(x \otimes b).$$

□

Remark 2.2.5. Note that from the definition of $\mathfrak{F}_0$ and $\mathfrak{F}_1$, it follows that $g = f \otimes 1$ if and only if g is in the kernel of $\mathfrak{F}_0 - \mathfrak{F}_1$.

2.2.3 The Descent Datum and some examples

Let $\theta : A \to B$ be a homomorphism of commutative rings. We begin with a general construction for any four B-modules M, N, L, K. Consider the tensor products $M \otimes_A N$ and $L \otimes_A K$ as $B \otimes_A B$-modules. Let $f : M \otimes_A N \to L \otimes_A K$ be a homomorphism of $B \otimes_A B$-modules. Write $f(m \otimes n) = \sum_i l_i \otimes k_i$, $f(m' \otimes n') = \sum_{i'} l'_{i'} \otimes k'_{i'}$ and $f(m'' \otimes n'') = \sum_{i''} l''_{i''} \otimes k''_{i''}$. Define homomorphisms

$$f_1 : B \otimes_A M \otimes_A N \to B \otimes_A L \otimes_A K$$

$$f_2 : M \otimes_A B \otimes_A N \to L \otimes_A B \otimes_A K$$

$$f_3 : M \otimes_A N \otimes_A B \to L \otimes_A K \otimes_A B$$

by the formulas

$$f_1(b \otimes m \otimes n) = b \otimes \sum_i l_i \otimes k_i,$$

$$f_2(m' \otimes b \otimes n') = \sum_{i'} l'_{i'} \otimes b \otimes k'_{i'},$$

$$f_3(m'' \otimes n'' \otimes b) = (\sum_{i''} l''_{i''} \otimes k''_{i''}) \otimes b.$$

This means that the f_i are obtained from f by tensoring it with the identity map on B in ith position.

Definition 2.2.6. Let M a B-module and $g : B \otimes_A M \to M \otimes_A B$ a $B \otimes_A B$-module homomorphism. We apply the construction above to g, to obtain the following three $B \otimes_A B \otimes_A B$-module homomorphisms

$$g_1 : B \otimes_A B \otimes_A M \to B \otimes_A M \otimes_A B$$
$$g_2 : B \otimes_A B \otimes_A M \to M \otimes_A B \otimes_A B$$
$$g_3 : B \otimes_A M \otimes_A B \to M \otimes_A B \otimes_A B,$$

where g_i is obtained by tensoring g with the identity map on B in position i. If g is an isomorphism of $B \otimes_A B$-modules and $g_2 = g_3 \circ g_1$ then we call it a **descent datum for M over B.**

Example 2.2.7. Suppose that M is already of the form $N \otimes_A B$ for some A-module N, i.e. $M = N \otimes_A B$. Then $B \otimes_A M = B \otimes_A N \otimes_A B$ and $M \otimes_A B = N \otimes_A B \otimes_A B$. We claim that the isomorphism τ appearing in the diagram in the statement of the Theorem of Faithfully Flat Descent for modules is a descent datum. Recall that

$$B \otimes_A N \otimes_A B \xrightarrow{\tau} N \otimes_A B \otimes_A B$$
$$a \otimes b \otimes c \mapsto b \otimes a \otimes c$$

First, we check that $g = \tau$ is a $B \otimes_A B$-module homomorphism. In fact, for any $b, b' \in B$ and $x, z \in B$, $y \in N$, we have

$$(b \otimes b') \cdot g(x \otimes y \otimes z) = (b \otimes b') \cdot (y \otimes x \otimes z) = y \otimes bx \otimes b'z = g(bx \otimes y \otimes b'z) = g((b \otimes b') \cdot (x \otimes y \otimes z)).$$

It remains to check the cocycle condition, i.e. $g_2 = g_3 g_1$. In fact, for any $n \in N$ and $a, b, c \in B$, we have

$$g_2(a \otimes b \otimes m \otimes c) = m \otimes a \otimes b \otimes c = g_3(a \otimes m \otimes b \otimes c) = g_3 g_1(a \otimes b \otimes m \otimes c),$$

so $g_2 = g_3 g_1$.

Example 2.2.8. Let A be a commutative ring and $\alpha_1, \ldots, \alpha_n$ a set of n elements of A such that $A = A\alpha_1 + \ldots + A\alpha_n$. Denote the localization of A with respect to the multiplicative set $\{\alpha^k \mid k \geq 0\}$ by A_α. Write $S_i = \{\alpha_i^{k_i} \mid k_i \geq 0\}$ for the multiplicative sets corresponding to the generators of A. Let $B = \bigoplus_{i=1}^n A_{\alpha_i}$.

We claim that B is faithfully flat over A. In fact, localizations A_{α_i} are flat and since tensor products commute with direct sums, B is also flat. In order to check that B is faithfully flat over A it suffices to show that for any A-module N, if $N \otimes_A B = 0$, then $N = 0$. Suppose that $N \otimes_A B = 0$. Then

$$0 = N \otimes_A B = \bigoplus_{i=1}^n N \otimes_A A_{\alpha_i} \cong \bigoplus_{i=1}^n N_{\alpha_i},$$

where N_{α_i} denotes the localization of N with respect to the multiplicative set S_i. Thus, $N_{\alpha_i} = 0$ for all i. Let $x \in N$. By definition of localization, for each i, there is $t_i = \alpha_i^{k_i} \in S_i$ such that $t_i x = 0$. Let I be the ideal generated by all t_i. Note that $I = A$. In fact, if $I \subsetneq A$, then I is contained in some maximal ideal $\mathfrak{m}$ of A. Then $t_i = \alpha_i^{k_i} \in \mathfrak{m}$. Since $\mathfrak{m}$ is a prime ideal, we have $\alpha_i \in \mathfrak{m}$ for all i, so $A \subset \mathfrak{m}$, a contradiction. Hence $I = A$. We can write $1 = \sum_i a_i t_i$ for some $a_i \in A$. Then $x = 1 \cdot x = \sum_i a_i t_i x = 0$. Hence, $N = 0$ which proves that B is faithfully flat.

We identify $A_{\alpha_i} \otimes_A A_{\alpha_j}$ with $A_{\alpha_i \alpha_j}$. Then $B \otimes_A B = \bigoplus_{1 \leq i,j \leq n} A_{\alpha_i \alpha_j}$. Suppose that for each i, M_i is an A_{α_i}-module. Then $M = \bigoplus_{i=1}^n M_i$ is a B-module. We have

$$B \otimes_A M = \bigoplus_{i,j} A_{\alpha_i} \otimes_A M_j$$

and

$$M \otimes_A B = \bigoplus_{i,j} M_i \otimes_A A_{\alpha_j}.$$

A descent datum $g : B \otimes_A M \to M \otimes_A B$ consists of a collection of $A_{\alpha_i\alpha_j}$-module isomorphisms

$$A_{\alpha_i} \otimes_A M_j \xrightarrow{g_{ij}} M_i \otimes_A A_{\alpha_j}$$

where $(i,j) \in I_n^2$. The identity $g_2 = g_3 \circ g_1$ is equivalent to the statement that the diagram of $A_{\alpha_i\alpha_j\alpha_k}$-module homomorphisms

$$\begin{array}{ccc} A_{\alpha_i} \otimes_A A_{\alpha_j} \otimes_A M_k & \xrightarrow{\quad g_{ik}\otimes 1 \quad} & M_i \otimes_A A_{\alpha_j} \otimes_A A_{\alpha_k} \\ & {\scriptstyle g_{jk}\otimes 1} \searrow \qquad \nearrow {\scriptstyle g_{ij}\otimes 1} & \\ & A_{\alpha_i} \otimes_A M_j \otimes_A A_{\alpha_k} & \end{array}$$

commutes for all triples $(i,j,k) \in I_n^3$. If a descent datum exists, then by Theorem 2.0.4, there is an A-module N and for each i an isomorphism $M_i \cong N \otimes_A A_{\alpha_i}$ of A_{α_i}-modules.

Example 2.2.9. Let A be a commutative ring with 1 such that $A = A\alpha_1 + \ldots + A\alpha_n$. For each i let $S_i = \{\alpha_i^{k_i} \mid k_i \geq 0\}$ be the corresponding multiplicative set. Denote by $A_{\alpha_i} = S_i^{-1}A$ the localization. We saw that $B = \bigoplus_{i=1}^n A_{\alpha_i}$ is faithfully flat over A. For all i,j we identify $A_{\alpha_i} \otimes_A A_{\alpha_j} \cong A_{\alpha_i\alpha_j}$. Then the Amitsur complex $\mathcal{C}^\bullet(B/A)$ looks as follows

$$0 \to A \xrightarrow{\theta} \bigoplus_i A_{\alpha_i} \xrightarrow{d^0} \bigoplus_{i,j} A_{\alpha_i\alpha_j} \xrightarrow{d^1} \cdots$$

Since $\mathcal{C}^\bullet(B/A)$ is exact, any element $y \in A$ is completely determined by the set of local data $x = (x_1, \ldots, x_n) \in B$ with $x_i = x_j$ in $A_{\alpha_i\alpha_j}$ where $x_i = \frac{a_i}{\alpha^{k_i}}$, $x_j = \frac{a_j}{\alpha_j^{k_j}}$ for some $a_i, a_j \in A$ and $k_i, k_j \geq 0$. The element y can be constructed from the local data x and the elements α_i. For some $p \geq 0$, there exist $a_1, \ldots, a_n \in A$ such that $x_i = \frac{a_i}{\alpha_i^p}$. Assuming that $d^0(x) = 0$, there exists $q \geq 0$ such that for all i,j, we have

$$(\alpha_i\alpha_j)^q(a_i\alpha_j^p - a_j\alpha_i^p) = 0 \Leftrightarrow a_i\alpha_i^q\alpha_j^{q+p} = a_j\alpha_i^{q+p}\alpha_j^q.$$

Since $A = A\alpha_1^{q+p} + \ldots + A\alpha_n^{q+p}$, we can write $1 = g_1\alpha_1^{q+p} + \ldots + g_n\alpha_n^{q+p}$ for some $g_i \in A$. Set

$$y = g_1\alpha_1^q a_1 + \ldots + g_n\alpha_n^q a_n.$$

We claim that $y = \frac{a_j}{\alpha_j^p} = x_j$ in A_{α_j} for all j, so that $\theta(y) = x$. It suffices to show this for $j = 1$. The condition $y = x_1$ is equivalent to showing that there is $r \geq 0$ such that

$$\alpha_1^r(a_1g_1\alpha_1^{q+p} + a_2g_2\alpha_1^p\alpha_2^q + \ldots + a_ng_n\alpha_1^p\alpha_n^q - a_1) = 0.$$

We can multiply the identity $1 = g_1\alpha_1^{q+p} + \ldots + g_n\alpha_n^{q+p}$ by a_1 so that after replacing the term $a_1g_1\alpha_1^{q+p} - a_1$, the condition above can we rewritten as follows

$$\alpha_1^r(a_2g_2\alpha_1^p\alpha_2^q - a_1g_2\alpha_2^{q+p} + a_3g_3\alpha_3^q\alpha_1^p - a_1g_3\alpha_3^{q+p} + \ldots + a_ng_n\alpha_n^q\alpha_1^p - a_1g_n\alpha_n^{q+p}) = 0.$$

It is easy to see that for $r = q$ we get this equality.

2.2.4 Proof of the Theorem of Faithfully Flat Descent for Modules

Proof. (of Theorem 2.0.4):

- **Existence**: Set $N = \{x \in M \mid x \otimes 1 = g(1 \otimes x)\}$ and let $\nu : N \otimes_A B \to M$ be the multiplication map $\nu(x \otimes b) = xb$. We show that N and ν have the desired properties. Notice that N is the kernel of the A-module homomorphism $ge_0 - e_1 : M \to M \otimes_A B$, where the maps $e_0 : M \to B \otimes_A M$, $e_1 : M \to M \otimes_A B$ are defined as usual. In fact,

$$\ker(ge_0 - e_1) = \{x \in M \mid (ge_0 - e_1)(x) = 0\} = \{x \in M \mid g(1 \otimes x) - x \otimes 1 = 0\} = N.$$

In particular, $0 \in N$ and N is an A-module. Hence the sequence

$$0 \to N \to M \xrightarrow{ge_0 - e_1} M \otimes_A B \tag{2.3}$$

is exact.

The proof will follow from:

(1) The commutativity of the diagram

$$\begin{array}{ccc} B \otimes_A N \otimes_A B & \xrightarrow{1 \otimes \nu} & B \otimes_A M \\ {\scriptstyle\tau}\downarrow & & \downarrow{\scriptstyle g} \\ N \otimes_A B \otimes_A B & \xrightarrow{\nu \otimes 1} & M \otimes_A B \end{array} \tag{2.4}$$

where $\tau(a \otimes b \otimes c) = b \otimes a \otimes c$

(2) The commutativity of the diagram

$$\begin{array}{ccc} B \otimes_A M & \xrightarrow{1 \otimes e_1} & B \otimes_A M \otimes_A B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_3 = g \otimes 1} \\ M \otimes_A B & \xrightarrow{1 \otimes e_1 = e_2} & M \otimes_A B \otimes_A B \end{array} \tag{2.5}$$

(3) The commutativity of the diagram

$$\begin{array}{ccc} B \otimes_A M & \xrightarrow{1 \otimes ge_0} & B \otimes_A M \otimes_A B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_3 = g \otimes 1} \\ M \otimes_A B & \xrightarrow{1 \otimes e_0 = e_1} & M \otimes_A B \otimes_A B \end{array} \tag{2.6}$$

(4) Combine the tensored exact sequence (2.3) with diagrams (2.5) and (2.6) into one commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & B \otimes_A N & \longrightarrow & B \otimes_A M & \xrightarrow{1 \otimes (ge_0 - e_1)} & B \otimes_A M \otimes_A B \\ & & \vdots{\scriptstyle \exists \phi} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_3} \\ 0 & \longrightarrow & M & \xrightarrow{1 \otimes \theta} & M \otimes_A B & \xrightarrow{1 \otimes (e_0 - e_1)} & M \otimes_A B \otimes_A B. \end{array}$$

Note however, that ϕ has not been defined yet.

(5) The rows of this diagram are exact. The upper row is exact, because B is faithfully flat, and therefore, tensoring the exact sequence (2.3), preserves exactness. The bottom row is exact by the properties of the Amitsur complex for faithfully flat algebras (see Proposition 2.2.3, (2)). Note that ϕ is the required isomorphism, which must be constructed using the usual diagram chasing. One uses only the characterization of the descent datum: g must be an isomorphism and consequently g_3 is also an isomorphism.

(6) The diagram chasing. This is usual homological algebra. Assume we are given a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C \\ & & \vdots{\scriptstyle \phi} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} \\ 0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' \end{array}$$

where γ and δ are isomorphisms and the rows are exact. Looking at the diagram we see that $\gamma(\mathrm{im}(\alpha)) = \mathrm{im}(\alpha')$, so one defines $\phi(a) = a'$ where a' is an element of A' such that

$$\alpha'(a') = \gamma(\alpha(a)).$$

One easily checks that this is a correct definition and it yields an isomorphism. In fact, suppose that there is another $a'' \in A'$ such that $\alpha'(a'') = \gamma \circ \alpha(a)$. Then $a' - a'' \in \ker(\alpha') = \{0\}$, so $a' = a''$, which proves that ϕ is in fact, well-defined. Now suppose that $\phi(a) = 0$. Then $\phi(a) = a' = 0$, so $\alpha(a) \in \ker(\gamma) = \{0\}$, and $a \in \ker(\alpha) = \{0\}$. Hence $a = 0$, which proves that ϕ is injective. Finally, we prove surjectivity of ϕ. Let $\tilde{a} \in A'$. Then $\alpha'(\tilde{a}) \in B'$ and since γ is an isomorphism, there is $b \in B$ such that $\alpha'(\tilde{a}) = \gamma(b)$. By the commutativity of the right square

$$\delta \circ \beta(b) = \beta' \circ \gamma(b) = \beta' \circ \alpha'(\tilde{a}) = 0,$$

because $\beta' \circ \alpha' = 0$ by exactness. Therefore, $\beta(b) \in \ker(\delta) = \{0\}$, so $b \in \ker(\beta) = \mathrm{im}(\alpha)$. Thus, $b = \alpha(a)$ for some $a \in A$. Note that by construction we have

$$\alpha'(\tilde{a}) = \gamma(b) = \gamma \circ \alpha(a),$$

so $\phi(a) = \tilde{a}$, as required, which proves that ϕ is surjective.

(7) In order to complete the proof we only need to check the commutativity of the diagrams in steps (1), (2) and (3).

Commutativity of (1): Over $B \otimes_A B$, the module $B \otimes_A N \otimes_A B$ is generated by elements of the form $1 \otimes x \otimes 1$, for $x \in N$. The diagram commutes because

$$g \circ (1 \otimes \nu)(1 \otimes x \otimes 1) = g(1 \otimes x) = x \otimes 1 = (\nu \otimes 1)(x \otimes 1 \otimes 1) = (\nu \otimes 1) \circ \tau(1 \otimes x \otimes 1).$$

Commutativity of (2): The diagram of B-module homomorphisms

$$\begin{array}{ccc} B \otimes_A M & \xrightarrow{1 \otimes e_1} & B \otimes_A M \otimes_A B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_3 = g \otimes 1} \\ M \otimes_A B & \xrightarrow{1 \otimes e_1 = e_2} & M \otimes_A B \otimes_A B \end{array} \tag{2.7}$$

commutes, since

$$g_3 \circ (1 \otimes e_1)(b \otimes x) = g_3(b \otimes x \otimes 1) = g(b \otimes x) \otimes 1 = e_2 \circ g(b \otimes x).$$

Commutativity of (3): Since $g_2 = g_3 \circ g_1$, it follows that

$$g_3 \circ (1 \otimes g e_0)(b \otimes x) = g_3\big(b \otimes g(1 \otimes x)\big) = g_3 \circ g_1(b \otimes 1 \otimes x) = g_2(b \otimes 1 \otimes x) = e_1 \circ g(b \otimes x).$$

Therefore, the diagram of B-module homomorphisms

$$\begin{array}{ccc} B \otimes_A M & \xrightarrow{1 \otimes g e_0} & B \otimes_A M \otimes_A B \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_3 = g \otimes 1} \\ M \otimes_A B & \xrightarrow{1 \otimes e_0 = e_1} & M \otimes_A B \otimes_A B \end{array} \tag{2.8}$$

commutes.

- **Uniqueness**: Suppose K is another A-module satisfying the assumptions of the theorem and let $\rho : K \otimes_A B \to M$ the corresponding B-module isomorphism. Consider the commutative diagram

$$\begin{array}{ccccc} B \otimes_A K \otimes_A B & \xrightarrow{1 \otimes \rho} & B \otimes_A M & \xleftarrow{1 \otimes \nu} & B \otimes_A N \otimes_A B \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \tau} \\ K \otimes_A B \otimes_A B & \xrightarrow{\rho \otimes 1} & M \otimes_A B & \xleftarrow{\nu \otimes 1} & N \otimes_A B \otimes_A B. \end{array} \tag{2.9}$$

In the notation of Proposition 2.2.4, the commutativity of diagram (2.9) means that

$$\Big(\tau \circ (1 \otimes \nu)^{-1} \circ (1 \otimes \rho) \circ \tau^{-1}\Big)(a \otimes b \otimes c) = \tau \circ (1 \otimes \nu^{-1}\rho)(b \otimes a \otimes c) = \tau(b \otimes \nu^{-1}\rho(a \otimes c)) = \mathfrak{F}_0(\nu^{-1}\rho)(a \otimes b \otimes c)$$

is equal to

$$\Big((\nu\otimes 1)^{-1}\circ(\rho\otimes 1)\Big)(a\otimes b\otimes c) = \big((\nu^{-1}\rho)(a\otimes b)\big)\otimes c = \mathfrak{F}_1(\nu^{-1}\rho)(a\otimes b\otimes c).$$

By Proposition 2.2.4, there exists $\lambda\in \mathrm{Hom}_A(K,N)$ such that $\nu^{-1}\rho = \mathfrak{F}(\lambda) = \lambda\otimes 1$. Since B is faithfully flat over A and $\nu^{-1}\rho$ is an isomorphism, $\lambda : K\to N$ is an A-module isomorphism. Lastly, $\rho = \nu(\lambda\otimes 1)$.

□

2.2.5 Proof of theorem of descent for algebras

In this section, we will prove the theorem of faithfully flat descent for algebras (see Theorem 2.0.5). Let us recall the setting. Let A be a commutative ring, B a faithfully flat commutative A-algebra and M a B-algebra together with multiplication map denoted by $\mu : M\otimes_B M\to M$. Let N be an A-module and $\nu : N\otimes_A B\to M$ the isomorphism of B-modules provided by the theorem of faithfully flat descent for modules. The B-module $N_B := N\otimes_A B$ has a multiplication operation which is defined by a B-module homomorphism $\overline{\mu} : N_B\otimes_B N_B\to N_B$. More precisely, we define a multiplicative structure $\overline{\mu}$ on $N\otimes_A B$ by the (module) isomorphism ν so that the following diagram is commutative:

$$\begin{array}{ccc} N_B\otimes_B N_B & \xrightarrow{\overline{\mu}} & N_B \\ {\scriptstyle \nu\otimes_B\nu}\big\downarrow & & {\scriptstyle\nu}\big\downarrow \\ M\otimes_B M & \xrightarrow{\mu} & M. \end{array} \qquad (*)$$

Note that the commutativity of this diagram means precisely that ν is a homomorphism of algebras. We can now ask the following question.

Question: When does $\overline{\mu}$ descend onto N?

This question means that we want to have a multiplicative structure on N which is inherited from $N\otimes_A B$. The answer is given by Theorem 2.0.5.

If we identify $N_B\otimes_B N_B$ with $N\otimes_A N\otimes_A B$, then $\overline{\mu}$ belongs to $\mathrm{Hom}_B(N\otimes_A N\otimes_A B, N\otimes_A B)$. By Proposition 2.2.4, the homomorphism $\overline{\mu}$ descends to a unique A-module homomorphism $N\otimes_A N\to N$ if and only if $\mathfrak{F}_0(\overline{\mu})$ and $\mathfrak{F}_1(\overline{\mu})$ induce equal multiplication operations on $N\otimes_A B\otimes_A B$. Recall that we have the following exact sequence.

$$0 \longrightarrow \mathrm{Hom}_A(N\otimes_A N, N) \xrightarrow{\mathcal{F}} \mathrm{Hom}_B(N\otimes_A N\otimes_A B, N\otimes_A B)$$

$$\xrightarrow{\mathcal{F}_0-\mathcal{F}_1} \mathrm{Hom}_{B\otimes_A B}(N\otimes_A N\otimes_A B\otimes_A B, N\otimes_A B\otimes_A B)$$

We need to check that $\overline{\mu}$ is in the kernel of $\mathcal{F}_0-\mathcal{F}_1$. Along with the existence and uniqueness of the A-module N and the B-module isomorphism $\nu : N\otimes_A B\to M$ are guaranteed by Theorem 2.0.4, the following diagram

$$\begin{array}{ccc} B\otimes_A N\otimes_A B & \xrightarrow{1\otimes\nu} & B\otimes_A M \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ N\otimes_A B\otimes_A B & \xrightarrow{\nu\otimes 1} & M\otimes_A B \end{array} \qquad (2.10)$$

commutes, where $\tau(a\otimes b\otimes c) = b\otimes a\otimes c$. The following diagram

$$\begin{array}{ccc} B\otimes_A (N\otimes_A N)\otimes_A B & \xrightarrow{1\otimes(\nu\otimes_B\nu)} & B\otimes_A (M\otimes_A M) \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle g\otimes_B g} \\ (N\otimes_A N)\otimes_A B\otimes_A B & \xrightarrow{(\nu\otimes_B\nu)\otimes 1} & (M\otimes_A M)\otimes_A B \end{array} \qquad (2.11)$$

is the counterpart of square (2.10) for $N \otimes_A N \otimes_A B \cong M \otimes_B M$ and it commutes. In fact, let us check this.

$$\begin{aligned}
(g \otimes_B g) \circ (1 \otimes (\nu \otimes_B \nu))(b \otimes n \otimes n' \otimes b') &= (g \otimes_B g)(b \otimes (\nu \otimes_B \nu)(n \otimes n' \otimes b')) \\
&= (g \otimes_B g)(b \otimes n \otimes_B n'b') \\
&= g(b \otimes n) \otimes_B g(1 \otimes n'b') \\
&= (b \otimes 1)g(1 \otimes n) \otimes_B (1 \otimes b')g(1 \otimes n') \\
&= (b \otimes 1)(n \otimes 1) \otimes_B (1 \otimes b')(n' \otimes 1) \\
&= bn \otimes 1 \otimes_B n' \otimes b'
\end{aligned}$$

which we can identify with $n \otimes_B bn' \otimes b'$. On the other hand, we have

$$\begin{aligned}
((\nu \otimes_B \nu) \otimes 1) \circ \tau(b \otimes n \otimes n' \otimes b') &= ((\nu \otimes_B \nu) \otimes 1)(n \otimes n' \otimes b \otimes b') \\
&= (\nu \otimes_B \nu)(n \otimes n' \otimes b) \otimes b' \\
&= n \otimes_B n'b \otimes b' \\
&= n \otimes_B bn' \otimes b'
\end{aligned}$$

so the diagram (2.11) commutes. Since g is a $B \otimes_A B$-algebra isomorphism, the diagram

$$\begin{array}{ccccc}
B \otimes_A M \otimes_A M & = & (B \otimes_A M) \otimes_B (B \otimes_A M) & \longrightarrow & B \otimes_A M \\
 & & \big\downarrow{\scriptstyle g \otimes_B g} & & \big\downarrow{\scriptstyle g} \\
M \otimes_A M \otimes_A B & = & (M \otimes_A B) \otimes_B (M \otimes_A B) & \longrightarrow & M \otimes_A B
\end{array} \tag{2.12}$$

commutes, where the horizontal arrows are the multiplication maps. We can apply $\mathfrak{F}_0$ and $\mathfrak{F}_1$ to the commutative diagram $(*)$ which defines $\overline{\mu}$, to obtain the following two commutative diagrams

$$\begin{array}{ccc}
(N \otimes_A N) \otimes_A B \otimes_A B & \xrightarrow{\mathfrak{F}_0(\overline{\mu})} & N \otimes_A B \otimes_A B \\
{\scriptstyle (\nu \otimes_B \nu) \otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle \nu \otimes 1} \\
M \otimes_B M \otimes_A B & \xrightarrow{\mathfrak{F}_0(\mu)} & M \otimes_A B.
\end{array} \tag{2.13}$$

and

$$\begin{array}{ccc}
(N \otimes_A N) \otimes_A B \otimes_A B & \xrightarrow{\mathfrak{F}_1(\overline{\mu})} & N \otimes_A B \otimes_A B \\
{\scriptstyle (\nu \otimes_B \nu) \otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle \nu \otimes 1} \\
M \otimes_B M \otimes_A B & \xrightarrow{\mathfrak{F}_1(\mu)} & M \otimes_A B.
\end{array} \tag{2.14}$$

From these two diagrams we cannot deduce yet that $\mathfrak{F}_0(\overline{\mu})$ and $\mathfrak{F}_0(\overline{\mu})$ induce equal multiplications because the multiplications $\mathfrak{F}_0(\mu)$ and $\mathfrak{F}_1(\mu)$ are different. However, we can proceed as follows. Combine the diagrams (2.10), (2.11), (2.12), (2.13) and (2.14) to get the commutative diagram

$$\begin{array}{ccccccc}
(N \otimes_A N) \otimes_A B \otimes_A B & & & \xrightarrow{\mathfrak{F}_i(\overline{\mu})} & & & N \otimes_A B \otimes_A B \\
 & \searrow{\scriptstyle (1 \otimes (\nu \otimes_B \nu))\tau^{-1}} & & & & \swarrow{\scriptstyle (1 \otimes \nu)\tau^{-1}} & \\
 & {\scriptstyle (\nu \otimes_B \nu) \otimes 1} & B \otimes_A M \otimes_A M & \longrightarrow & B \otimes_A M & {\scriptstyle \nu \otimes 1} & \\
 & \searrow & \big\downarrow{\scriptstyle g \otimes_B g} & & {\scriptstyle g}\big\downarrow & \swarrow & \\
 & & M \otimes_A M \otimes_A B & \longrightarrow & M \otimes_A B & &
\end{array} \tag{2.15}$$

The diagram commutes for both $\mathfrak{F}_0(\overline{\mu})$ or $\mathfrak{F}_1(\overline{\mu})$. Note that the map $\nu \otimes 1$ is invertible. Therefore, $\mathfrak{F}_i(\overline{\mu})$ induce equal multiplication operations on $(N \otimes_A N) \otimes_A B \otimes_A B$, as required.

Chapter 3

Flatness and Tor

Peter Abramenko

3.1 Notations

- R is a commutative ring.
- $\mathcal{C}$ is a category of R-modules.
- $- \otimes_R - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, defined by $(A, B) \mapsto A \otimes_R B$, is a (covariant) bifunctor.
- Let $f \colon A \to A'$ and $g \colon B \to B'$ be R-module homomorphisms. There exists a unique R-module homomorphism

$$\begin{aligned} f \otimes g \colon A \otimes_R B &\to A' \otimes_R B' \\ a \otimes b &\mapsto f(a) \otimes g(b). \end{aligned}$$

  Also, we have an isomorphism

$$\begin{aligned} A \otimes_R B &\cong B \otimes_R A \\ a \otimes b &\leftrightarrow b \otimes a. \end{aligned}$$

- For a fixed $A \in \mathcal{C}$, we have a functor

$$\begin{aligned} T := T_A := A \otimes_R - \colon \mathcal{C} &\to \mathcal{C} \\ B &\mapsto A \otimes_R B. \end{aligned}$$

  Also, for $g \in \mathrm{Hom}_R(B, B')$ we have

$$T(g) := 1_A \otimes g \in \mathrm{Hom}_R(T(B), T(B')) \colon A \otimes_R B \to A \otimes_R B'.$$

  Remark: For fixed $B \in \mathcal{C}$, the functor $- \otimes_R B \colon \mathcal{C} \to \mathcal{C}$ has similar properties as $A \otimes_R -$; so we may concentrate on the latter.

3.2 Important Properties of $T = A \otimes_R -$

1. T commutes with direct sums, i.e., there exists a natural isomorphism

$$\begin{aligned} A \otimes_R \bigoplus_{i \in I} B_i &\xrightarrow{\sim} \bigoplus_{i \in I} (A \otimes_R B_i) \\ a \otimes (b_i)_{i \in I} &\mapsto (a \otimes b_i)_{i \in I}. \end{aligned}$$

   In other words, T is an additive functor.

2. T commutes with direct limits (of direct systems), i.e.,

$$A \otimes_R \left(\varinjlim_I B_i \right) \cong \varinjlim_I \left(A \otimes_R B_i \right).$$

   More generally: T commutes with colimits.

3. T is right exact, i.e., if

$$B_1 \xrightarrow{g_1} B_2 \xrightarrow{g_2} B_3 \to 0$$

   is an exact sequence of R-modules, then so is

$$A \otimes_R B_1 \xrightarrow{1_A \otimes g_1} A \otimes_R B_2 \xrightarrow{1_A \otimes g_2} A \otimes_R B_3 \to 0.$$

   Moreover, A is flat if T is exact, i.e., T also preserves injectivity.

Lemma 3.2.1. *For a family of R-modules $(A_i)_{i \in I}$,*

$$\bigotimes_{i \in I} A_i \text{ is flat} \quad \text{if and only if} \quad A_i \text{ is flat for all } i \in I.$$

Proof. (Sketch) Given $0 \to B' \xrightarrow{g} B$, consider

$$\begin{array}{ccc}
\left(\bigoplus_{i \in I} A_i \right) \otimes_R B' & \xrightarrow{1 \otimes g} & \left(\bigoplus_{i \in I} A_i \right) \otimes_R B \\
\downarrow \wr & \circlearrowleft & \downarrow \wr \\
\bigoplus_{i \in I} (A_i \otimes_R B') & \xrightarrow{(1_{A_i} \otimes g)} & \bigoplus_{i \in I} (A_i \otimes_R B)
\end{array}$$

Then $1 \otimes g$ is injective if and only if $1_{A_i} \otimes g$ is injective for all $i \in I$. □

Corollary 3.2.2. *If A is projective, then A is flat.*

Remark 3.2.3.

1. So we have in general: free $\implies$ projective $\implies$ flat.

2. If R is an integral domain, we also have: flat $\implies$ torsion free.

   Reason: Let K be the field of fractions of R and $\epsilon \colon R \hookrightarrow K$ be an embedding. Consider $1_A \otimes \epsilon \colon A \otimes_R R \cong A \to A \otimes_R K$. We have $\ker(1_A \otimes \epsilon) = t(A) \otimes_R R \cong t(A)$, where $t(A)$ is the torsion submodule of A.

Corollary 3.2.4. *Let R be a PID and A be a finitely generated R-module. We have:*

$$\textit{free} \implies \textit{projective} \implies \textit{flat} \implies \textit{torsion free}.$$

<u>Note</u>: For general integral domains, torsion free $\not\Rightarrow$ flat (even if A is f.g.). Moreover, flat $\not\Rightarrow$ projective (e.g. $\mathbb{Q}$ is a flat, but not a projective $\mathbb{Z}$-module).

Remark 3.2.5. For any (commutative) R, and any multiplicatively closed subset $S \subset R$, $S^{-1}R$ is a flat R-module.

Lemma 3.2.6. *If all finitely generated submodules of A are flat, then A is flat.*

Corollary 3.2.7. *If R is a PID, then torsion free $\iff$ flat.*

Generalization of the lemma: If $(A_i)_{i\in I}$ is a direct system of R-modules, and all A_i are flat, then also $\varinjlim_I A_i$ is flat.

We now get back to the fact that $T = A\otimes_R -$ is always right exact but not nescessarily exact (e.g. if R is an integral domain and $t(A)\neq 0$). A main motivation for considering the Tor functors is the sequence of a <u>long exact sequence</u> associated (naturally) to any exact sequence

$$0\to B' \xrightarrow{f} B \xrightarrow{g} B'' \to 0$$

of R-modules, namely

$$\ldots \to \mathrm{Tor}_n(A,B') \xrightarrow{f_*} \mathrm{Tor}_n(A,B) \xrightarrow{g_*} \mathrm{Tor}_n(A,B'') \xrightarrow{\omega_n} \mathrm{Tor}_{n-1}(A,B') \to \ldots$$

$$\to \mathrm{Tor}_1(A,B'') \xrightarrow{\omega_1} A\otimes_R B' \xrightarrow{1_A\otimes f} A\otimes_R B \xrightarrow{1_A\otimes g} A\otimes_R B'' \to 0$$

(so in particular $\ker(1_A\otimes f) = \mathrm{im}\ \omega_1$).

The sequence $(\mathrm{Tor}_n^R(A,-)\colon \varphi\to\varphi)_{n\geq 0}$ is an example of <u>left derived functors</u> which we will now briefly discuss.

Assumptions: $T\colon \varphi\to\varphi$ is a covariant, additive, right exact functor. Then one defines a sequence of left derived functors $(L_nT\colon \varphi\to\varphi)_{n\geq 0}$ as follows:

Given $B\in\varphi$, choose a <u>projective resolution</u> $\underline{\underline{P}}$ of B, i.e., an (infinite) chain complex

$$\underline{\underline{P}} = \ldots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{\partial_1} P_0 \to 0$$

such that all P_n are projective R-modules, $\mathrm{im}\ \partial_{n+1} = \ker\partial_n$ for all $n\geq 1$, and there exists a homomorphism $\epsilon\colon P_0\to B$ such that $P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} B \to 0$ is exact, implying that $B\cong P_0/\mathrm{im}\ \partial_1 = \mathrm{coker}\ \partial_1$ (and the whole sequence $\ldots\to P_n \xrightarrow{\partial_n} P_{n-1}\to\ldots\to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} B\to 0$ is exact).

Notation: $\underline{\underline{P}} \overset{\epsilon}{\twoheadrightarrow} B$ is a projective resolution.

Definition 3.2.8. $L_nT(B) := H_n(T(\underline{\underline{P}}))\in\varphi$.

Facts:

(a) If $\underline{\underline{P'}} \overset{\epsilon'}{\twoheadrightarrow} B$ is another projective resolution of B, then there exists a <u>homotopy equivalence</u> $f\colon \underline{\underline{P}}\to \underline{\underline{P'}}$ which implies that (since T is additive) $T(f)\colon T(\underline{\underline{P}})\to T(\underline{\underline{P'}})$ is a homotopy equivalence, so $H_n(T(\underline{\underline{P}}))\cong H_n(T(\underline{\underline{P'}}))$ for all n.

(b) L_nT is a <u>functor</u> (covariant): Given $\varphi\in\mathrm{Hom}_R(B,B')$, choose projective resolutions $\underline{\underline{P}} \overset{\epsilon}{\twoheadrightarrow} B$, $\underline{\underline{P'}} \overset{\epsilon'}{\twoheadrightarrow} B'$, implying that there exists a chain homomorphism $f\colon \underline{\underline{P}}\to\underline{\underline{P'}}$ (unique up to homotopy) such that

$$\begin{array}{ccc} \underline{\underline{P}} & \xrightarrow{\ \epsilon\ } & B \\ {\scriptstyle f}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \varphi} \\ \underline{\underline{P'}} & \xrightarrow{\ \epsilon'\ } & B' \end{array}$$

commutes. In more detail:

$$\begin{array}{ccccccccc} \longrightarrow & P_n & \xrightarrow{\ \partial_n\ } & P_{n-1} & \longrightarrow \ldots \longrightarrow & P_0 & \xrightarrow{\ \epsilon\ } & B \\ & {\scriptstyle f_n}\downarrow & \circlearrowleft & \downarrow{\scriptstyle f_{n-1}} & & {\scriptstyle f_0}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \varphi} \\ \longrightarrow & P'_n & \xrightarrow{\ \partial'_n\ } & P'_{n-1} & \longrightarrow \ldots \longrightarrow & P'_0 & \xrightarrow{\ \epsilon'\ } & B' \end{array}$$

There exists a chain homomorphism $T(f)\colon T(\underline{\underline{P}})\to T(\underline{\underline{P'}})$

$$\varphi_* \text{ or } \varphi_{*,n} = L_nT(\varphi) := T(f)_*\colon H_n(T(\underline{\underline{P}})) = L_nT(B)\to H_n(T(\underline{\underline{P'}})) = L_nT(B').$$

Theorem 3.2.9. *The sequence $(L_nT)_{n\geq 0}$ has the following properties:*

(a) $L_0T \cong T$ *(naturally equivalent).*

(b) *P projective $\Rightarrow$ $L_nT(P) = 0$ for all $n \geq 1$.*

(c) *Let $0 \to B' \xrightarrow{\varphi} B \xrightarrow{\psi} B'' \to 0$ be an exact sequence of R-module. Then there exist natural exact sequences*

$$\ldots \to L_nT(B') \xrightarrow{\varphi_*} L_nT(B) \xrightarrow{\psi_*} L_nT(B'') \xrightarrow{\omega_n} L_{n-1}T(B') \to \ldots$$

$$\begin{array}{ccccccc} \to L_1T(B'') & \xrightarrow{\omega_1} & L_0T(B') & \xrightarrow{\varphi_0} & L_0T(B) & \xrightarrow{\psi_0} & L_0T(B'') \\ & & \| & & \| & & \\ & & \to T(B') & \xrightarrow{T(\varphi)} & T(B) & \xrightarrow{T(\psi)} & T(B'') \end{array}$$

Proof. **(a)** follows from the def. of L_0T (including $L_0T(\varphi)$) and the right exactness of T.

(b) is trivial; choose $\ldots 0 \to P_0 = P \xrightarrow{1_P} P \to 0$ as projective resolution of P.

(c) requires some work ("horseshoe lemma"; long exact homotopy sequence for short exact sequences of R-modules).

□

These properties determine $(L_nT)_{n\geq 0}$ uniquely, up to natural equivalence, due the following:

Proposition 3.2.10. *Let $F_n, G_n \colon \mathcal{C} \to \mathcal{C}$ $(n \geq 0)$ be two sequences of covariant additive functors. If*

(a) $F_0 \simeq G_0$.

(b) *$F_n(P) = 0 = G_n(P)$ for all projective P and all $n \geq 1$.*

(c) *There exist natural long exact sequences for (F_n) as well as (G_n).*

Then $F_n \simeq G_n$ for all $n \geq 0$.

Proof. (Idea of proof) Use a presentation of B, i.e., a short exact sequence $0 \to K \to P \to B \to 0$, with projective P and construct inclusively isomorphisms $t_{n,B} \colon F_n(B) \xrightarrow{\sim} G_n(B)$

$n \geq 2:$

$$\begin{array}{ccccccccc} 0 & = & F_n(P) & \longrightarrow & F_n(B) & \xrightarrow{\sim} & F_{n-1}(K) & \longrightarrow & F_{n-1}(P) & \longrightarrow & 0 \\ & & & & \downarrow 2t_{n,B} & & \downarrow 2t_{n-1,K} & & & & \\ 0 & = & G_n(P) & \longrightarrow & G_n(B) & \xrightarrow{\sim} & G_{n-1}(K) & \longrightarrow & G_{n-1}(P) & = & 0 \end{array}$$

$n = 1:$ requires a separate argument.

□

Definition 3.2.11. For $A \in \mathcal{C}$, $\mathrm{Tor}_n(A, -) := L_nT \colon \mathcal{C} \to \mathcal{C}$ for $T = A \otimes_R - \colon \mathcal{C} \to \mathcal{C}$.

The above theorem now yields

Theorem 3.2.12. **(a)** *$Tor_0(A, -) \simeq A \otimes_R -$.*

(b) *If A is flat or B is projective, then $Tor_n(A, B) = 0$ for all $n \geq 1$.*

(c) *For any short exact sequence $0 \to B' \to B \to B'' \to 0$, we get a long exact sequence*

$$\ldots \to Tor_n(A, B') \to Tor_n(A, B) \to Tor_n(A, B'') \xrightarrow{\omega_n} Tor_{n-1}(A, B') \to$$
$$\ldots \to Tor_1(A, B'') \xrightarrow{\omega_1} A \otimes B' \to A \otimes B \to A \otimes B'' \to$$

The additional statement in (b) follows from the construction of $\mathrm{Tor}_n(A,-)$:

If $\underline{\underline{P}} \overset{\epsilon}{\twoheadrightarrow} B$ is a projective resolution, then, by flatness of A, $A \otimes_R \underline{\underline{P}} \xrightarrow{1_A \otimes \epsilon} A \otimes_R B$ is exact, and so $\mathrm{Tor}_n(A,B) = H_n(A \otimes_R \underline{\underline{P}}) = 0$ for all $n \geq 1$.

Corollary 3.2.13. *For $A \in \mathcal{C}$, the following are equivalent:*

(i) *A is flat.*

(ii) *$Tor_n(A,B) = 0$ for all $B \in \mathcal{C}$ and all $n \geq 1$.*

(iii) *$Tor_1(A,B) = 0$ for all $B \in \mathcal{C}$.*

Proof. $(iii) \Rightarrow (i)$ follows from the long exact sequence: If $0 \to B' \to B \to B'' \to 0$ is exact, then so is

$$0 = \mathrm{Tor}_1(A, B'') \to A \otimes_R B' \to A \otimes_R B \to A \otimes_R B'' \to 0$$

So $A \otimes_R -$ is exact if and only if A is flat.

□

We are now going to resolve the symmetry in (b) above, using the proposition on the previous page. But first we observe:

Lemma 3.2.14 (Properties of $\mathrm{Tor}_n(-,B)$ for fixed $B \in \mathcal{C}$)**.**

(a) *$Tor_0(-,B) \simeq - \otimes_R B$.*

(b) *$Tor_n(P,B) = 0$ for all projective ($\Rightarrow$ flat) P and all $n \geq 1$.*

(c) *For any short exact sequence $0 \to A' \to A \to A'' \to 0$, there exists a natural long exact sequence $\ldots \to Tor_n(A',B) \to Tor_n(A,B) \to Tor_n(A'',B) \to Tor_{n-1}(A',B) \to \ldots$.*

This all can quickly be verified, e.g.,

Proof. **(c)** Fix a projective resolution $\underline{\underline{P}} \overset{\epsilon}{\twoheadrightarrow} B$. Then, since all P_n are projective $\Rightarrow$ flat,

$$0 \to A' \otimes_R \underline{\underline{P}} \to A \otimes_R \underline{\underline{P}} \to A'' \otimes_R \underline{\underline{P}} \to 0$$

is an exact sequence of chain complexes. This gives rise to a long exact homology sequence

$$\begin{array}{ccccccccc} \ldots \longrightarrow & H_n(A' \otimes_R \underline{\underline{P}}) & \longrightarrow & H_n(A \otimes_R \underline{\underline{P}}) & \longrightarrow & H_n(A'' \otimes_R \underline{\underline{P}}) & \xrightarrow{\omega_n} & H_{n-1}(A' \otimes_R \underline{\underline{P}}) & \longrightarrow \ldots \\ & \| & & \| & & \| & & \| & \\ & \mathrm{Tor}_n(A',B) & & \mathrm{Tor}_n(A,B) & & \mathrm{Tor}_n(A'',B) & & \mathrm{Tor}_{n-1}(A',B) & \end{array}$$

□

Corollary 3.2.15. *$Tor_n(-,B) \simeq Tor_n(B,-)$ for all $B \in \mathcal{C}$ and all $n \geq 0$. In particular, we have $Tor_n(A,B) \cong Tor_n(B,A)$ for all $A,B \in \mathcal{C}$ and all $n \geq 0$.*

Corollary 3.2.16. *For $B \in \mathcal{C}$, the following are equivalent:*

(i) *B is flat.*

(ii) *$Tor_n(A,B) = 0$ for all $A \in \mathcal{C}$ and all $n \geq 1$.*

(iii) *$Tor_1(A,B) = 0$ for all $A \in \mathcal{C}$.*

Remark 3.2.17. Using tensor products of chain complexes, one can give a more concrete argument for the symmetry of Tor:

Choose projective resolutions $\underline{\underline{P}} \twoheadrightarrow A, \underline{\underline{Q}} \twoheadrightarrow B$. Then

$$\mathrm{Tor}_n(A,-)(B) = H_n(A \otimes_R \underline{\underline{Q}}) \cong H_n(P \otimes_R \underline{\underline{Q}}) \cong H_n(P \otimes_R \underline{\underline{B}}) = \mathrm{Tor}_n(-,B)(A),$$

where the above two equivalences will later be proved.

Hence $\mathrm{Tor}_n(A,B) = H_n(A \otimes_R \underline{\underline{Q}}) = H_n(\underline{\underline{Q}} \otimes_R A) = \mathrm{Tor}_n(B,A)$.

Some connections with torsion for integral domains R:
Let A be an R-module. Then $t(A) := \{a \in A \colon \exists r \in R - \{0\} \colon ra = 0\}$ is a torsion submodule and $t \colon \mathcal{C} \to \mathcal{C}, A \mapsto t(A)$, is a functor.

Facts:

1. If K is the field of fractions of R, then $\mathrm{Tor}_1^R(K/R, -) \simeq t$.
2. $\mathrm{Tor}_n^R(A,B) = t(\mathrm{Tor}_n^R(A,B))$ for all $A, B \in \mathcal{C}$ and all $n \geq 1$.
3. If R is a PID, then $\mathrm{Tor}_1^R(A,B) = \mathrm{Tor}_1^R(t(A), t(B))$ for all $A, B \in \mathcal{C}$ and $\mathrm{Tor}_n^R(A,B) = 0$ for all $A, B \in \mathcal{C}$ and all $n \geq 2$.

Proof. 3. For any $B \in \mathcal{C}$, there exists a free ($\Rightarrow$ projective) resolution $0 \to F_1 \hookrightarrow F_0 \xrightarrow{\epsilon} B \to 0$ with $F_1 = \ker \epsilon$. □

Finally, we want to discuss an application for local rings: In the following, R is a (commutative) local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

We do not assume that R is noetherian or an integral domain. For an R-module M, we consider the following statements:

(i) M is a free R-module.

(ii) M is a projective R-module.

(iii) M is a flat R-module.

(iv) $\mathrm{Tor}_1^R(M,k) = 0$.

We know that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$, and want to show $(iv) \Rightarrow (i)$ provided that M is finitely presented.

Recall: M is finitely generated if there exists a finitely generated free R-module F and a surjection $F \to M \to 0$.

Definition 3.2.18. M is called **finitely presented** if there exists a short exact sequence $0 \to K \to F \to M \to 0$ with finitely generated R-modules K and F such that F is free.

Fact: If M is finitely presented and $0 \to K' \to F' \to M \to 0$ is another exact sequence with finitely generated free F', then K' is also finitely generated. This follows from Schanuel's Lemma which implies $K \oplus F' \cong K' \oplus F \Rightarrow K' \cong K \oplus F'/F$ is finitely generated.

Corollary 3.2.19. *If the ideal $I \triangleleft R$ is not finitely generated, then R/I is a finitely generated (cyclic) but not a finitely presented R-module.*

We also recall a standard application of Nakayama's Lemma for local rings: If N is a finitely generated R-module with $\mathfrak{m}N = N$, then $N = 0$.

Corollary 3.2.20. *If $m_1, \ldots, m_\ell \in M$ are such that the cosets $m_i + \mathfrak{m}M$ for $1 \leq i \leq \ell$ generate $M/\mathfrak{m}M$ for some finitely generated R-module M, then $m_1, \ldots, m_\ell$ generate M.*

Proof. Set $L = \sum_{i=1}^{\ell} Rm_i \leq M$. Then $L + \mathfrak{m}M = M$ by assumption, so

$$\mathfrak{m} \cdot M/L = (\mathfrak{m}M + L)/L = M/L \Rightarrow M/L = 0 \Rightarrow L = M.$$

□

Proof. ($(iv) \Rightarrow (i)$ provided that M is a finitely presented R-module) Since M is in particular finitely generated, $M/\mathfrak{m}M$ is a finitely generated k-vector space (recall: $k = R/\mathfrak{m}$ is a field). Choose a k-basis $x_1, \ldots, x_s$ of $M/\mathfrak{m}M$ and preimages $m_1, \ldots, m_s \in M$, i.e., $x_i = m_i + \mathfrak{m}M$ for all $1 \le i \le s$. By the corollary, $M = \sum_{i=1}^{s} Rm_i$. Set $F = R^s$ and consider

$$0 \to K \xrightarrow{\iota} F \xrightarrow{\varphi} M \to 0, \quad e_i \mapsto m_i.$$

Here $\{e_i \colon 1 \le i \le s\}$ is the standard basis of R^s and $K := \ker \varphi$. Since M is finitely presented, K is finitely generated by the Fact above.

Using the long exact sequence for $\mathrm{Tor}_*(-, k)$, where $k = R/\mathfrak{m}$ is considered as an R-module, we get the exact sequence

$$0 = \mathrm{Tor}_1(M, k) \to K \otimes_R k \xrightarrow{\iota \otimes 1_k} F \otimes_R k \xrightarrow{\varphi \otimes 1_k} M \otimes_R k \to 0 \qquad (*)$$

And $\varphi \otimes 1_k \colon F \otimes_R k \to M \otimes_R k$ is an isomorphism:

1. $F \otimes_R k = R^s \otimes_R k \cong (R \otimes_R k)^s \cong k^s$ has k-basis $\{e_i \otimes 1 \colon 1 \le i \le s\}$.

2. $M \otimes_R k = M \otimes_R R/\mathfrak{m} \cong M/\mathfrak{m}M$, $m \otimes (1 + \mathfrak{m}) \Leftrightarrow m + \mathfrak{m}M$ has k-basis $\{m_i \otimes 1 \colon 1 \le i \le s\}$, since $M/\mathfrak{m}M$ has k-basis $\{x_i = m_i + \mathfrak{m}M \colon 1 \le i \le s\}$ by construction. So $\varphi \otimes 1_k$ sends a k-basis of $F \otimes_R k$ to a k-basis of $M \otimes_R k$ since $\varphi \otimes 1_k(e_i \otimes 1) = m_i \otimes 1$ for all $1 \le i \le s$, so $\varphi \otimes 1_k$ is an isomorphism of k-vector spaces, implying that $\varphi \otimes 1_k$ is an isomorphism of R-modules. In particular, $\varphi \otimes 1_k$ is injective. So the exact sequence $(*)$ yields the exact sequence $0 \to K \otimes_R k \xrightarrow{\iota \otimes 1_k} F \otimes_R k \xrightarrow{\varphi \otimes 1_k} M \otimes_R k$, where $\mathrm{im}\ \iota \otimes 1_k = \ker \varphi \otimes 1_k = 0$. Hence $0 \to K \otimes_R k \to 0$ is exact, implying $K \otimes_R k = 0$.

   But $K \otimes_R k = K \otimes_R R/\mathfrak{m} \cong K/\mathfrak{m}K$. So we have $K = \mathfrak{m}K$, and it follows from Nakayama's lemma (since K is finitely generated!) that $K = 0$. Hence $\varphi \colon F \xrightarrow{\sim} M$ is an isomorphism, and M is a finitely generated free R-module.

□

Remark 3.2.21. Assuming still that R is a local ring and M an R-module, the following can be shown:

(a) $(i) \Leftrightarrow (ii)$, i.e., M is free $\Leftrightarrow$ M is projective (Kaplansky).

(b) For $(i) \Leftrightarrow (iii)$, we only need that M is finitely generated (not necessarily finitely presented). So M is finitely generated $\Rightarrow$ $\{M$ is free $\Leftrightarrow$ M is flat$\}$.

(c) For a finitely generated M, $(iv) \Rightarrow (iii)$ is in general not true, i.e., if a finitely generated R-module M satisfies $\mathrm{Tor}_1(M, k) = 0$, it need not be flat.

Remark 3.2.22. For any ring R and a finitely presented R-module M, we have M is flat $\Leftrightarrow$ M is projective.

Question: Does finitely generated and flat already imply projective?

Answer: Not in general. Counter-examples can be obtained as follows: Let R be an absolutely flat ring, i.e., every R-module is flat. If R has an ideal I which is not finitely generated, then R/I is flat (since it is an R-module), finitely generated (cyclic) but neither finitely presented nor projective.

Note:

(a) Every finitely generated projective module is finitely presented: If $0 \to K \to F \to P \to 0$ is a short exact sequence with a finitely generated free R-module F, then it splits since P is projective, implying that $F \cong K \oplus P \Rightarrow K \cong F/P$ is finitely generated.

(b) If I is an ideal of R such that R/I is projective, then I is principal: $0 \to I \to R \to R/I \to 0$ splits $\Rightarrow R \cong I \oplus R/I \Rightarrow I \cong R/(R/I)$ is cyclic, i.e., I is a principal ideal.

Question: How do we get absolutely flat rings R which are not noetherian?
Let's first discuss a criterion for absolute flatness:

Proposition 3.2.23. *For a commutative ring R, the following are equivalent:*

(i) *R is absolutely flat.*

(ii) *$I^2 = I$ for every principal ideal I of R.*

(iii) *Every finitely generated ideal I of R is a direct summand of R.*

Proof. $(i) \Rightarrow (ii):$ Tensor the exact sequence $0 \to (x) \xrightarrow{\iota} R$ with $R/(x)$ (for $x \in R$) to get the exact sequence $0 \to (x) \otimes_R R/(x) \xrightarrow{\iota \otimes 1_{R/(x)}} R \otimes_R R/(x)$. But the image of $\iota \otimes 1_{R/(x)}$ is $0:$
$rx \otimes (1 + (x)) = 1 \otimes rx(1 + (x)) = 1 \otimes 0 = 0$ in $R \otimes_R R/(x)$. So $0 \to (x) \otimes_R R/(x) \to 0$ is exact, implying $0 = (x) \otimes_R R/(x) \cong (x)/(x)(x)$, so $(x) = (x)^2$.

$(ii) \Rightarrow (iii):$ It follows from (ii) that each principal ideal (x) of R is generated by an idempotent: $(x) = (x^2) \Rightarrow \exists a \in R$ with $x = ax^2 \Rightarrow (ax) \subset (x) \subset (ax) \Rightarrow (x) = (e)$ with $e = ax$. We also have $e^2 = (ax)^2 = a(ax^2) = ax = e$.

In general, any ideal generated by finitely many idempotents is a principal ideal generated by an idempotent. Consider first $I = (e, f)$ with two idempotents e and f. Then $I = (e + f - ef)$ since $e(e + f - ef) = e^2 = e$ and $f(e + f - ef) = f^2 = f$. And we get

$$(e + f - ef)^2 = e(e + f - ef) + f(e + f - ef) - ef(e + f - ef) = e + f - ef.$$

The general case now easily follows by induction on the number of generators of I. Finally, it is easy to check that $R = (e) \oplus (1 - e)$ if e is an idempotent.

$(iii) \Rightarrow (i):$ Let M be an R-module and I a finitely generated ideal of R. By (iii), $R = I \oplus J$ for some other ideal J of R. Note that this implies that I and J are projective $\Rightarrow$ flat R-modules. But $R/I \cong J$, and so also R/I is flat. It follows that $\mathrm{Tor}_1(M, R/I) = 0$. By a homological criterion for flatness (to be discussed below) this implies that M is flat.

□

Example 3.2.24. Let $(k_j)_{j \in J}$ be any (finite or infinite) family of fields. Then, by (ii) above, $R = \prod_{j \in J} k_j$ is absolutely flat: It is easy to check that for any $x \in R$, there exists $J' \subset J$ with $(x) = \prod_{j \in J'} k_j = (x)^2$. And if J is infinite, then R has ideals which are not finitely generated, e.g., $I = \bigoplus_{j \in J} k_j$. Hence R/I is a flat cyclic R-module which is not finitely presented or projective. We finally discuss the criterion for flatness used in $(iii) \Rightarrow (i)$ above.

Proposition 3.2.25. *For any R-module M, the following are equivalent:*

(i) *M is flat.*

(ii) *$\mathrm{Tor}_1(M, R/I) = 0$ for every finitely generated ideal I of R.*

(iii) *The canonical map $M \otimes_R I \to M \otimes_R R = M$ is injective for all finitely generated ideals I of R.*

Proof. $(i) \Rightarrow (ii):$ See Corollary 3.2.13.

$(ii) \Rightarrow (iii):$ After tensoring the short exact sequence $0 \to I \xrightarrow{\iota} R \to R/I \to 0$, we get the long exact sequence which ends in

$$\mathrm{Tor}_1(M, R/I) \to M \otimes_R I \xrightarrow{1_M \otimes \iota} M \otimes_R R \to M \otimes_R R/I \to 0.$$

So if $\mathrm{Tor}_1(M, R/I) = 0$, then $1_M \otimes \iota$ is injective.

$(iii) \Rightarrow (i):$ Assuming (iii), we have to show that $M \otimes_R N' \to M \otimes_R N$ is injective for any R-module N and any submodule $N' \leq N$. If an element of $M \otimes_R N'$ becomes 0 in $M \otimes_R N$, then there exists a finitely generated submodule N_0 of N such that this element is already 0 in $M \otimes_R (N' + N_0)$. So it suffices to show that $M \otimes_R N' \to M \otimes_R N$ is surjective if $N = N' + F$. Assume that $N_0 = \sum_{i=1}^{\ell} Rn_i$. By an easy induction on ℓ, we may reduce further to the case $\ell = 1$, i.e., $N = N' + Rn$.

We now consider the "conductor" ideal of N/N', i.e., $I := \{r \in R \colon rn \in N'\}$. There is a surjective homomorphism $R \to N/N', r \mapsto rn + N'$ with kernel I, and so $N/N' \cong R/I$. Therefore, we get a short exact sequence $0 \to N' \to N \to R/I \to 0$ with associated long exact sequence ending in

$$\mathrm{Tor}_1(M, R/I) \to M \otimes_R N' \to M \otimes_R N \to M \otimes_R R/I \to 0.$$

So to show that $M \otimes_R N' \to M \otimes_R N$ is injective, it suffices to show that $\mathrm{Tor}_1(M, R/I) = 0$ for all ideals I of R.

We first note that (iii) implies the injectivity of $M \otimes_R I \to M \otimes_R R$ for all ideals I of R. The argument is the same as above: If an element of $M \otimes_R I$ becomes 0 in $M \otimes_R R = M$, then it is also an element of $M \otimes_R I_0$ for some finitely generated subideal I_0 of I which becomes 0 in M. But by assumption, this element must be 0 in $M \otimes I_0$ and hence also in $M \otimes I$.

Now consider, for any ideal I of R, the short exact sequence $0 \to I \to R \to R/I \to 0$ with associated long exact sequence

$$0 = \mathrm{Tor}_1(M, R) \to \mathrm{Tor}_1(M, R/I) \to M \otimes_R I \to M \otimes_R R \to M \otimes_R R/I \to 0,$$

where the first equality holds since R is free. So $\mathrm{Tor}_1(M, R/I)$ injects into the kernel of the map $M \otimes_R I \to M \otimes_R R$, which is 0. Hence $\mathrm{Tor}_1(M, R/I) = 0$ for all ideals I of R, and we are done. □

Chapter 4

Étale morphisms

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4.1 Introduction

We want to incorporate the theory of manifolds to scheme theory. However, we cannot do this directly. To illustrate this difficulty, let

$$E \ : \ y^2 = f(x)$$

be an elliptic curve, that is, $f(x) \in \mathbb{C}[x]$ is a separable cubic polynomial. The separability condition implies that

$$\left(\frac{\partial f}{\partial x}\Big|_{x=x_0}, \frac{\partial f}{\partial y}\Big|_{y=y_0} \right) \neq 0$$

for all $(x_0, y_0) \in E \subset \mathbb{C}^2$, so the Implicit Function Theorem (IFT) will imply that E is a submanifold of $\mathbb{C}^2$. More precisely, if (in particular) $\partial f/\partial y \neq 0$, then the IFT tells us that:

(*i*) $\exists\, U \subset E$ and open neighborhood of (x_0, y_0),

(*ii*) $\exists g(x) \in \mathbb{C}[[x]]$ a power series,

such that

$$f(x, y) = 0 \qquad \Longleftrightarrow \qquad y = g(x) \ \ \forall (x, y) \in U,$$

and thus

$$\begin{array}{ccc} E \longrightarrow \mathbb{C} & \qquad & \mathbb{C} \longrightarrow E \\ (x,y) \longmapsto x & \qquad & x \longmapsto (x, g(x)) \end{array}$$

are analytic isomorphisms when restricted to U.

There are two immediate problems with this approach if we want to generalize to Scheme Theory:

(*i*) The open neighborhood U is usually too small for the Zariski topology, more precisely, the usual topology on $E \subset \mathbb{C}^2$ is stronger than the Zariski topology so the U given in (*i*) is not generally open in the Zariski topology.

(*ii*) The power series g need not be polynomial and so the map $x \mapsto (x, g(x))$ is not generally regular.

The above two hueristics don't actually prove that E is not a closed subscheme of $\mathbb{A}^2_{\mathbb{C}}$ of dimension 1 in the sense that E is locally isomorphic to $\mathbb{A}^1_{\mathbb{C}}$. The choice of E being an elliptic curve was not idle, in fact, no such isomorphism can exist since E has genus 1 (by Riemann-Roch) and $\mathbb{A}^1_{\mathbb{C}}$ is of genus 0.

So we must be clever in order to apply the theory of smooth manifolds to Scheme Theory. In what follows, we will describe the theory of differential forms from the point of view of commutative algebra in order to define smooth maps between schemes. This will lead us to study *étale* morphisms between schemes.

4.2 Differential Forms

Throughout this section, we assume the following notation:

- R is a commutative ring with 1,
- A is a commutative R-algebra, and its elements will be denoted with $f, g, h, \ldots$.
- M is an A-module.

Definition 4.2.1. An *R-derivation from A to M* is a map $d : A \to M$ that satisfies:

(*i*) d is R-linear,

(*ii*) (Product rule) For all $f, g \in A$, then $d(fg) = f \cdot d(g) + g \cdot d(f)$.

Remarks 4.2.2. (about derivations)

4.2.2.1 The set of R-derivations from A to M is naturally an A-module and we denote it by

$$\mathrm{Der}_R(A, M) := \{d : A \to M \mid d \text{ is an } R\text{-derivation}\}.$$

4.2.2.2 Derivations vanish on constants. More precisely,

$$d(r \cdot 1_A) = 0 \qquad \forall r \in R. \tag{4.1}$$

This can be shown by applying the product rule to $1_A = 1_A \cdot 1_A$.

4.2.2.3 We have the classical formula:

$$d(f^n) = nf^{n-1}d(f) \qquad \forall n > 0, f \in A, \tag{4.2}$$

which is obtained by repeatedly applying the product formula.

4.2.2.4 By combining (4.1) and (4.2), we see that if A is generated as an R-algebra by a set $\{x_i \mid i \in I\}$, then any R-derivation $d \in \mathrm{Der}_R(A, M)$ is completely determined by its values on the generators, more precisely, if $d, d' \in \mathrm{Der}_R(A, M)$ and $d(x_i) = d'(x_i)$ for all $i \in I$, then $d = d'$. This follows from the fact that derivations are R-linear and from (4.1) and (4.2).

4.2.2.5 Given any R-derivation $d : A \to M$ and any A-module homomorphism $\varphi : M \to N$, then $\varphi \circ d$ is an R derivation from A to N. Indeed, R-linearity follows from the fact that φ is, in particular, R-linear and the product rule follows immediately from the product rule and the fact that φ is A linear: for all $f, g \in A$ we have

$$(\varphi \circ d)(fg) = \varphi(d(fg)) = \varphi(fd(g) + gd(f)) = f\varphi(d(g)) + g\varphi(d(f)) = f(\varphi \circ d)(g) + g(\varphi \circ d)(f).$$

We can summarize this as follows: every $\varphi \in \mathrm{Hom}_A(M, N)$ induces a map

$$\Phi_\varphi : \mathrm{Der}_R(A, M) \longrightarrow \mathrm{Der}_R(A, N) \quad \text{defined by} \quad d \mapsto \varphi \circ d.$$

4.2.2.6 Let $\psi : A \to B$ be an R-algebra homomorphism and let M be a B-module. By restriction of scalars, M is an A-module. More precisely, if $x \in M$ and $f \in A$, then $f \cdot x = \psi(f)x$ is an A-module structure on M; we denote this A-module as $M/_A$. Now, if $d \in \mathrm{Der}_R(B, M)$ is an arbitrary R-derivation then, similarly as above, $d \circ \psi$ is an R-derivation from A to $M/_A$. The product rule follows from

$$(d \circ \psi)(fg) = d(\psi(fg)) = d(\psi(f)\psi(g)) = \psi(f)d(\psi(g) + \psi(g)d(\psi(f)) = \psi(f)(d \circ \psi)(g) + \psi(g)(d \circ \psi)(f)$$
$$\therefore \qquad (d \circ \psi)(fg) = f \cdot (d \circ \psi)(g) + g \cdot (d \circ \psi)(f)$$

We can summarize this as follows: every $\psi \in \mathrm{Hom}_{R\text{-alg}}(A, B)$ induces a map

$$\Phi_\psi : \mathrm{Der}_R(B, M) \longrightarrow \mathrm{Der}_R(A, M/_A) \quad \text{defined by} \quad d \mapsto d \circ \psi.$$

for every B-module M.

Examples 4.2.3. (of derivations)

4.2.3.1 Let A be the polynomial ring $A = R[x_i : i \in I]$. Given any family $(m_i)_{i\in I} \subset M$, there exists an R-derivation $d \in \mathrm{Der}_R(A, M)$ such that $d(x_i) = m_i$, in fact, for $p \in A$,

$$d(p) = \sum_{i\in I} \frac{\partial p}{\partial x_i} m_i$$

where $\partial p/\partial x_i$ is defined to be the "formal derivative" of the polynomial p with respect to x_i. Furthermore, by Remark **4.2.2.4**, this derivation is uniquely determined. This construction is analogous to the construction of linear maps between vector spaces by defining them on basis elements.

4.2.3.2 Consider $A \otimes_R A$ as a ring and consider the *multiplication map*:

$$\mu : A \otimes_R A \longrightarrow A \quad \text{defined by} \quad \mu(f \otimes g) = fg.$$

The multiplication map is a surjective ring homomorphism, so it is determined by the ideal

$$\mathcal{I} := \ker \mu.$$

Since $\mathcal{I}$ is an ideal of $A \otimes_R A$, it is an $(A \otimes_R A)$-submodule of $A \otimes_R A$ and thus $\mathcal{I}/\mathcal{I}^2$ is an $(A \otimes_R A)$-module. However, $\mathcal{I}/\mathcal{I}^2$ can be viewed as an A-module in three equivalent ways:

- Since multiplication by $\mathcal{I}$ annihilates $\mathcal{I}/\mathcal{I}^2$, i.e. $\mathcal{I} \subseteq \mathrm{Ann}_{A\otimes A}(\mathcal{I}/\mathcal{I}^2)$ then $\mathcal{I}/\mathcal{I}^2$ naturally inherits a $(A \otimes_R A)/\mathcal{I}$-module structure; since $(A \otimes_R A)/\mathcal{I}$ is naturally isomorphic to A via μ, then $\mathcal{I}/\mathcal{I}^2$ is an A-module.
- The ring $A \otimes_R A$ is an A-algebra with structure morphism

  $$\iota_1 : A \longrightarrow A \otimes_R A \quad \text{defined by} \quad f \mapsto f \otimes 1.$$

  So the $(A \otimes_R A)$-module $\mathcal{I}/\mathcal{I}^2$ can be given the structure of an A modulo by "restricting scalars" via ι_1. That is, for a general element $(\sum f_i \otimes g_i) + \mathcal{I}^2 \in \mathcal{I}/\mathcal{I}^2$ then the action of $h \in A$ is given by

  $$h \cdot \Big(\sum f_i \otimes g_i + \mathcal{I}^2\Big) = (h \otimes 1) \sum f_i \otimes g_i + \mathcal{I}^2 = \sum h f_i \otimes g_i + \mathcal{I}^2.$$

- Analogously, $\mathcal{I}/\mathcal{I}^2$ is an A-module by restricting scalars via

  $$\iota_2 : A \longrightarrow A \otimes_R A \quad \text{defined by} \quad f \mapsto 1 \otimes f.$$

With the structure of an A-module, we can define

$$\mathfrak{d} : A \longrightarrow \mathcal{I}/\mathcal{I}^2 \qquad \text{defined by} \qquad f \mapsto (1 \otimes f - f \otimes 1) + \mathcal{I}^2.$$

Notice that $f \mapsto (1 \otimes f - f \otimes 1)$ maps A into $\mathcal{I}$, so $\mathfrak{d}$ is well-defined. Furthermore, since the tensor product is over R, then $\mathfrak{d}$ is R-linear. Finally, if $f, g \in A$, then

$$\begin{aligned}
g \cdot \mathfrak{d}(f) + f \cdot \mathfrak{d}(g) - \mathfrak{d}(fg) &= g \cdot (1 \otimes f - f \otimes 1) + f \cdot (1 \otimes g - g \otimes 1) - \big((1 \otimes fg) - (fg \otimes 1)\big) + \mathcal{I}^2 \\
&= (g \otimes 1)(1 \otimes f - f \otimes 1) + (f \otimes 1) \cdot (1 \otimes g - g \otimes 1) - 1 \otimes fg + fg \otimes 1 + \mathcal{I}^2 \\
&= g \otimes f - gf \otimes 1 + f \otimes g - fg \otimes 1 - 1 \otimes fg + fg \otimes 1 + \mathcal{I}^2 \\
&= g \otimes f - gf \otimes 1 + f \otimes g - 1 \otimes fg + \mathcal{I}^2 \\
&= \underbrace{\big(1 \otimes g - g \otimes 1\big)\big(1 \otimes f - f \otimes 1\big)}_{\in \mathcal{I}^2} + \mathcal{I}^2
\end{aligned}$$

and thus $\mathfrak{d}$ is an R-derivation of A to $\mathcal{I}/\mathcal{I}^2$.

4.2.3.3 Let X be a smooth manifold and $p \in X$. Let $A = C_p(X, \mathbb{R})$ be the $\mathbb{R}$-algebra of germs of smooth functions around p, i.e.

$$C_p(X, \mathbb{R}) := \{f : U \to \mathbb{R} \mid U \subset X \text{ is open, } p \in U, \ f \text{ is smooth}\}/_{\sim}$$

where two smooth functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ are equivalent if there exists an open neighborhood W of p such that $W \subset U \cap V$ and $f|_W = g|_W$. In fact,

$$C_p(X, \mathbb{R}) = \varinjlim_{U \ni p} C(U, \mathbb{R})$$

where the direct limit is taken over all open neighborhoods of p and $C(U, \mathbb{R})$ is simply the R-algebra of smooth functions $U \to \mathbb{R}$.

Clearly, $A = C_p(X, \mathbb{R})$ is a commutative $\mathbb{R}$-algebra since it contains all the constant functions. Furthermore, evaluation at p induces a ring homomorphism $A \to \mathbb{R}$ defined by $f \mapsto f(p)$. Thus, if we set $M = \mathbb{R}$, then M is an A-module via $f \cdot x := f(p)x$.

Next we define an $\mathbb{R}$-derivation from $C_p(X, \mathbb{R})$ to $\mathbb{R}$. Let $\alpha : (-1, 1) \to X$ be a smooth curve in X passing through p, that is $\alpha(0) = p$, and take $f : U \to \mathbb{R}$ a germ in $C_p(X, \mathbb{R})$; we may assume without loss of generality, that the image of α is contained in U. Then the composition $f \circ \alpha : (-1, 1) \to \mathbb{R}$ is a smooth function at $t = 0$ and thus its derivative is well-defined. We can therefore define:

$$d : C_p(X, \mathbb{R}) \longrightarrow \mathbb{R} \quad \text{with} \quad d(f) := \left.\frac{d}{dt}\right|_{t=0} (f \circ \alpha)(t)$$

This function is clearly well-defined since derivatives are defined locally, and it is an $\mathbb{R}$-derivation by the basic properties of derivatives. The $\mathbb{R}$-derivation above is called the *directional derivative* in the direction of $\alpha'(0)$.

Derivations occur naturally in geometry, but what makes them useful in commutative algebra, and therefore in algebraic geometry, is the fact that they admit a universal object.

Theorem 4.2.4. *Let A be an R-algebra. There exists an A-module $\mathcal{O}^1_{A/R}$ and an R-derivation $d_{A/R} : A \to \mathcal{O}^1_{A/R}$ that satisfy the following universal property: for any A-module M and any R-derivation $d \in \mathrm{Der}_R(A, M)$, there exists a unique A-module morphism $\varphi : \Omega^1_{A/R} \to M$ the following diagram commutes:*

$$\begin{array}{ccc} A & \xrightarrow{\quad d \quad} & M \\ {\scriptstyle d_{A/R}}\big\downarrow & \nearrow_{\varphi} & \\ \Omega^1_{A/R} & & \end{array} \tag{4.3}$$

Remark 4.2.5. The universal property is equivalent to saying that there is a natural bijection

$$\mathrm{Hom}_A(\Omega^1_{A/R}, M) \xrightarrow{\sim} \mathrm{Der}_R(A, M) \quad \text{defined by} \quad \varphi \mapsto \varphi \circ d_{A/R}.$$

In fact, it is equivalent to saying that the functor

$$\mathrm{Der}_R(A, -) : {}_A\mathbf{Mod} \longrightarrow {}_A\mathbf{Mod} \quad \text{defined by} \quad M \mapsto \mathrm{Der}_R(A, M)$$

is *representable* and it is represented by $\Omega^1_{A/R}$.

Proof. (of Theorem 4.2.4) First we assume that A is a polynomial ring, i.e. $A = R[X]$ where $X = \{x_i\}_{i \in I}$. In this case, set $\Omega^1_{A/R} = A^{(I)}$, the free A-module generated by some set $(f_i)_{i \in I}$. By Example **4.2.3.1**, there is a unique derivation, say $d_{A/R} : A \to A^{(I)}$, such that $d_{A/r}(x_i) = f_i$.

The pair $(A^{(I)}, d_{A/R})$ satisfies the universal property. Indeed, if M is any A-module, and $d \in \mathrm{Der}_R(A, M)$, then

$$\varphi : A^{(I)} \longrightarrow M \qquad \text{defined on generators by} \qquad f_i \mapsto d(x_i),$$

is the required A-module homomorphism because, by construction, (4.3) commutes when restricted to the R-algebra generators x_i of A, and thus it still commutes when extended to any element of A because of the product rule and (4.1).

The general case, when A is not necessarily a polynomial ring, follows from the fact that "derivations descend to quotients". More precisely, if A is any R-algebra, then A is a homomorphic image of some polynomial ring $R[x_i : i \in I]$ and thus the proof of Theorem 4.2.4 follows from the following lemma. $\square$

Lemma 4.2.6. *Let $\pi : A \twoheadrightarrow B$ be a surjective ring homomorphism and let $\mathfrak{a} := \ker \pi$. If the R-algebra, A admits a solution $(\Omega^1_{A/R}, d_{A/R})$ to the universal property in Theorem 4.2.4, then $d_{A/R}$ induces a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{d_{A/R}} & \Omega^1_{A/R} \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow \\ B & \overset{d_{B/R}}{\dashrightarrow} & \dfrac{\Omega^1_{A/R}}{\mathfrak{a}\Omega^1_{A/R}+A\cdot d_{A/R}(\mathfrak{a})} \end{array}$$

where $(\mathcal{O}^1_{B/R}, d_{B/R})$ solves the universal property for B. In particular

$$\mathcal{O}^1_{B/R} \cong \frac{\Omega^1_{A/R}}{\mathfrak{a}\Omega^1_{A/R} + A \cdot d_{A/R}(\mathfrak{a})}$$

Proof. For simplicity, write $N = \mathfrak{a}\Omega^1_{A/R}+A\cdot d_{A/R}(\mathfrak{a})$ and $\nu : \Omega^1_{A/R} \twoheadrightarrow \Omega^1_{A/R}/N$ as its natural projection. By definition, $\Omega^1_{A/R}/N$ is an A-module. If $f \in \mathfrak{a}$ and $x \in \Omega^1_{A/R}$, then clearly $fx \in N$ so $f(x+N) = 0$ and thus $\mathfrak{a} \subseteq \operatorname{Ann}_A(\Omega^1_{A/R}/N)$. This implies that $\Omega^1_{A/R}/N$ is naturally a $A/\mathfrak{a} \cong B$-module with the following structure: if $g \in B$, there exists $f \in A$ such that $\pi(f) = g$ and if $y \in \Omega^1_{A/R}/N$, there exists $x \in \Omega^1_{A/R}$ such that $\nu(x) = y$; the B-module structure on $\Omega^1_{A/R}/N$ is given by

$$g \cdot y = \pi(f) \cdot \nu(x) = \nu(fx). \tag{4.4}$$

By Remark **4.2.2.5**, the composition

$$A \xrightarrow{d_{A/R}} \Omega^1_{A/R} \overset{\nu}{\twoheadrightarrow} \Omega^1_{A/R}/N$$

is an R-derivation of A to $\Omega^1_{A/R}/N$. Furthermore, its kernel contains $\mathfrak{a}$ since $A \cdot d_{A/R}(\mathfrak{a}) \subseteq N$. Thus the above composition factors through the projection $\pi : A \twoheadrightarrow B$ (which is R-linear if B is given the R-algebra structure induced by π), that is there exists an R-linear map $d_{B/R} : B \to \Omega^1_{A/R}/N$ such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\nu\circ d_{A/R}} & \Omega^1_{A/R}/N \\ \big\downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle d_{B/R}} & \\ B & & \end{array} \qquad \nu \circ d_{A/R} = d_{B/R} \circ \pi \tag{4.5}$$

We show that the pair $(\Omega^1_{A/R}/N, d_{B/R})$ satisfies the universal property.

Let M be a B-module and $d \in \operatorname{Der}_R(B, M)$. By Remark **4.2.2.6**, the composition $d \circ \pi$ is an R-derivation of A to M, the latter considered as an A-module via restriction of scalars. Since $(\Omega^1_{A/R}, d_{A/R})$ satisfies the universal property (4.3), the R-derivation $d \circ \pi$ factors through $d_{A/R}$. That is, there exists a unique A-module homomorphism $\varphi : \Omega^1_{A/R} \to M$ such that

$$\varphi \circ d_{A/R} = d \circ \pi. \tag{4.6}$$

We record what the A-module structure of M means for φ: if $f \in A$ and $x \in \Omega^1_{A/R}$, then being A-linear means

$$\varphi(fx) = f \cdot \varphi(x) = \pi(f)\varphi(x) \tag{4.7}$$

Now let $x \in N$, then $x = f_1 y + g d_{A/R}(f_2)$ where $f_1, f_2 \in \mathfrak{a} = \ker \pi$, $g \in A$ and $y \in \Omega^1_{A/R}$. Thus

$$\varphi(x) = \varphi(f_1 y + g d_{A/R}(f_2)) \overset{(4.7)}{=} \pi(f_1)\varphi(y) + \pi(g)\varphi(d_{A/R}(f_2)) \overset{(4.6)}{=} \pi(f_1)\varphi(y) + \pi(g)d(\pi(f_2)) = 0.$$

This means that $\varphi|_N = 0$ and thus factors through a unique A-module homomorphism $\overline{\varphi} : \Omega^1_{A/R}/N \to M$, i.e.

$$\overline{\varphi} \circ \nu = \varphi \quad \Longrightarrow \quad \overline{\varphi} \circ d_{B/R} \circ \pi \overset{(4.5)}{=} \overline{\varphi} \circ \nu \circ d_{A/R} = \varphi \circ d_{A/R} \overset{(4.6)}{=} d \circ \pi$$

Since π is an epimorphism, the above implies that $\overline{\varphi} \circ d_{B/R} = d$ as required. Now $\overline{f}$ is unique by construction and it is a B-module homomorphism because of the following: arbitrary elements of B and $\Omega^1_{A/R}/N$ are of the form $\pi(f)$ and $\nu(x)$ for some $f \in A$ and $x \in \Omega^1_{A/R}$, so that

$$\overline{\varphi}(\pi(f)\nu(x)) \overset{(4.4)}{=} \overline{\varphi}(\nu(fx)) = \varphi(fx) \overset{(4.7)}{=} \pi(f)\varphi(x) = \pi(f)\overline{\varphi}(\nu(x)).$$

This finishes the proof.

□

Definition 4.2.7. Given any R-algebra A, the A-module $\Omega^1_{A/R}$ is called the *module of relative differentials* of A over R and $d_{A/R}$ is called the *exterior differential* of A over R.

Remarks 4.2.8. (about the module of relative differentials)

4.2.8.1 Since the pair $(\Omega^1_{A/R}, d_{A/R})$ solves a universal property, then the pair is unique up to unique isomorphism.

4.2.8.2 The proof of Theorem 4.2.4, together with Lemma 4.2.6 implies that if A is generated by $\{x_i\}_{i\in I}$ as an R-algebra, then $\Omega^1_{A/R}$ is generated by $\{d_{A/R}(x_i)\}_{i\in I}$ as an A-module. Furthermore, if A is the polynomial ring $A = R[x_i : i \in I]$, then $\{d_{A/R}(x_i)\}_{i\in I}$ is a free generating set.

Examples 4.2.9. (of modules of relative differentials)

4.2.9.1 Let L/K be a finite separable extension, then $\Omega^1_{L/K} = 0$. Indeed, by the Primitive Element Theorem, $L = K(\theta)$ for some $\theta \in L$. Let $f(x) \in K[x]$ be the minimal polynomial of θ; note that separability implies that $f'(\theta) \neq 0$. By Remark **4.2.8.2**, $\Omega^1_{L/K}$ is a one dimensional vector space generated by $d_{L/K}(\theta)$. However,

$$0 = d(0) = d(f(\theta)) = f'(\theta)d(\theta) \quad \Longrightarrow \quad d(\theta) = 0$$

and thus $\Omega^1_{L/K} = 0$ as required.

4.3 The Sheaf of Differentials

In this section we review the construction of the sheaf of differentials of a scheme X. To do so, we review some basics of sheaves of modules.

We begin with affine schemes. Let A be a ring and $X = \operatorname{Spec} A$ be the prime spectrum of A with the Zariski Topology. The open sets

$$D(f) := \{\mathfrak{p} \in X \mid f \notin \mathfrak{p}\}$$

form a basis for the Zariski Topology. If $D(f) \subseteq D(g)$, then $f^n = ga$ for some $a \in A$ and $n \in \mathbb{N}$. The natural localization map

$$\ell_f : A \longrightarrow A_f \quad \text{defined by} \quad h \mapsto \frac{h}{1}$$

thus factors through the localization map $\ell_g : A \to A_g$ and we get a natural ring homomorphism

$$\ell_f^g : A_g \longrightarrow A_f \quad \text{defined by} \quad \frac{h}{g^m} = \frac{ha^n}{f^{nm}}$$

The above construction yields a sheaf on the basic open sets of X which in turn induces the following sheaf on X: let $\mathrm{Top}(X)$ be the category of open sets on X whose only morphisms are inclusions, then

$$\mathcal{O}_X : \mathrm{Top}(X) \longrightarrow \mathbf{Rings}, \quad \mathcal{O}_X(U) := \varprojlim_{D(f)\subseteq U} A_f, \quad \mathcal{O}_X(D(f) \hookrightarrow D(g)) = \ell_f^g$$

is a sheaf of rings on X and it is called the *structure sheaf* of X.

Definition 4.3.1. Let $(X, \mathcal{O}_X)$ be a ringed space. An $\mathcal{O}_X$*-module* is a sheaf $\mathcal{F}$ on X such that for every open $U \subseteq X$, the abelian group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and such that for every inclusion $V \hookrightarrow U$ the associated restriction map $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$. A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves is a *morphism of* $\mathcal{O}_X$*-modules* if each map $\mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism.

Remark 4.3.2. The phrase *compatible* in the previous definition means the following. Since every $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, then there is a structure map $\theta_U : \mathcal{O}_X(U) \times \mathcal{F}(U) \to \mathcal{F}(U)$ acting like $(a, x) \mapsto a \cdot x$. Then being *compatible* means that for every inclusion $\iota : V \hookrightarrow U$ we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{O}_X(U) \times \mathcal{F}(U) & \xrightarrow{\theta_U} & \mathcal{F}(U) \\ {\scriptstyle \mathcal{O}_X(\iota)\times\mathcal{F}(\iota)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(\iota)} \\ \mathcal{O}_X(V) \times \mathcal{F}(V) & \xrightarrow{\theta_V} & \mathcal{F}(V) \end{array}$$

Remarks 4.3.3. (about $\mathcal{O}_X$-modules)

4.3.3.1 Most operations allowed in the category of A-modules is also allowed for $\mathcal{O}_X$-modules. More precisely

4.3.3.1.*i* If $(\mathcal{F}_i)_{i\in I}$ is a family of $\mathcal{O}_x$-modules, then

$$\bigoplus_{i\in I} \mathcal{F}_i, \quad \prod_{i\in I} \mathcal{F}_i, \quad \varinjlim_{i\in I} \mathcal{F}_i, \quad \text{and} \quad \varprojlim_{i\in I} \mathcal{F}_i \quad \text{are all } \mathcal{O}_X\text{-modules.}$$

4.3.3.1.*ii* If $\mathcal{F}'$ is a subsheaf of $\mathcal{O}_X$-modules of $\mathcal{F}$, then the quotient $\mathcal{F}/\mathcal{F}'$ is an $\mathcal{O}_X$-module.

4.3.3.1.*iii* If $U \subseteq X$ is an open subset and $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}|_U$ is an $\mathcal{O}_X|_U$-module.

4.3.3.1.*iv* If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, then the *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is an $\mathcal{O}_X$-module, where $\mathcal{F} \otimes \mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

4.3.3.1.*v* If $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{O}_X$-modules, then $U \mapsto \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}|_U, \mathcal{G}|_U)$ is an $\mathcal{O}_X$-module where $\mathrm{Hom}_{\mathcal{O}_X}$ denotes the group of sheaf morphisms.

4.3.3.2 The $\mathcal{O}_X$-module structure can be transferred across morphisms of ringed spaces. More precisely, if $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, i.e. we have a pair $(f, f^\#)$ where $f : X \to Y$ is continuous and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings (where $f_*\mathcal{O}_X$ is the *direct image sheaf* defined by $(f_*\mathcal{O}_X)(V) := \mathcal{O}_X(f^{-1}(V))$), then we have the following:

4.3.3.2.*i* If $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $f_*\mathcal{F}$ is a $\mathcal{O}_Y$-module in the following manner: firstly, $f_*\mathcal{F}$ is an $f_*\mathcal{O}_X$-module, secondly the sheaf homomorphism $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ given to us by definition, makes $f_*\mathcal{O}_x$ into a $\mathcal{O}_Y$-module by restriction of scalars.

4.3.3.2.*ii* If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, then the *inverse image sheaf* $f^{-1}\mathcal{G}$, defined as the sheaf associated to the presheaf

$$U \mapsto \varprojlim_{V\supseteq f(U)} \mathcal{G}(V),$$

is a $f^{-1}\mathcal{O}_Y$-module.

4.3.3.2.*iii* The adjoint property of f_* and f^{-1} tells us that we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}X}(f^{-1}\mathcal{O}_Y, \mathcal{O}_X) \cong \mathrm{Hom}_{\mathrm{Sh}Y}(\mathcal{O}_Y, f_*\mathcal{O}_X).$$

Therefore, the morphism $f^{\#}$ gives us a natural morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$; this gives $\mathcal{O}_X$ a structure of an $f^{-1}\mathcal{O}_Y$-module. Furthermore, if $\mathcal{G}$ is an $\mathcal{O}_Y$-module, by **4.3.3.2.*ii***, it is an $f^{-1}\mathcal{O}_Y$-module and thus we can extend it to an $\mathcal{O}_X$ module via extension by scalars. More precisely, **4.3.3.1.*iv*** allows us to define

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

Below we describe a very important way to construct $\mathcal{O}_X$-modules in the affine case. Let $X = \mathrm{Spec}\, A$ be an affine scheme with structure sheaf $\mathcal{O}_X$ and let M be an A-module. Then for any $f \in A$, we may localize M at f:

$$M_f = M \otimes_A A_f.$$

This means that the functor $D(f) \mapsto M_f$ on objects and which sends the inclusion $D(f) \hookrightarrow D(g)$ to $\mathrm{id}_M \otimes \ell_f^g : M \otimes_A A_g \to M \otimes_A A_f$, is a sheaf on the category of basic open sets of the affine scheme $X = \mathrm{Spec}\, A$. Therefore, it extends uniquely to a sheaf on X. Furthermore, this sheaf is an $\mathcal{O}_X$-module because $M_f = M \otimes_A A_f$ is naturally an A_f-module with action $(h/f^m) \cdot (x \otimes h') = x \otimes (hh'/f^m)$ and thus the following diagram is commutative:

$$\begin{array}{ccc} A_f \times (M \otimes_A A_f) & \longrightarrow & M \otimes_A A_f \\ {\scriptstyle \ell_f^g \times (\mathrm{id}_M \otimes \ell_f^g)} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_M \otimes \ell_f^g} \\ A_g \times (M \otimes_A A_g) & \longrightarrow & M \otimes_A A_g \end{array}$$

This leads to the following definition.

Definition 4.3.4. Let $(X, \mathcal{O}_X)$ be an affine scheme with $X = \mathrm{Spec}\, A$ for some ring A and let M be an A-module. The $\mathcal{O}_X$-*module sheaf associated to* M is the $\mathcal{O}_X$-module $\widetilde{M}$ defined on basic open sets by

$$\widetilde{M}(D(f)) = M \otimes_A A_f.$$

Remarks 4.3.5. (about $\widetilde{M}$)

4.3.5.1 If $X = \mathrm{Spec}(A)$, we have $\mathcal{O}_X \cong \widetilde{A}$.

4.3.5.2 The functor $M \mapsto \widetilde{M}$ gives an exact fully faithful functor from ${}_A\mathbf{Mod}$ to the category of $\mathcal{O}_X$-modules.

4.3.5.3 If M and N are A-modules, then $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.

4.3.5.4 If $(M_i)_{i \in I}$ is a family of A-modules, then $\widetilde{\oplus_i M_i} \cong \oplus_i \widetilde{M_i}$.

4.3.5.5 If B is an A-algebra and $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is the induced map, then for any B-module N we have

$$f_*(\widetilde{N}) \cong \widetilde{N/_A},$$

where $N/_A$ is the A-module obtained by restriction of scalars.

4.3.5.6 For any A-module M, we have $f^*(\widetilde{M}) \cong \widetilde{M \otimes_A B}$.

Next we review properties of quasi-coherent and coherent $\mathcal{O}_X$-modules.

Definition 4.3.6. Let $(X, \mathcal{O}_X)$ be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. We say that $\mathcal{F}$ is *quasi-coherent* if there exists an open cover $X = \bigcup_{i \in I} U_i$ that satisfies: for every $i \in I$ we have

(i) U_i is affine, i.e. there exists a ring A_i such that $U_i \cong \mathrm{Spec}\, A_i$, (*ii*) there exists an A_i-module M_i such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$.

Furthermore, if each M_i can be taken to be finitely generated as an A_i-module, then we say that $\mathcal{F}$ is *coherent.*

Remarks 4.3.7. (about quasi-coherent and coherent $\mathcal{O}_X$-modules)

4.3.7.1 $\mathcal{F}$ is quasi-coherent if and only if for every open affine subset $U = \operatorname{Spec} A$ of X, there is an A-module M such that $\mathcal{F}|_U \cong \widetilde{M}$. We can replace "quasi-coherent" with "coherent" if we required M to be finitely generated and that X be a noetherian scheme.

4.3.7.2 Let $X = \operatorname{Spec} A$. The functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between A-modules and the category of quasi-coherent $\mathcal{O}_X$-modules; its inverse functor is the *global sections* functor $\Gamma(X, \mathcal{F})$. Furthermore, if A is noetherian, $M \mapsto \tilde{M}$ is an equivalence between the category of finitely generate A-modules and the category of coherent $\mathcal{O}_X$-modules.

4.3.7.3 If $X = \operatorname{Spec} A$ is an affine scheme and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $H^1(X, \mathcal{F}) = 0$, i.e. the global sections functor $\Gamma(X, -)$ is exact.

4.3.7.4 Let $f : \mathcal{F} \to \mathcal{G}$ a morphism of $\mathcal{O}_X$-modules. Then $\ker(f)$, $\operatorname{coker}(f)$ and $\operatorname{im}(f)$ are all quasi-coherent. If X is noetherian, then the same is true if we replace "quasi-coherent" with coherent.

4.3.7.5 If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-modules where $\mathcal{F}'$ and $\mathcal{F}''$ are quasi-coherent, then $\mathcal{F}$ is quasi-coherent. If X is noetherian, then the same is true if we replace "quasi-coherent" with coherent.

4.3.7.6 Let $f : X \to Y$ be a morphism of schemes and let $\mathcal{F}$ and $\mathcal{G}$ be a $\mathcal{O}_X$-module and a $\mathcal{O}_Y$-module respectively. Then:

(*i*) If $\mathcal{G}$ is quasi-coherent, then $f^*\mathcal{O}_Y$ is a quasi-coherent $\mathcal{O}_X$-module.

(*ii*) If X is noetherian and $\mathcal{F}$ is quasi-coherent, then $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module

Definition 4.3.8. Let $(X, \mathcal{O}_X)$ be a scheme and $(Y, \mathcal{O}_Y)$ a closed subscheme. If $(i, i^\#)$ is the inclusion morphism $Y \hookrightarrow X$, then the *ideal sheaf* of Y is defined as

$$\mathcal{I}_Y := \ker(i^\# : \mathcal{O}_Y \to i_*\mathcal{O}_X).$$

Proposition 4.3.9. *Let X be a scheme. For any closed subscheme Y, the ideal sheaf $\mathcal{I}_Y$ is quasi-coherent. Conversely, if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, there exists a unique closed subscheme Y of X such that $\mathcal{F} \cong \mathcal{I}_Y$.*

4.3.1 The Sheaf of Relative Differentials

Let $\sigma : X \to S$ be an S-scheme and let $\Delta : X \to X \times_S X$ be the diagonal embedding. If $X = \operatorname{Spec} A$ is affine then this diagonal embedding corresponds to the multiplication map $A \otimes A \to A$ which is surjective and thus, in the affine case, Δ is a closed immersion. Thus, in the general case, Δ is a locally closed immersion, i.e. there exists an open subscheme $W \subset X \times_S X$ such that $\Delta(X) \subseteq W$ and the corestriction $\Delta| : X \to W$ is a closed immersion.

By Proposition 4.3.9, there is a quasi-coherent ideal sheaf $\mathcal{I}$ associated to $\Delta(X)$. By Remark **4.3.7.4**, $\mathcal{I}/\mathcal{I}^2$ is a quasi-coherent $\mathcal{O}_X$-module. With this, we can define:

Definition 4.3.10. Let $X \to S$ be an S-scheme. The *sheaf of relative differentials forms* of X/S is the $\mathcal{O}_X$-module

$$\Omega^1_{X/S} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$$

where $\mathcal{I}$ is the quasi-coherent ideal sheaf associated to the locally closed immersion $\Delta : X \to X \times_S X$.

Remarks 4.3.11. (about $\Omega^1_{X/S}$)

4.3.11.1 The definition of $\Omega^1_{X/S}$ is independent of choice of open set W containing $\Delta(X)$ (because the inverse image sheaf is compatible with restricting to open sets, i.e. $\Delta^*(\mathcal{J}/\mathcal{J}^2)|_U = \Delta|_U^*(\mathcal{J}/\mathcal{J}^2|_W)$ if $\Delta(U) \subseteq W$).

4.3.11.2 Remark **4.3.7.6** guarantees that $\Omega^1_{X/S}$ is quasi-coherent.

Next we describe $\Omega^1_{X/S}$ locally. Let $U = \operatorname{Spec} A$ and $V = \operatorname{Spec} R$ be open affine subschemes of X and S respectively where $\sigma(U) \subseteq V$. The restriction $\sigma| : U \to V$ corresponds to a ring homomorphism $R \to A$ and thus A is an R-algebra. Define

$$\mathcal{I} := \ker(\mu : A \otimes_R A \longrightarrow A)$$

which is an A-module. We have

$$\mathcal{J}|_{U \times_S U} \cong \widetilde{\mathcal{I}}$$

because the diagonal map Δ is locally the multiplication map. By Remark **4.3.5.6** we have

$$\Delta^*(\mathcal{J}/\mathcal{J}^2)|_U \cong \Delta^*|_U(\mathcal{J}/\mathcal{J}^2|_{U\times U}) \cong \Delta^*|_U(\widetilde{\mathcal{I}}/\widetilde{\mathcal{I}^2}) = \Delta^*(\widetilde{\mathcal{I}}/\widetilde{\mathcal{I}^2}) \cong \widetilde{\mathcal{I}/\mathcal{I}^2 \otimes_{A\otimes A} A}$$

Since $(A \otimes A)/\mathcal{I} \cong A$ via μ, then the above is simply

$$\Delta^*(\mathcal{J}/\mathcal{J}^2)|_U \cong \widetilde{\mathcal{I}/\mathcal{I}^2} \cong \widetilde{\Omega^1_{A/R}}.$$

This observation means that we can also construct $\Omega^1_{X/S}$ by gluing the the quasi-coherent $\mathcal{O}_X$-modules $\widetilde{\Omega^1_{A/R}}$ along all affine open sets using the standard scheme gluing:

Lemma 4.3.12. *Let $(X_i)_{i\in I}$ be a family of schemes. Furthermore, for each pair $i, j \in I$, there is an isomorphism $\varphi_{ij} : X_{ij} \to X_{ji}$ where X_{ij} and X_{ji} are open subschemes of X_i and X_j respectively. Suppose that these objects are subject to the following conditions:*

(i) $X_{ii} = X_i$, $\varphi_{ii} = \mathrm{id}$ *and* $\varphi_{ji} \circ \varphi_{ij} = \mathrm{id}$.

(ii) For each pair $i, j \in I$ and each index $k \in I$, the isomorphism φ_{ij} restricts to an isomorphism

$$\varphi_{ij}^{(k)} : X_{ij} \cap X_{ik} \longrightarrow X_{ji} \cap X_{jk}$$

such that $\varphi_{ij}^{(k)}$ satisfies the cocycle condition

$$\varphi_{ik}^{(i)} = \varphi_{jk}^{(i)} \circ \varphi_{ij}^{(k)}.$$

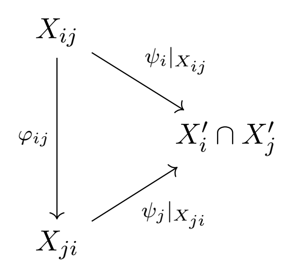
If all of these conditions hold, there is a scheme X, unique up to canonical isomorphism, together with a family of morphisms $(\psi_i : X_i \to X)$ that satisfy the following:

(a) For each $i \in I$ there is an open subscheme $X_i' \subseteq X$ such that ψ restricts to an isomorphism $\psi_i : X_i \to X_i'$,

(b) (X_i') is an open cover of X,

(c) for all $i, j \in I$ we have $\psi_i(X_{ij}) = X_i' \cap X_j'$,

(d) for all $i, j \in I$ we have the following commutative diagram:



Chapter 5

Henselian rings

CRAIG HUNEKE

Topics:

(1) Hensel's Lemma and Applications

(2) Henselian Rings

(3) Henselizations

There are not many references on Henselian rings. One source written in French is "Anneaux Locaux Henséliens" by Michel Raynaud [4].

5.1 Hensel's Lemma and Applications

Notation: By (R, m, κ) we mean a local ring R with maximal ideal m and residue field κ.

Example 5.1.1. Let k be a field and R be the ring of polynomials $R = k[x_1, \dots x_n]$. Then the completion of R with respect to the maximal ideal $(x_1, x_2, \dots x_n)$ is the ring of power series $\hat{R} = k[[x_1, \dots x_n]]$.

Definition 5.1.2. Let (R, m, κ) be a local ring with the m-adic topology. The ring R is called **separated** if $\cap_{n\geq 1} m^n = 0$. This is because the collection $\{m^n\}_{n\in\mathbb{N}}$ is a fundamental system of neighborhoods of zero. Their intersection being zero ensures that R is Hausdorff (separated) as a topological ring.

Notation: By overscoring an algebraic object (e.g. $\overline{R}$) we mean "modulo m" ($R \rightsquigarrow R/m$).

Lemma 5.1.3 (Hensel's Lemma). *Let (R, m, κ) be a local, complete ($R = \hat{R}$), separated ring and $F \in R[x]$ be a monic polynomial. Suppose $\overline{F} = gh$ in $\kappa[x]$ where g, h are monic and relatively prime. Then there exist monic polynomials G, $H \in R[x]$ such that $F = GH$ with $\overline{G} = g$ and $\overline{H} = h$.*

Aside: Since g, h are relatively prime, $(g, h) = \kappa[x]$. If $deg\ f < deg\ g + deg\ h$, then $f = ag + bh$ for some polynomials a, b where we claim $deg\ a < deg\ h$ and $deg\ b < deg\ g$. This because if $f = a'g + b'h$ we can apply the division algorithm in $\kappa[x]$ to obtain remainder b: $b' = dg + b$ where $deg\ b < deg\ g$. Then $f = (a' + dh)g + bh$ and $a := a' + dh$ must have degree less than $deg\ h$.

Proof of Hensel's Lemma: (By induction) We claim that there exist monic polynomials G_n, H_n such that $F - G_nH_n \in m^nR[x]$ and $\overline{G_n} = g$, $\overline{H_n} = h$. We will also construct them to satisfy $G_n - G_{n-1}$, $H_n - H_{n-1} \in m^{n-1}R[x]$. Then we set $G = lim_n G_n$ and $H = lim_n H_n$ which are elements of $R[x]$ because R is complete and $F - GH \in \cap_n m^n R[x] = 0$ because R is separated.

i) Begin by lifting g and h to monic polynomials G_1 and H_1. Note $\overline{F - G_1H_1} = \overline{F} - gh = 0$ so $F - G_1H_1 \in mR[x]$.

ii) Choose $y_1, ...y_l \in m^n$ such that $F - G_n H_n = \sum_{i=1}^{l} y_i L_i$ where each $L_i \in R[x]$ and $deg\ L_i < degF = deg\ g + deg\ h$. Write $\overline{L_i} = a_i\, g + b_i\, h$ (choose a_i, b_i satisfying the aside) and lift them to $a_i \rightsquigarrow A_i$ and $b_i \rightsquigarrow B_i$ keeping the degrees the same.

Note that $L_i - A_i\, G_n - B_i\, H_n \in mR[x]$ so that $F - G_n\, H_n - \sum_i y_i(A_i\, G_n + B_i\, H_n) \in m^{n+1}R[x]$. We thus set $G_{n+1} = G_n - \sum_i y_i\, B_i$ and $H_{n+1} = H_n - \sum_i y_i\, A_i$ so that

$$G_{n+1}\, H_{n+1} = G_n\, H_n - \textstyle\sum_i y_i(A_i\, G_n + B_i\, H_n) + \sum_{i,\,j} y_i\, y_j\, A_i\, B_j$$

The last term $\sum_{i,\,j} y_i\, y_j\, A_i\, B_j \in m^{2n}R[x]$. We have that $F - G_{n+1}\, H_{n+1} \in m^{n+1}R[x]$ and $G_{n+1} \equiv G_n\ mod\ m^n R[x]$, $H_{n+1} \equiv H_n\ mod\ m^n R[x]$. Finally, because $deg\ G_{n+1} = deg\ G_n$ and $deg\ H_{n+1} = deg\ H_n$ we have a coherent sequence of polynomials, being elements of our (complete) ring which provide the desired factorization.

Corollary 5.1.4. *Let (R, m, κ) be a local, complete, and separated ring. Suppose $F \in R[x]$ is a monic polynomial and there is some $r \in R$ such that $F(r) \in m$ but $F'(r) \notin m$ (r is a non-repeated root of F mod m). Then there exists $a \in R$ such that $a \equiv r\ mod\ m$ and $F(a) = 0$.*

Proof: The polynomial $\overline{F}$ factors into $(x - \overline{r})h(x)$ where $((x - \overline{r}),\ h(x)) = R[x]$ (because $\overline{r}$ is not a repeated root). Apply Hensel's Lemma.

Notation: Let $k = \overline{k}$ be an algebraically closed field of characteristic 0. We denote the ring of Laurent series by $k((t)) = k[[t]][t^-1]$.

Theorem 5.1.5 (Newton-Puiseaux))**.** *The completion of the ring of Laurent Series, $\overline{k((t))} = \cup_{n\geq 1} k[[t^{1/n}]]$.*

Proof: It suffices to show that the right-hand side is algebraically closed; denote the RHS by L. Let $F(x) \in S[x]$ where $S = \cup_{n\geq 1} k[[t^{1/n}]]$ be monic, of degree at least two. It suffices to show that F splits in L.

Then $F = a_1 x^{n-1} + ... + a_n$ where each coefficient $a_i \in k[[t^{1/m_i}]]$ and without loss of generality we may assume each $m_i = 1$ by changing variables $t^{1/lcm\{m_i\}} \rightsquigarrow t$. Furthermore, we may change variables $x' = x + \frac{1}{n}a_1$ (recall $char\ k \neq 0$) which allows us to assume $a_1 = 0$.
Next, we want to show one of $a_2,\ ...a_n$ is a unit iff there is some a_i such that $a_i(0) \neq 0$. Assume this is not the case and let $r = \min\{ord_t(a_i(t))\,/i\}$ so that r is a positive, rational number. Then for each $i = 2, ...n$, $t^{ri}|a_i(t)$ so define $b_i = a_i(t)/t^{r_i}$ and $X = t^r Y$. Then F becomes $F(X) = (t^r Y)^n + ... + t^{ri} b_i (t^r Y)^{n-i} + ... + t^{rn} b_n = t^{rn}(Y^n + ... + b_i Y^{n-i} + ... + b_n)$. The polynomial $G(Y) = Y^n + ... + b_i Y^{n-i} + ... + b_n$ has at least one coefficient a unit. By renormalizing the power of t to assume $F \in k[[t]]$, $a_1 = 0$, and at least one a_i is a unit for $i = 2, ...n$.
Now we use Hensel's Lemma. It suffices by Hensel's Lemma to prove that $\overline{F}$ factors into two relatively prime polynomials. Since $k = \overline{k}$, F factors into linear factors, there is no problem unless $\overline{F} = (x - \alpha)^n$ for some $\alpha \in k$. But $0 = \overline{a_1} = n\alpha \implies \alpha = 0$ so $\overline{F} = x^n$ would mean $\overline{a_2} = ... = \overline{a_n} = 0$ contradicts one is nonzero.

5.2 Henselian Rings

Let (R, m, κ) be a local ring. The following are equivalent:

Theorem 5.2.1.
1. *If $F(x) \in R[x]$ is monic and there is some $r \in R$ such that $F(r) \in m$, $F'(r) \notin m$, then there is some $a \in R$ where $a \equiv r\ mod\ m$ such that $F(a) = 0$.*
2. *Every elementary (pointed) étale neighborhood $R \to S$ is an isomorphism.*
3. *If $F_1, \dots,\ F_n \in R[x_1, \dots,\ x_n]$ and $\underline{r} \in R^n$ satisfying $F_i(\underline{r}) \in m$ for all $i = 1, \dots,\ n$ and $|\partial F_i/\partial x_j|_{\underline{r}} \notin m$, then there exists a $\underline{s} \in R^n$ such that $\underline{s} - \underline{r} \in mR^n$ such that $F_i(\underline{s}) = 0$ for all $i = 1, \dots,\ n$.*
4. *If $F \in R[x]$ is monic, $\overline{F} = g\, h$, where g and h are monic and relatively prime, then $F = G\, H$ with G, H monic $\overline{G} = g$ and $\overline{H} = h$.*
5. *Every module-finite extension $R \to S$ (S is a finite R-mod) is a (finite) product of local rings.*

Definition 5.2.2. A local ring satisfying these definitions is called Henselian.

<u>*Examples/Remarks*</u>:

1. If R is local, complete, and separated, then R is Henselian.
2. Quotients of Henselian rings are Henselian ($R \implies R/I$).
3. A ring R is Henselian iff $R/\sqrt{(0)}$ is Henselian.
4. The ring $\mathbb{C} << x_1, ..., x_n >>$ (convergent power series) is Henselian.

Definition 5.2.3. An elementary étale neighborhood is an étale morphism between local rings $\phi : R \to S$ where $S/m_S \simeq R/m_R$.

Theorem 5.2.4 (Structure Theorem for Elementary Étale Neighborhood)**.** *If $\phi : R \to S$ is an elementary étale neighborhood, then $S \simeq (\frac{R[x]}{(F(x))})_Q$ for some $a \in R$, $F \in R[x]$ is monic, $Q = (x - a, mR[x])$, and $F'(a) \notin m$.*

<u>*Proof*</u>: We will first prove the equivalence of 1. through 4. Recall (R, m, κ) is a local ring.
<u>(4. $\implies$</u> 1.) DONE

(1. $\implies$ 2.) Let $\phi : R \to S$ be an elementary étale neighborhood. By the Structure Theorem, $S \simeq (\frac{R[x]}{(F(x))})_Q$ for some $r \in R$, $F \in R[x]$ is monic, $Q = (x - r, mR[x])$, and $F'(r) \notin m$. As assumed in 1, there is some $a \in R$ such that $a \equiv r \ mod\ m$ and $F(a) = 0$ so F factors as $F(x) = (x - a)G(x)$. Because $F'(r) \notin m$, $G(x) \notin Q$ and becomes a unit when localized with respect to Q. Thus $S \simeq (\frac{R[x]}{((x-a)G(x))})_Q \simeq (\frac{R[x]}{(x-a)})_Q \simeq R$. We remark most of the work of this part is done by the Structure Theorem.

(2. $\implies$ 3.) Recall the set up of 3: let $F_1, ..., F_n \in R[x_1, ..., x_n]$ and $\underline{r} \in R^n$ satisfying $F_i(\underline{r}) \in m$ for all $i = 1, ..., n$ and $|\partial F_i/\partial x_j|_{\underline{r}} \notin m$. Set $S = (\frac{R[x_1, ..., x_n]}{(F_1, ..., F_n)})_Q$ where $Q = (x_i - r_i, mR[x_1, ..., x_n])$ because of the Jacobian condition of the F_i, $\phi : R \to S$ is étale and $S_Q/QS_Q = \kappa$. So S is an elementary étale neighborhood. By 2 we know that ϕ is an isomorphism so there are s_i where $\phi(s_i) = x_i$ and $F_i(s_1, ..., s_n)$ for all $i = 1, ..., n$. Moreover $s_i \equiv r_i \ mod\ m$.

(3. $\implies$ 4.) Let F be a monic polynomial in $R[x]$ where $\overline{F} = gh$, g and h are monic and relatively prime. Then write $F = x^n + r_{n-1}x^{n-1} + ... r_0$, $g = x^a + \alpha_{a-1}x^{a-1} + ... + \alpha_0$, $h = x^b + \beta_{n-1}x^{n-1} + ... + \beta_0$ where the coefficients $r_i \in R$, $\alpha_i, \beta_i \in \kappa$. Note our equation $\overline{F} = gh$ means that $\alpha_0\beta_0 = \overline{r_0}$, $[\alpha_1\beta_0 + \alpha_0\beta_1 = \overline{r_1}]$, ... $[\beta_a + \alpha_{a-1}\beta_1 + ... + \alpha_0\beta_a = \overline{r_a}]$, ... $[\alpha_{a-1} + \beta_{b-1} = \overline{r_{n-1}}]$. Consider the following system of equations (of the variables $Y_0, Y_1, ... Y_{a-1}, Z_0, Z_1, ... Z_{b-1}$):

$$F_0(...) = Y_0Z_0 - r_0$$
$$F_1(...) = Y_1Z_0 + Y_0Z_1 - r_1$$
$$...$$
$$F_a(...) = Z_0 + Y_{a-1}Z_1 + ... + Y_0Z_a - r_a$$
$$...$$
$$F_{n-1}(...) = Y_{a-1} + Z_{b-1} - r_{n-1}$$

So each $F_j \in R[Y_0, Y_1, ... Y_{a-1}, Z_0, Z_1, ... Z_{b-1}]$ are functions of n variables. We want to find a solution lifting $\alpha_0, ... \alpha_{a-1}, \beta_0, ... \beta_{b-1}$. Then defining G, H in the obvious way gives $F = GH$. Note that there is a solution $mod\ m$. We need to prove that $|\partial F_i/\partial Y_i$ or $\partial Z_i|_((\alpha_0, ... \alpha_{a-1}, \beta_0, ... \beta_{b-1}) \neq 0$ in κ to be able to use the assumption from 3. Consider the matrix J:

$$\begin{array}{c|ccccccccc|}
\partial Z_0 & Y_0 & Y_1 & ... & Y_{a-1} & 1 & 0 & 0 & ... & 0 \\
\partial Z_1 & 0 & Y_0 & ... & Y_{a-2} & Y_{a-1} & 1 & 0 & ... & 0 \\
\vdots & \vdots & \ddots & \ddots & ... & \ddots & \ddots & \ddots & ... & \vdots \\
\partial Z_{b-1} & 0 & 0 & ... & Y_0 & Y_1 & ... & ... & Y_{a-1} & 1 \\
\partial Y_0 & Z_0 & Z_1 & ... & Z_{b-1} & 1 & 0 & 0 & ... & 0 \\
\partial Y_1 & 0 & Z_0 & ... & Z_{b-2} & Z_{b-1} & 1 & 0 & ... & 0 \\
\vdots & \vdots & \ddots & \ddots & ... & \ddots & \ddots & \ddots & ... & \vdots \\
\partial Y_{a-1} & 0 & 0 & ... & Z_0 & Z_1 & ... & ... & Z_{b-1} & 1
\end{array}$$

Assume that $det(J) = 0$ when evaluated at $(\alpha_0, ... \alpha_{a-1}, \beta_0, ... \beta_{b-1})$. Then there exists a nonzero solution to $(t_0, ... t_{a-1}, s_0, ... s_{b-1})J(\alpha_0, ... \alpha_{a-1}, \beta_0, ... \beta_{b-1}) = 0$. Said a different way, there are $(t_0, ... t_{a-1}, s_0, ... s_{b-1})$ such that $[t_0\alpha_0 + s_0\beta_0 = 0]$, $[t_0\alpha_1 + t_1\alpha_0 + s_0\beta_1 + s_1\beta_0 = 0]$, ... $[t_{b-1} + s_{a-1} = 0]$. So define the polynomials $t(x) = t_{b-1}x^{b-1} + ... + t_0$ and $s(x) = s_{a-1}x^{a-1} + ... s_0$. The previous equations imply $t(x)g + s(x)h = 0$. But $deg\ t = b - 1$ and $deg\ s = a - 1$, so g and h cannot be relatively prime because if they were, $h|t$. □

This shows 1. through 4. are equivalent, but before proving the equivalence of 5. we must first discuss splittings.

Definition 5.2.5. A commutative ring S is said to **decompose** if $S \equiv S_1 \times S_2 \times ... \times S_l$ where (S_i, m_i) is a local ring.

Examples:

1. Any local ring decomposes.

2. Any Artinian ring decomposes.
   Proof: If S is an Artinian ring, then $Spec(S) = \{M_1, ..., M_l\}$ and each $M_i \subset (M_1 \cap ... \cap M_l)\sqrt{(0)}$. Then there is some $n \in \mathbb{N}$ such that $M_1^n...M_l^n = 0$ and because M_i^n is comaximal to M_j^n for $i \neq j$, the Chinese Remainder Theorem tells us $S = S/(M_1^n...M_l^n) = S/M_1^n \times ... \times S/M_l^n$ where each S/M_i^n is a local ring.

3. Any homomorphic image of a decomposable ring decomposes.
   Proof: Note that any ideal of a product of rings $R_1 \times ... \times R_l$ is of the form $I_1 \times ... \times I_l$ where I_j is an ideal of R_j. The statement follows.

4. If R is Henselian and $F(x) \in R[x]$ is monic, then $R[x]/(F(x))$ decomposes.
   Proof: Factor $\overline{F} = g_1 ... g_l$ with each $(g_i, g_j) = 1$ for $i \neq j$. Since R is Henselian, the g_i lift to G_i which are monic since $F = G_1 ... G_l$. Then $R[x]/(F) \equiv R[x]/(G_1) \times ... \times R[x]/(G_l)$. Since $mS \subset Jac(S)$ (S is module-finite over R) and $mod\ m\ \sqrt{(g_i)}$ is maximal. The product follows using the Chinese Remainder Theorem.

5. If S decomposes $S = S_1 \times ... \times S_l$ where (S_i, m_i) are local, then $Max(s) = \{M_1, ..., M_l\}$ where $M_i = (x_1, ..., x_{i-1}, m_i, x_{i+1}, ..., x_l)$ and $S_{M_i} = S_i$.

General Discussion of Idempotents: Setup: let (R, m, κ) be a local ring and $R \to S$ is finite as an $R-$module. Recall that $Idem(S) := \{e \in S : e^2 = e\}$.

Lemma 5.2.6. *The map* $Idem(S) \to Idem(\overline{S})$ *is* $1 - 1$.

Proof: Suppose e and f are idempotents of S and $\overline{e} = \overline{f}$ so $e - f \in mS$. Then $(e - f)^3 = e^3 - 3e^2f + 3ef^2 + -f^3 = e - f$ so $((e - f)^2 - 1)(e - f) = 0$. Because $(e - f)^2 - 1 \in mS$ so $(e - f)^2 - 1$ is a unit $\implies e = f$.

Proposition 5.2.7. *The following are equivalent:*

- *The map* $Idem(S) \to Idem(\overline{S})$ *is surjective.*
- *S decomposes*
- *If we write the (Artinian) ring* $\overline{S} = S_1 \times ... \times S_l$*, denote* $\overline{e_i} = (0, ..., 1, ..., 0)$*, then the* $\overline{e_i}$ *lift to* $Idem(S)$.

Proof: The fact that 1. $\iff$ 3. is follows from $Idem(\overline{S}) = \{\overline{e_{i_1}} + ... + \overline{e_{i_m}}\}$ since the only idempotents in a local ring are 0 and 1.

(2. $\implies$ 3.) Lift each $\overline{e_i}$ to $e_i \in Idem(S)$ and note e_ie_j lifts $\overline{e_ie_j} = 0$ for $i \neq j$ so by injectivity $e_ie_j = 0$. Now $e_1 + ... + e_l \in Idem(S)$ lifts $1 = \overline{e_1} + ... \overline{e_l}$ so by the lemma $1 = e_1 + ... + e_l$. We thus have a set of orthonormal idempotents which shows $S \equiv Se_1 \times ... \times Se_l$ and $\overline{Se_i} = S_i$ is local hence Se_i is local.

(3. $\implies$ 2.) This follows from the (not proven here) fact that decompositions are essentially unique. $\square$
We are now ready to prove the equivalence of statement 5. of a Henselian ring.

Theorem 5.2.8. *A local ring R is Henselian iff every (module) finite extension $R \to S$ decomposes.*

Proof: ($\implies$) Assume R is Henselian, $R \to S$ is a module-finite map, then $\overline{S}$ is Artinian hence decomposes and $Max(S) = \{M_1, \dots M_l\}$. (Using the Jacobson radical and going *mod* M_1, we obtain a finite-dimensional vector space. Then $\overline{S} \equiv (\overline{S})_{M_1} \times \dots \times (\overline{S})_{M_l}$.) By the previous proposition and without loss of generality, it suffices to lift $\overline{e} = \overline{e_1} = (1, 0, \dots, 0)$ to an idempotent in S. Choose any $c \in S$ such that $\overline{c} = \overline{e}$. Then $c \in M_2 \cap \dots \cap M_l$ and $c - 1 \in M_1$.

Consider the maps $R \overset{R}{\hookrightarrow} [c] \hookrightarrow S$ and $R[x]/(F(x)) \to R[c]$ where because S is finite over R, we let $F(x)$ be a minimal polynomial such that $F(c) = 0$. Because $R[x]/(F(x))$ decomposes and surjects onto $R[c]$, so does $R[c]$, being a homomorphic image.

The maximal ideals of $R[c]$ are exactly $M_i \cap R[c]$ so we write $Q = M_1 \cap R[c]$. Notice since $c \in M_2 \cap \dots \cap M_l$, $Q' := M_i \cap R[c] = (c, mR[c])$ for all $i = 2, \dots l$. So there are only two maximal ideals: $R[c] \equiv R[c]_Q \times R[c]_{Q'}$ containing an idempotent $\epsilon := (1, 0)$. Notice $\epsilon \in M_2 \cap \dots \cap M_l \subset S$ so $\overline{\epsilon} = (1, 0, \dots, 0) \in \overline{S}$. So ϵ lifts e.

Exercise: The other direction is easier.

Remark: If (R, m, k) is a Henselian ring and a domain, $p \in Spec(R)$ which is neither 0 nor m, then R_p is never Henselian.

Proof: Choose $z \in m - p$ and $a \in p$, $a \neq 0$ such that $a \notin m^n$ for some $n \in \mathbb{N}$ with $n \cdot 1 \notin p$. Suppose by way of contradiction that R_p is Henselian. Consider the monic polynomial $f(T) = T^n - (a + z^n)$ so going mod pR_p we obtain $T^n - \overline{z}^n$ which has a root $\overline{z}$. Also, $f'(\overline{z}) = n\overline{z}^{n-1} \neq 0$ so there is a root u of $f(T)$ in R_p. Without loss of generality take $u \in R$ so that $u^n - (a + z^n) = 0 \implies a = u^n - z^n \in m^n$ which is a contradiction.

5.3 Henselizations

Definition 5.3.1. Let (R, m, k) be a local ring. A **Henselization** (R^h, m, k) of (R, m, k) is a Henselian ring (containing R by way of a local ring map $\iota : R \to R^h$) that satisfies the following universal property. Given a Henselian ring S and a local map $\phi : R \to S$, there exists a unique local map $\Phi : R^h \to S$.

Remark: Since R^h is Henselian, it must contain all roots of monic polynomials over R with simple roots mod m, i.e. elementary étale neighborhoods must map to R^h because for example:

$$\frac{R[T]}{(f(T))} \to S \text{ sending } T \mapsto \text{root of } f(T).$$

Conversely, if R^h is henselian then we proved every een S of R^h is trivial i.e. the map $R^h \to S$ is the identity. Our idea is to try to use the direct limit of the elementary étale neighborhoods.

Proposition 5.3.2. *Suppose that S, T are een of (R, m, k)*

1. *If $I \subset R$, then $R/I \to S/IS$ is an elementary étale neighborhood.*
2. *Let $Q = ker(S \otimes_R T \to k \otimes_R k)$, then $(S \otimes_R T)_Q$ is an een of R.*
3. *There exist at most one R-homomorphism from $S \to T$. If such a homomorphism exists, then T is an een of S.*
4. *If there is an R-homomorphism $S \to T$ and another R-homomorphism $T \to S$, then $S \simeq T$.*

Proof: 1. follows from base change for étale maps and the fact that their residue fields are the same.

2. By the Structure Theorem for een, $S \simeq (\frac{R[X]}{(f(X))})_{Q_1}$ where f is a monic polynomial, $Q_1 = (X-a, mR[X])$, and $f'(a) \notin m$. Also, $T \simeq (\frac{R[Y]}{(g(Y))})_{Q_2}$, g is monic, $Q_2 = (Y-b, mR[Y])$, and $g'(Y) \notin m$. Then $\frac{R[X]}{(f(X))} \otimes \frac{R[Y]}{(g(Y))} \simeq \frac{R[X,Y]}{(f,g)}$ and the Jacobian of (f,g) is just $\begin{pmatrix} f'(X) & 0 \\ 0 & g'(Y) \end{pmatrix}$ with determinant not in $mR[X,Y]$. So $R \to (S \otimes T)_Q$ is étale and by construction $\frac{(S\otimes T)_Q}{Q(S\otimes T)_Q} \simeq k$.

3. Suppose we have two R-homomorphisms $\alpha, \beta : S \to T$. This gives a map $\phi : S \otimes_R S \to T$ defined by $\phi = [\alpha \otimes \beta]$. Let $Q = \phi^{-1}(m_T)$. It is enough to show that $(S \otimes_R S)_Q \simeq S$ by multiplication.

We use the exact sequence $0 \to \Theta \to S \otimes S \to S \to 0$ where by assumption $0 = \Omega_{S/R} \simeq \Theta/\Theta^2$. Note Θ is finitely generated and $\Theta = \Theta^2$ so Θ is generated by an idempotent. Therefore $(\Theta)_Q = 0$ ($\neq 1$) $\implies (S \otimes S)_Q \simeq S$. Then $0 = 1 \otimes S - S \otimes 1$ is mapped to $\alpha(s) - \beta(s)$. Use localization.

For the second part of 3. we use the Jacobi-Zariski sequence; $R \to S \to T$ induces:

$$0 = \Gamma_{T/R} \to \Gamma_{T/S} \to T \otimes \Omega_{S/R} \to \Omega_{T/R} \to \Omega_{T/S} \to 0.$$

(Recall that $Gamma_{T/S} = I/I^2$). Because the first, third, and fourth terms of the above exact sequence are 0 (by étaleness) we have that $\Gamma_{T/S} = 0$ and $\Omega_{T/S} = 0 \implies T$ is étale. Finally, because $T/m_T = S/m_S$ we have that T is an een of S.

Lastly, 4. follows from 3.

To form the diect limit of elementary étale neighborhoods of R, we first need to define a partial ordering on the een. We define $S \leq T$ if there is some R-homomorphism $\phi_{ST} : S \to T$. Under this partial ordering we get a directed system. For instance, by 3. of Proposition 5.3.2 if we have two een S, T we get an upper bound $R \rightrightarrows S, T \rightrightarrows (S \otimes T)_Q$. By taking isomorpism classes of een of R, we obtain a set of een.

Define $B = \varinjlim S$ over all S an een of R (see Milne's notes 2.8.8).

Proposition 5.3.3. *Consider a family of local rings $(A_{\lambda, m_\lambda, k_\lambda})_\lambda$ with local morphisms $\phi_{\lambda\mu}$ which makes it into a directed system of local rings. Let $A = \varinjlim A_\lambda$. Then:*

1. *The ring A is local with maximal ideal $\varinjlim m_\lambda$ and residue field $\varinjlim k_\lambda$.*
2. *If $m_\mu = m_\lambda$ for all $\mu \geq \lambda$, then we have $m = m_\lambda A$ for all λ.*
3. *If A_μ is flat over A_λ for all $\mu \geq \lambda$, then A is flat over λ for all λ.*
4. *If 2. and 3. hold, and A_λ are Noetherian, so is A.*

The proofs of 1. - 3. are relatively easy and are left as exercises. We give the following hint for 4. If you have an ascending chain of ideals in A, they contract to stabilizing chains in each A_λ. By faithful flatness, if they restrict to the same, they are the same.

Theorem 5.3.4. *The following hold:*

1. *The ring B is Henselian.*
2. *The ring B is a Henselization of R.*
3. *The map $R \to B$ is flat and unramified.*
4. *If $R \to T$ is an een, then $R^h = T^h = B$.*
5. *The compeletions are equal: $\hat{R} \simeq \hat{R^h}$.*
6. *If R is Noetherian, then R^h is Noetherian (by prop 5.3.3 part 4.).*

One way to think of Henselizations are that "the Henselization is the algebraic part of the completion".

These will be proven shortly, but first note some implications of Prop 5.3.3. Part 1. $\implies$ B is a local ring with residue field k. Part 2. $\implies m_B = mB$. Part 3. $\implies$ B is flat over R (and all of S as well). Part 4. implies that B is Noetherian if R is Noetherian.

The next theorem is an extension of the results of this talk. For certain rings, we can completely characterize the Henselization.

Theorem 5.3.5. *If (R, m, k) is an excellent, normal, Noetherian, local ring, then $R^h = \overline{R}_{\hat{m}\cap\overline{R}}$ (meaning R^h is an approximation ring) where $\overline{R}$ is the integral closure of R in $\hat{R}$. For $p \in Spec(R)$ define $k(p) = R_p/pR_p$, then $k(p) \otimes_R \hat{R}$ is regular.*

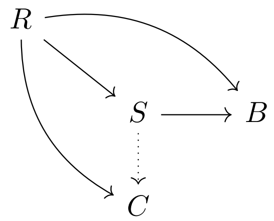
Note: if R is not an excellent ring, then R^h doesn't need to be an approximation ring.

Proof: Let $F(X)$ be a monic polynomial in $B[X]$ and assume there is some $b \in B$ with $F(b) \in m_B$, $F'(b) \notin m_B$, and there exists some elementary étale neighborhood S of R such that $F(X) \in S[X]$ and $b \in S$. Then $S \to (\frac{S[X]}{(F(X))})_{(x-b,\, mS[X])} \to B$ is an een of S where the second map exists from the fact S is a direct limit. Then the image of X gives a root $b' \in B$ of F such that $b' - b \in mB = m_B$.

We now return to prove Theorem 5.3.4.

Proof 1 follows from the previous discussions.

Proof 2: Consider the commutative diagram between two Henselizations S, B of R and another Henselian ring C:



Then C has no een and by the Structure Theorem $S \simeq (\frac{R[X]}{(F[X])})_{(x-b,mR[X])}$ and $F'(b) \notin m$ so we have $S_C = (\frac{C[X]}{(F(X))})_{(x-b,mR[X])} = C$

Proof 3: Recall that B is flat by 3. of Prop 5.3.3 and the result follows because $\Omega_{\varinjlim S/R} = \varinjlim \Omega_{S/R} = 0$.

Proof 4: This is done.

Proof 5: By base change, $(R/m^n)^h = R^h/(mR^h)^h$ because R/m^n is complete and therefore it is Henselian. Because $(R/m^n)^h = R/m^n \implies \hat{R} = \varinjlim R/m^n = \varinjlim R^h/(mR^h)^h = \hat{R^h}$.

Proof 6: This is done by part 4. of Prop 5.3.3.

Some further topics of study include the Artin Approximation Theorem. In the literature, Henselizations allow us to construct counterexamples to certain statements by starting with a counterexample in some ring and pushing it down a chain to counterexamples in different rings. For example, if we start in a ring R we might proceed as follows:

$$R \to \hat{R} \to R^h \to S \to S_A \to S_k$$

where S is a finite extension of R^h over $\mathbb{C}$, A is a finitely generated $\mathbb{Z}$, and k has charactersitic p.

Exercise : Let R be a local ring and $R \to S$ be module-finite with $max(S) = \{n_1, \dots, n_k\}$, then $S \otimes_R R^h \simeq (S_{n_1})^h \times \dots \times (S_{n_k})^h$.

We conclude this section with one final theorem (without proof).

Theorem 5.3.6. *Let R be an excellent, Henselian, Noetherian, local ring. If $f_i(x_1, \dots, x_n) = 0$ is a finite system of polynomial equations over R with a solution if $\hat{R}$, they have a solution in R. (Also, we obtain solutions that approximate power series solutions).*

Chapter 6

Étale fundamental group

Kian Cheong Aik

Most of the material here is based on Szamuely's Galois Groups and Fundamental Groups Chapter 5. Missing proofs and details can be found there.

6.1 Motivation

We begin with a classical theorem, that is, Grothendieck's categorical reformulation of Galois Theory.

Theorem 6.1.1. *Let k be a field and $G = \mathrm{Gal}(k_s/k)$ be the absolute Galois Group. Then we have the following contravariant equivalence of categories:*

$$\begin{Bmatrix} \textit{finite separable} \\ \textit{extensions of } k \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \textit{finite sets with} \\ \textit{transitive action of } G \end{Bmatrix}$$

$$L/k \longmapsto \mathrm{Hom}_k(L/k_s)$$

There is a very similar theorem in algebraic topology on covering spaces:

Theorem 6.1.2. *Let X be a connected and locally simply connected topological space, $x \in X$ a base point. Then we have the following equivalence of categories:*

$$\begin{Bmatrix} \textit{finite connected covering} \\ \textit{spaces of } X \end{Bmatrix} \longleftrightarrow \begin{Bmatrix} \textit{left } \pi_1(X,x) \textit{ sets} \\ \textit{with transitive action} \end{Bmatrix}$$

We state some immediate observations:

1. In the first theorem we have a contravariant equivalence, while in the second we have a covariant equivalence. This is resolved by taking Spec of the fields.

2. The choice of basepoint in algebraic topology really corresponds to the choice of k_s in Galois theory. In our case, this is really just choosing a geometric point of $\operatorname{Spec} k$.

6.2 The Fundamental Group

We begin with our setup and some basic notions. Let S be a scheme, and Fet_S denote the category of finite étale covers of S (and morphisms are morphisms of schemes over S). We define a geometric point of a scheme S as a morphism $\bar{s} : \operatorname{Spec}\Omega \to S$ where Ω is algebraically closed. Then by definition, the image of $\bar{s}$ is a point s such that Ω is an algebraically closed extension of $k(s)$.

Example 6.2.1. $\operatorname{Spec}\Omega \to \operatorname{Spec} k \iff \Omega/k$ is an algebraically closed extension.

Given a morphism $\phi : X \to S$ and a geometric point $\overline{s} : \operatorname{Spec}\Omega \to S$, define the geometric fibre $X_{\overline{s}}$ of ϕ over $\overline{s}$ as the fibre product $X \times_S \operatorname{Spec}\Omega$. Let $\operatorname{Fib}_{\overline{s}}(X)$ denote the underlying set of $X \times_S \operatorname{Spec}\Omega$, and given a morphism of S schemes $X \to Y$, we get an induced morphism $X_\sigma \to Y_\sigma$. Then we can consider the functor $\operatorname{Fib}_\sigma : \operatorname{Fet}_S \to \operatorname{Set}$, and we call this the fibre functor at the geometric point σ.

Definition 6.2.2. Given a scheme S and a geometric point σ, we define the algebraic (or étale) fundamental group $\pi_1(S, \sigma)$ as the automorphism group of the fibre functor $\operatorname{Fib}_\sigma$ on Fet_S.

We can now state the fundamental theorem of étale funcamental groups, which we will prove in the last section.

Theorem 6.2.3 (Grothendieck)**.** *Let S be a connected scheme, $\sigma : \operatorname{Spec}\Omega \to S$ a geometric point.*

1. $\operatorname{Fib}_\sigma$ *is pro-representable.*

2. *The group $\pi_1(S, \sigma)$ is profinite, and its action on $\operatorname{Fib}_\sigma(X)$ is continuous for every X ins Fet_S.*

3. *The functor $\operatorname{Fib}_\sigma$ induces an equivalence of Fet_S with the category of finite continuous left $\pi_1(S, \sigma)$-sets. In other words, we have the following equivalence of categories:*

$$\{\operatorname{Fet}_\sigma\} \longleftrightarrow \begin{Bmatrix} \textit{finite continuous left} \\ \pi(S,\sigma)\textit{-sets} \end{Bmatrix}$$

   Furthermore, connected covers correspond to transitive action, and Galois covers correspond to finite quotients of $\pi_1(S, \sigma)$.

We end this section with an example showing that classical Galois Theory is a special case of our setup.

Example 6.2.4. Let $S = \operatorname{Spec} k$. Then a finite étale S-scheme X is the spectrum of finite product of finite separable extensions of K. To see this, observe that X is finite implies $X = \operatorname{Spec} L$ where L is a finite k-algebra, which means L is Artinian and so it decomposes into local rings. Unramified then means that all these local rings are fields. If X is connected, then X is simply $\operatorname{Spec} L$ where L/k is finite separable. For a geometric point σ, the fibre functor maps X to $\operatorname{Spec}(K \otimes_k \Omega)$, which is a finite set indexed by $\operatorname{Hom}_k(L, \Omega)$, so in this case $\operatorname{Fib}_\sigma(X) \simeq \operatorname{Hom}_k(L, k_s)$ and we get Galois Theory.

6.3 Galois Theory for finite étale covers

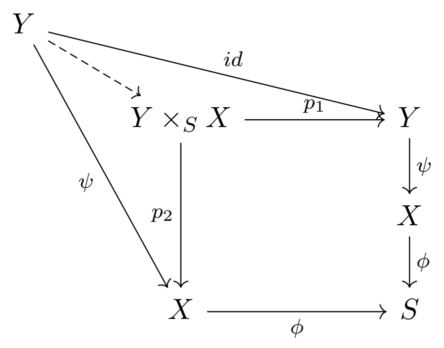
This section will be a collection of results that we will need to prove the fundamental theorem.

Lemma 6.3.1. *Let $\phi : X \to S$ and $\psi : Y \to X$ be morphisms of schemes. Then*

1. *If $\phi \circ \psi$ is finite and ϕ is separated, then ψ is finite.*

2. *If moreover $\phi \circ \psi$ and ϕ are finite étale, then so is ψ.*

Proof. We will only prove the second statement, the first is similar. We have



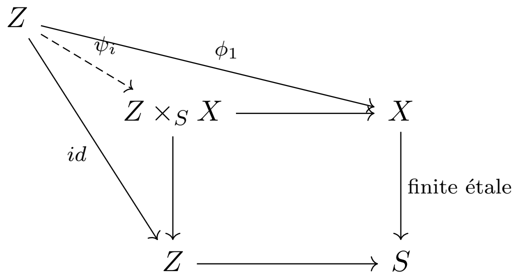
i.e. a map $\Gamma_\psi : Y \to Y \times_S X$. By base change property, p_2 is finite étale, and since Γ_ψ is the base change of the diagonal morphism, we have that $\psi = p_2 \circ \Gamma_\psi$ is finite étale. ☐

Proposition 6.3.2. *Let $\phi : X \to S$ be a finite étale cover $s : S \to X$ be a section of ϕ (i.e. $\phi \circ s = id_S$). Then s induces an isomorphism of S with an open and closed subscheme of X. In particular, if S is connected, then s maps S isomorphically onto a whole connected component of X.*

Proof. By lemma 6.3.1, S is finite étale, and hence its image is opened and closed in X. □

Corollary 6.3.3. *If $Z \to S$ is a connected S-scheme and $\phi_1, \phi_2 : Z \to X$ are two S-morphisms to a fintie étale S-scheme X with $\phi_1 \circ \overline{z} = \phi_2 \circ \overline{z}$ for some geometric point $\overline{z} : \mathrm{Spec}(\Omega) \to Z$, then $\phi_1 = \phi_2$.*

Proof.



If $\phi_1 \circ \overline{z} = \phi_2 \circ \overline{z}$, then $\psi_1 \circ \overline{z} = \psi_2 \circ \overline{z}$. By the base change property, we have $Z \times_S X \to Z$ is finite étale. If $\psi_1 = \psi_2$, then $\phi_1 = \phi_2$, so we may take $Z \to Z \times_S X \to Z$ as our new candidates. By proposition 6.3.2 $Z \to Z \times_S X$ is an isomorphism onto an open and closed subscheme, and since Z is connected, we have that $Z \simeq C$ for some connected component of $Z \times_S X$. In particular, ψ_1 and ψ_2 map to the same connected component since they agree on $\overline{z}$. On the other hand, there can only be one morphism $Z \to C$ over Z. □

Given a morphism of schemes $\phi : X \to S$, define $\mathrm{Aut}(X \mid S)$ to be the group of scheme autmorphisms of X preserving ϕ. For a geometric point $\sigma : \mathrm{Spec}(\Omega) \to S$ there is a natural left action of $\mathrm{Aut}(X \mid S)$ on X_σ by base change from its action on X.

Corollary 6.3.4. *If $\phi : X \to S$ is a connected finite étale cover, the nontrivial elements of $\mathrm{Aut}(X \mid S)$ act without fixed points on each geometric fibre. Hence $\mathrm{Aut}(X \mid S)$ is finite.*

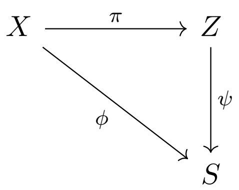
Proof. Apply previous corollary to $\phi_1 = \phi$, $\phi_2 = \phi \circ \lambda$ and we get the first statement. Then the permutation representation of $\mathrm{Aut}(X \mid S)$ on the underlying sets of geometric fibres is faithful. But the underlying sets are finite. □

Definition 6.3.5. We call a connected finite étale cover $X \to S$ Galois if its $S-$automorphism group acts transitively on geometric fibres.

Remark 6.3.6. If $S = \mathrm{Spec}(k)$, then $X = \mathrm{Spec}(L)$ where L/k finite separable. Then $X \to S$ Galois implies that $\mathrm{Aut}(L \mid k)$ acts transitively on $\mathrm{Hom}(L, k_s) \to \mathrm{Hom}(L, k_s)$, which means that $|\mathrm{Aut}(L \mid k)| \geq |\mathrm{Hom}(L, k_s)| = [L : k]$, which means that L/k is Galois.

The following is the analogue of the usual Galois Theory:

Proposition 6.3.7. *Let $\phi : X \to S$ be a finite étale Galois cover. If $Z \to S$ is a connected finite étale cover fitting into a commutative diagram*



Then $\pi : X \to Z$ is a finite étale Galois cover, and actually $Z \simeq H/X$ for some subgroup H of $G = \mathrm{Aut}(X \mid S)$. In this way, we get a bijection between subgroups of G and intermediate covers of Z as above. The cover $\psi : Z \to S$ is Galois if and only if H is a normal subgroup of G, in which case $\mathrm{Aut}(Z \mid S) \simeq G/H$.

Proof. Skipped, see Szamuely Chapter 5. □

The last proposition in this section is a generalization of a basic statement in field theory; that every finite separable field extension can be embedded in a finite Galois extension and there is a smalles such extension, the Galois closure.

Proposition 6.3.8 (Galois Closure)**.** *Let $\phi : X \to S$ be a connected finite étale cover. There is a morphism $\pi : P \to X$ such that $\phi \circ \pi : P \to S$ is a finite étale Galois cover, and moreoever every S-morphism from a Galois cover to X factors through P.*

Proof. We will break this into 4 steps.

Step 1: Construct P.

Fix a geometric point $\sigma : \mathrm{Spec}(\Omega) \to S$. Let $F = \{\overline{x_1}, \ldots, \overline{x_n}\}$ be the fintie set of the geometric gibre X_σ. Consider the n-fold fibre product $X^n := X \times_S \cdots \times_S X$ and let $\overline{x} := (\overline{x_1}, \overline{x_2}, \ldots \overline{x_n}) \in X^n$. Let P be the connected component of X^n containing $\overline{x}$, and let $\pi : P \to X$ be the map induced by first projection. Then P is a finite étale cover of S.

Step 2: Show P has distinct coordinates.

Consider the projection $\pi_{ij} : X^n \to X \times_S X$ to the i, jth component for some $1 \leq i, j, \leq n$. Then $\Delta(X)$ is open and closed and π_{ij} is continuous means that $\pi_{ij}^{-1}(\Delta(X))$ is open and closed. Since $P \subset X^n$ is connected, $\pi_{ij}^{-1}(\Delta(X)) \cap P$ is either empty or all of P. But $\overline{x} \in P$ means the intersection is empty, and so P has distinct coordinates.

Step 3: P is Galois.

Let $\overline{x_\sigma} := (\overline{x_{\sigma(1)}}, \ldots, \overline{x_{\sigma(n)}}) \in P$. By step 2, $\sigma \in S_n$, and σ induces an S-automorphism ϕ_σ of X^n by permuting its components. Then $\phi_\sigma \circ \overline{x} = \overline{x_\sigma} \implies \phi_\sigma(P) \cap P$ is not empty. But P and ϕ_σ are connected, so $\phi_\sigma(P) = P \implies \phi_\sigma \in \mathrm{Aut}(P \mid S)$, which proves the induced action is transitive.

Step 4: If $q : Q \to X$ is a Galois S-morphism, then it factors through P.

Choose $\overline{y}$ such that $q(\overline{y}) = \overline{x_1}$. Since q is surjective (as X connected), by composing with appropriate elements of $\mathrm{Aut}(Q \mid S)$ such that $q_i : Q \to X$, $\overline{y} \mapsto \overline{x_i}$. Then we get a $S-$morphism $\gamma := \prod q_i : Q \to X^n$, $\overline{y} \mapsto (\overline{x_1}, \ldots, \overline{x_n}) = \overline{x}$. Connectedness of Q implies that $\gamma : Q \to P$, and $q = \pi \circ \gamma$ because they both map $\overline{y}$ to $\overline{x_1}$. □

6.4 Grothendieck's Fundamental Theorem

Now we return to the Fundamental Theorem, and begin by proving the that the fibre functor is pro-representable.

Proposition 6.4.1. Fib_σ *is pro-representable.*

Proof. We first construct the inverse system. Let Λ denote the indexing set, given by the finite étale Galois covers $P_\alpha \to S$, and we say $P_\beta \geq P_\alpha$ if there exists $\phi : P_\beta \to P_\alpha$. This is directed due to 6.3.8. The objects of our inverse system will be given by $\mathrm{Hom}(P_\alpha, X)$, and morphisms are those given by precomposition with a morphism $P_\alpha \to P_\beta$. By construction, there exists at least one such morphism if $P_\beta \geq P_\alpha$, but this is not unique in general. To rigidy the situation, we fix an arbitrary element $\overline{p_\alpha} \in \mathrm{Fib}_\sigma(P_\alpha)$ for each α. By definition of Galois covers, there is a transitive action on the $\mathrm{Fib}_\sigma(P_\alpha)$, and so we can find a $\phi_{\alpha\beta} : P_\beta \to P_\alpha$ such that $\mathrm{Fib}_\sigma(\phi_{\alpha\beta})(\overline{p_\beta}) = \overline{p_\alpha}$.

Now, for every $X \in \mathrm{Fet}_S$ and $P_\alpha \in \Lambda$, we define a map $\mathrm{Hom}(P_\alpha, X) \to \mathrm{Fib}_\sigma(X)$, $\phi \mapsto \mathrm{Fib}_\sigma(\phi)(\overline{p_\alpha})$.

This map is compatible with the $\phi_\alpha\beta$ by construction, and so we get a functorial map $\varinjlim \operatorname{Hom}(P_\alpha, X) \to \operatorname{Fib}_\sigma(X)$. Finally, to show this is an isomorphism, it suffices to construct an inverse. We may assume X is connected (otherwise just take disjoint unions). By 6.3.8, there exists $\pi : P \to X$ the Galois closure, and so by definition $P = P_\alpha \in \Lambda$. By the transitive property, for each $\overline{x} \in \operatorname{Fib}_\sigma(X)$, we can find a unique S-automorphism of P_α and compose it with π such that it maps $\overline{p_\alpha} \mapsto \overline{x}$ (where $\overline{p_\alpha}$ is our distinguished element we fixed earlier). Hence, we may assume that our original π maps $\overline{p_\alpha}$ to $\overline{x}$, and then our inverse sends $\overline{x} \in \operatorname{Fib}_\sigma(X)$ to $[\pi] \in \varinjlim \operatorname{Hom}(P_\alpha, X)$. □

An inspection of the proof shows that the maps $\phi_{\alpha\beta}$ in the system pro-representing the functor $\operatorname{Fib}_\sigma$ depend on the choice of the system of geometric points $\{\overline{p_\alpha}\}$. Once such a system is fixed, the pro-representing system becomes unique.

Corollary 6.4.2. *The automorphism groups* $\operatorname{Aut}(P_\alpha)^{\mathrm{op}}$ *form an inverse system whose limit is* $\pi_1(S, \sigma)$.

Proof. If $P_\beta \geq P_\alpha$, then there exists $\phi_{\alpha\beta} : P_\beta \to P_\alpha$. Since the covers are Galois, there is a natural surjective group homomorphism $\operatorname{Aut}(P_\beta \mid S) \twoheadrightarrow \operatorname{Aut}(P_\alpha \mid S)$, i.e. given $\lambda_\beta \in \operatorname{Aut}(P_\beta \mid S)$, there exists a unique $\lambda_\alpha \in \operatorname{Aut}(P_\alpha \mid S)$ that maps $\overline{p_\alpha}$ to $\overline{p_\alpha}'$ fitting into the following diagram:

$$\begin{array}{ccccc}
\operatorname{Fib}_\sigma(P_\beta) & & \xrightarrow{\operatorname{Fib}_\sigma(\lambda_\beta)} & & \operatorname{Fib}_\sigma(P_\beta) \\
\Big\downarrow {\scriptstyle \operatorname{Fib}_\sigma(\phi_{\alpha\beta})} & \overline{p_\beta} & \longmapsto & \overline{p_\beta}' & \Big\downarrow {\scriptstyle \operatorname{Fib}_\sigma(\phi_{\alpha\beta})} \\
 & \big\downarrow & & \big\downarrow & \\
 & \overline{p_\alpha} & & \overline{p_\alpha}' & \\
\operatorname{Fib}_\sigma(P_\alpha) & & & & \operatorname{Fib}_\sigma(P_\alpha)
\end{array}$$

Then we have that $\lambda_\alpha \circ \phi_{\alpha\beta} = \phi_{\alpha\beta} \circ \lambda_\beta$, i.e. the diagram commutes, because the maps agree on a single geometric point (see 6.3.3).

Now, the elements of the inverse limit $\varprojlim \operatorname{Aut}(P_\alpha)$ are given by $\{(\lambda_\alpha) \in \prod \operatorname{Aut}(P_\alpha \mid S) \mid \lambda_\alpha \circ \phi_{\alpha\beta} = \phi_{\alpha\beta} \circ \lambda_\beta \text{ for } \beta \geq \alpha\}$. These are precisely the automorphisms of the system $(P_\alpha, \phi_{\alpha\beta})$. Now we construct the correspondence between $\pi_1(S, \sigma)$ and the automorphisms of $(P_\alpha, \phi_{\alpha\beta})$. Given $\phi \in \pi_1$, ϕ maps the set of distinguished elements $\{\overline{p_\alpha}\}$ to another set $\{\overline{p_\alpha}'\}$. Since P_α are Galois, for each α there is a unique $\lambda_\alpha \in \operatorname{Aut}(P_\alpha \mid S)$ such that $\operatorname{Fib}_\sigma(\lambda_\alpha)(\overline{p_\alpha}) = \overline{p_\alpha}'$. These are compatible with $\phi_{\alpha\beta}$ since ϕ is an automorphism of the fibre functor. On the other hand, given (λ_α) compatible automorphisms, we can define an automorphism of $\operatorname{Fib}_\sigma$ via its pro-representability. For each $X \in \operatorname{Fet}_S$, we need an autommorphism on $\operatorname{Fib}_\sigma(X)$. Now, each $\overline{x} \in \operatorname{Fib}_\sigma(X)$ corresponds to a class $[\pi] \in \varinjlim \operatorname{Hom}(P_\alpha, X)$. Then we send $[\pi] \mapsto [\pi \circ \lambda_\alpha]$. One checks that this is well-defined, natural, and an isomorphism. Hence, we get the following correspondences:

$$\begin{aligned}
\varprojlim \operatorname{Aut}(P_\alpha \mid S) &\iff (\lambda_\alpha) \text{ sequence of compatible automorphisms} \\
&\iff \text{automorphisms of } \operatorname{Fib}_\sigma, \text{ i.e. } \pi_1(S, \overline{s})
\end{aligned}$$

Then we have $\pi_1(S, \overline{s}) \simeq \varprojlim \operatorname{Aut}(P_\alpha \mid S)^{\mathrm{op}}$, where the opposite arises due to the contravariance of the Hom functor. □

Now we show the continuity of the π_1 action on $\operatorname{Fib}_\sigma(X)$: Each $\overline{x} \in \operatorname{Fib}_\sigma(X)$ comes from a class in $\operatorname{Hom}(P_\alpha, X)$, then the action of $\pi_1(S, \sigma) = \varprojlim \operatorname{Aut}(P_\alpha \mid S)^{\mathrm{op}}$ factors through $\operatorname{Aut}(P_\alpha \mid S)^{\mathrm{op}}$, i.e.

$$\begin{array}{ccc}
\varprojlim \operatorname{Aut}(P_\alpha \mid S)^{\mathrm{op}} \times \varinjlim \operatorname{Hom}(P_\alpha, X) & \xrightarrow{pr_\alpha \times id} & \operatorname{Aut}(P_\alpha \mid S)^{\mathrm{op}} \times \varinjlim \operatorname{Hom}(P_\alpha, X) \\
 & \searrow^{[\pi] \mapsto [\pi \circ \lambda_\alpha]} & \Big\downarrow \phi \\
 & & \varinjlim \operatorname{Hom}(P_\alpha, X)
\end{array}$$

Here $pr_\alpha \times id$ is continuous by definition of the profinite topology, and ϕ is continuous since everything has the discrete topology, and so their composition is continuous. Finally, we prove the last statement of the fundamental theorem. We shall focus on the case where we have connected covers, which correspond to transitive actions, the general case is similar.

Theorem 6.4.3. *The functor* Fib_σ *induces an equivalence of* Fet_S *with the category of finite continuous left* $\pi_1(S, \sigma)$*-sets.*

Proof. Faithful: Let $\psi_1, \psi_2 : X \to Y \in \mathrm{Hom}(X, Y)$ in Fet_S such that $\mathrm{Fib}_\sigma(\psi_1) = \mathrm{Fib}_\sigma(\psi_2)$. Since X is connected, morphisms are determined by a geometric point (by 6.3.3), so $\psi_1 = \psi_2$.

Full: Let $\phi : \mathrm{Fib}_\sigma(X) \to \mathrm{Fib}_\sigma(Y)$ be π_1-equivariant map. Then we want $\psi \in \mathrm{Hom}(X, Y)$ such that $\mathrm{Fib}_\sigma(\psi) = \phi$. Fix $\overline{x} \in \mathrm{Fib}_\sigma(X)$, since π_1 is transitive and ϕ is π_1-map, $\phi(\overline{x}) = \overline{y}$ determines ϕ. Find a Galois cover $Q \to S$, and $\overline{q} \in \mathrm{Fib}_\sigma(Q)$ such that

$$\pi_X : Q \to X, \;\; \mathrm{Fib}_\sigma(\pi_X)(\overline{q}) = \overline{x}$$

$$\pi_Y : Q \to Y, \;\; \mathrm{Fib}_\sigma(\pi_Y)(\overline{q}) = \overline{y}$$

By 6.3.7, $\mathrm{Aut}(Q \mid X) \setminus Q \simeq X$ (this is the quotient notation used in the text). If π_Y is constant on the orbits of $\mathrm{Aut}(Q \mid X)$, then π_Y factors through π_X. Let $h \in \mathrm{Aut}(Q \mid X)$, then $\mathrm{Fib}_\sigma(h)(\overline{q}) \in \mathrm{Fib}_\sigma(Q)$, by transitivity of π_1 action, there exists $f \in \pi_1(S, \overline{s})$ such that $f_Q(\overline{q}) = \mathrm{Fib}_\sigma(h)(\overline{q})$ (here f_Q denotes the induced map of f on $\mathrm{Fib}_\sigma(Q)$). Using the definitions, naturality, π_1-equivariance and other properties, we obtain the following equalities:

$$\begin{aligned}
\mathrm{Fib}_\sigma(\pi_Y \circ h)(\overline{q}) &= \mathrm{Fib}_\sigma(\pi_Y)(\mathrm{Fib}_\sigma(h)(\overline{q}) \\
&= \mathrm{Fib}_\sigma(\pi_Y)(f_Q(\overline{q})) \\
&= f_Y(\mathrm{Fib}_\sigma(\pi_Y)(\overline{q})) \\
&= f_Y(\overline{y}) \\
&= f_Y(\phi(\overline{x})) \\
&= \phi(f_X(\overline{x})) \\
&= \phi(f_X(\mathrm{Fib}_\sigma(\pi_X)(\overline{q}))) \\
&= \phi(\mathrm{Fib}_\sigma(\pi_X)(f_Q(\overline{q}))) \\
&= \phi(\mathrm{Fib}_\sigma(\pi_X)\mathrm{Fib}_\sigma(h)(\overline{q})) \\
&= \phi(\mathrm{Fib}_\sigma(\pi_X \circ h)(\overline{q})) \\
&= \phi(\mathrm{Fib}_\sigma(\pi_X)(\overline{q})) \\
&= \phi(\overline{x}) \\
&= \overline{y} = \mathrm{Fib}_\sigma(\pi_Y)(\overline{q})
\end{aligned}$$

By 6.3.3, $\pi_Y \circ h = \pi_Y$, i.e. π_Y is constant on $\mathrm{Aut}(Q \mid X)$ orbits as desired. Hence we have an S-morphism $\psi : X \to Y$ such that $\pi_Y = \psi \circ \pi_X$. By construction, $\mathrm{Fib}_\sigma(\psi)(\overline{x}) = \overline{y}$, so $\mathrm{Fib}_\sigma(\psi) = \phi$.

Essentially surjective: Let E be a transitive π_1-set. Fix $x \in E$, let U_x be its stabilizer under π_1. Then $\pi_1(S, \sigma)/U_x \simeq E$ (where $\overline{f} \mapsto f(x)$) as π_1-sets. Now U_x is open in π_1 by continuity, so $\exists \alpha$ such that $V_\alpha \subset U_x$, where $V_\alpha = \ker p_\alpha$ the canonical projection $p_\alpha : \pi_1 \to \mathrm{Aut}(P_\alpha \mid S)^{\mathrm{op}}$. Let $\overline{U} := p_\alpha(U_x)$. Consider

$$\phi : \pi_1(S, \sigma) \to \mathrm{Fib}_\sigma(P_\alpha), \;\; f \mapsto f_\alpha(\overline{p_\alpha})$$

This is surjective since π_1 acts transitively on connected covers. We then have

$$\pi_1(S, \sigma)/U_x \simeq \overline{U}^{\mathrm{op}} \setminus \mathrm{Fib}_\sigma(P_\alpha) \simeq \mathrm{Fib}_\sigma(\overline{U}^{\mathrm{op}} \setminus P_\alpha)$$

But this means $E \simeq \mathrm{Fib}_\sigma(\overline{U}^{\mathrm{op}} \setminus P_\alpha)$ as desired. □

Chapter 7

Galois categories

Miika Tuominen

7.1 Introduction

The notion of a Galois category, originating in [6][1], unifies some categorical properties of finite étale covers of a connected scheme in algebraic geometry and finite-sheeted covering spaces in algebraic topology. We define Galois categories and study a sequence of properties they possess, as well as their connected objects and Galois objects. We describe an arbitrary Galois category by means of the fundamental theorem of Galois category theory, which we prove, and we conclude our study of Galois categories with a detailed treatment of the example of finite-sheeted covering spaces. We prove many results in detail to demonstrate some techniques in the study of Galois categories, and we also accompany our study with an appendix on category theory to extend these notes' accessibility to as broad a spectrum of audience fluency in category theory as possible, as well as a table of standard categories to prevent notational confusion.

7.1.1 Acknowledgments

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Chapters 7.8 and 7.9 are independent addenda that were added by Eleftherios Chatzitheodoridis.

7.2 Definition, examples, and first properties of Galois categories

We begin with the definition of a Galois category and its associated fiber functor, followed by some examples:

Definition 7.2.1 (Galois category). An essentially small category[2] $\mathcal{C}$ is **Galois** with associated **fiber functor** $F : \mathcal{C} \to \mathrm{Set_f}$, where $\mathrm{Set_f}$ denotes the category of finite sets, if $\mathcal{C}$ satisfies the axioms below:

[1] [6] also goes by the name 'SGA I', for it is the first volume of notes from the Séminaire de Géométrie Algébrique (SGA) du Bois Marie, which took place at the Institut des Hautes Études Scientifiques (IHÉS) in Le Bois Marie, Bures-sur-Yvette, France from 1960 to 1969 and was run by Alexander Grothendieck. SGA I consists of notes taken in 1960-1961.

[2] A category is **essentially small** if the isomorphism classes of its objects form a set, rather than a proper class. Thus, every **small** category - one whose objects form a set, rather than a proper class - is essentially small, but the category $\mathrm{Set_f}$ of finite sets and the category $\mathbb{C}-\mathrm{Vect_{fd}}$ of finite-dimensional complex vector spaces both are essentially small but not small. All categories in our study are **locally small**: for every pair of objects, the morphisms between them form a set, rather than a proper class.

1. $\mathcal{C}$ has a terminal object $*$ and all fiber products (pullbacks);

2. $\mathcal{C}$ has all finite coproducts, including an initial object $\emptyset$ (empty coproduct), and all quotients by finite groups;

3. every morphism $f : X \to Y$ in $\mathcal{C}$ factors in $\mathcal{C}$ as:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \phi\big\downarrow & & \psi \cong \big\uparrow \\ Z & \xrightarrow{\iota_Z} & Z \coprod Z' \end{array}$$

   where ϕ is a strict epimorphism in $\mathcal{C}$, ι_Z is the coproduct morphism from Z to the coproduct $Z \coprod Z'$ and is required to be a monomorphism in $\mathcal{C}$, and ψ is an isomorphism in $\mathcal{C}$;

and $F : \mathcal{C} \to \mathrm{Set_f}$ satisfies the axioms below:

4. F preserves the terminal object $*$ of $\mathcal{C}$ - that is, $F(*)$ is a singleton set in $\mathrm{Set_f}$ - and F also preserves all fiber products (pullbacks) of $\mathcal{C}$;

5. F preserves all finite coproducts of $\mathcal{C}$ - in particular, F preserves the initial object $\emptyset$ of $\mathcal{C}$, sending it to the empty set in $\mathrm{Set_f}$ - and all quotients by finite groups in $\mathcal{C}$;

6. F preserves all strict epimorphisms in $\mathcal{C}$: if f is a strict epimorphism in $\mathcal{C}$, then $F(f)$ is a surjective set map of finite sets in $\mathrm{Set_f}$;

7. F reflects isomorphisms: f is an isomorphism in $\mathcal{C}$ if and only if $F(f)$ is a bijection of finite sets in $\mathrm{Set_f}$.

Remark 7.2.2. *Fiber functors are also sometimes called 'fundamental functors'.*

Remark 7.2.3. *Equivalently, every morphism $f : X \to Y$ in a Galois category $\mathcal{C}$ factors in $\mathcal{C}$ as:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle\phi}\searrow \quad \nearrow{\scriptstyle i} & \\ & Z & \end{array}$$

where ϕ is a strict epimorphism in $\mathcal{C}$ and $i := \psi\iota_Z$ is the composite of monomorphisms, thus a monomorphism, in $\mathcal{C}$ of the coproduct morphism ι_Z from Z to the coproduct $Z \coprod Z'$, which is required to be a monomorphism in $\mathcal{C}$, post-composed with the isomorphism ψ in $\mathcal{C}$.

Remark 7.2.4. *Since every Galois category has a terminal object and all fiber products, we infer that it has all finite products and all equalizers, thus all finite limits - in particular, all group quotients. Similarly, since every fiber functor F associated with a Galois category $\mathcal{C}$ preserves the terminal object and all fiber products of $\mathcal{C}$, we infer that it preserves all finite products and all equalizers of $\mathcal{C}$, thus all finite limits of $\mathcal{C}$ - in particular, all group quotients in $\mathcal{C}$ - as well as all monomorphisms in $\mathcal{C}$: if f is a monomorphism in $\mathcal{C}$, then $F(f)$ is an injective set map of finite sets in* $\mathrm{Set_f}$*. Lastly, recall that the category* $\mathrm{Set_f}$ *of finite sets has all finite limits and colimits, but neither all small limits nor all small colimits.*

Example 7.2.5. If G is a profinite group[3], then the category $G - \mathrm{Set_f}$ of finite and discrete G-spaces and G-equivariant continuous maps between them is a Galois category whose associated fiber functor is the forgetful functor $\Lambda : G - \mathrm{Set_f} \to \mathrm{Set_f}$. In section 5, the fundamental theorem of Galois category theory will inform us that this is the essentially universal example of a Galois category.

[3]A **profinite group** G is a topological group which is isomorphic to an inverse limit of finite and discrete topological groups. For example, for every prime p, the abelian topological group $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ of the p-adic integers is a profinite group. Less interestingly, every finite and discrete topological group is profinite, with the infinite abelian topological group of the p-adic integers, $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, being a counterexample to the converse implication.

Example 7.2.6. If S is a path-connected, locally path-connected, and semi-locally simply connected pointed space - that is, a pointed space satisfying the hypotheses in the Galois correspondence theorem for covering spaces - with basepoint s_0, then the category $\mathrm{Cov}_{\mathrm{f}}^{S}$ of finite-sheeted covering spaces of S and finite-sheeted covering space maps over S is a Galois category whose associated fiber functor $F : \mathrm{Cov}_{\mathrm{f}}^{S} \to \mathrm{Set}_{\mathrm{f}}$ sends each finite-sheeted covering space of S to its finite fiber at s_0 and each finite-sheeted covering space map over S to its induced set map of finite fibers at s_0. This example from algebraic topology will be our focus as an example of a Galois category in section 6.

Example 7.2.7. If S is a connected scheme, then the category $\mathrm{Fét}_S$ of finite étale covers of S is a Galois category from algebraic geometry. Note that finite étale covers of S are closed under base change.

We first observe that the fiber functor associated with a Galois category reflects more than isomorphisms:

Lemma 7.2.8. *The fiber functor F associated with a Galois category $\mathcal{C}$ preserves and reflects the initial object, terminal object, all strict epimorphisms, and all monomorphisms in $\mathcal{C}$.*

Proof. We prove all 4 claims in the order in which we stated them:

1. We know F preserves the initial object $\emptyset$ of $\mathcal{C}$. Conversely, suppose X is an object of $\mathcal{C}$ such that $F(X)$ is the empty set in $\mathrm{Set}_{\mathrm{f}}$. Then, the initial object morphism $f_X : \emptyset \to X$ of $\mathcal{C}$ is sent by F to the identity map of the empty set in $\mathrm{Set}_{\mathrm{f}}$. Because F reflects isomorphisms, this implies that f_X is an isomorphism in $\mathcal{C}$, so X is an initial object of $\mathcal{C}$, as required.

2. We know F preserves the terminal object $*$ of $\mathcal{C}$. Conversely, suppose X is an object of $\mathcal{C}$ such that $F(X)$ is a singleton set in $\mathrm{Set}_{\mathrm{f}}$. Then, the terminal object morphism $g_X : X \to *$ of $\mathcal{C}$ is sent by F to a bijection of singleton sets in $\mathrm{Set}_{\mathrm{f}}$. Because F reflects isomorphisms, this implies that g_X is an isomorphism in $\mathcal{C}$, so X is a terminal object of $\mathcal{C}$, as required.

3. We know F preserves all strict epimorphisms in $\mathcal{C}$. Conversely, suppose $f : X \to Y$ is a morphism in $\mathcal{C}$ such that $F(f) : F(X) \to F(Y)$ is a surjective set map of finite sets in $\mathrm{Set}_{\mathrm{f}}$. We factor f in $\mathcal{C}$ as:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\psi\,\cong}\uparrow \\ Z & \xrightarrow{\iota_Z} & Z\coprod Z' \end{array}$$

where ϕ is a strict epimorphism in $\mathcal{C}$, ι_Z is the coproduct morphism from Z to the coproduct $Z \coprod Z'$ and a monomorphism in $\mathcal{C}$, and ψ is an isomorphism in $\mathcal{C}$. We apply F to the above factorization of f to obtain the factorization of the surjective set map of finite sets $F(f)$ in $\mathrm{Set}_{\mathrm{f}}$ below:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle F(\phi)}\downarrow & & {\scriptstyle F(\psi)\,\approx}\uparrow \\ F(Z) & \xrightarrow{\iota_{F(Z)}} & F(Z)\coprod F(Z') \end{array}$$

because F preserves all finite coproducts in $\mathcal{C}$, where $F(\phi)$ is also a surjective set map of finite sets in $\mathrm{Set}_{\mathrm{f}}$ because F preserves all strict epimorphisms in $\mathcal{C}$, $\iota_{F(Z)}$ is the inclusion of $F(Z)$ in $F(Z) \coprod F(Z')$, and $F(\psi)$ is a bijection of finite sets in $\mathrm{Set}_{\mathrm{f}}$. Since the composite of set maps of finite sets $F(\psi)^{-1}F(f) = \iota_{F(Z)}F(\phi)$ in $\mathrm{Set}_{\mathrm{f}}$ is surjective, the inclusion $\iota_{F(Z)}$ is not only injective, but also surjective, thus a bijection of finite sets in $\mathrm{Set}_{\mathrm{f}}$. Because F reflects isomorphisms, this implies that the coproduct morphism ι_Z is an isomorphism in $\mathcal{C}$, so f factors as follows in $\mathcal{C}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & {\scriptstyle\psi\,\cong}\uparrow \\ Z & \xrightarrow{\iota_Z\,\cong} & Z\coprod Z' \end{array}$$

At last, we conclude that $f = \psi \iota_Z \phi$ is a strict epimorphism in $\mathcal{C}$ because ϕ is a strict epimorphism in $\mathcal{C}$ and both ι_Z and ψ are isomorphisms in $\mathcal{C}$.

4. We know F preserves all monomorphisms in $\mathcal{C}$. Conversely, suppose $f : X \to Y$ is a morphism in $\mathcal{C}$ such that $F(f) : F(X) \to F(Y)$ is an injective set map of finite sets in $\mathrm{Set}_{\mathrm{f}}$. We factor f in $\mathcal{C}$ as:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \phi}\big\downarrow & & \big\uparrow{\scriptstyle \psi\,\cong} \\ Z & \xrightarrow{\iota_Z} & Z \coprod Z' \end{array}$$

where ϕ is a strict epimorphism in $\mathcal{C}$, ι_Z is the coproduct morphism from Z to the coproduct $Z \coprod Z'$ and a monomorphism in $\mathcal{C}$, and ψ is an isomorphism in $\mathcal{C}$. We apply F to the above factorization of f to obtain the factorization of the injective set map of finite sets $F(f)$ in $\mathrm{Set}_{\mathrm{f}}$ below:

$$\begin{array}{ccc} F(X) & \xrightarrow{\ F(f)\ } & F(Y) \\ {\scriptstyle F(\phi)}\big\downarrow & & \big\uparrow{\scriptstyle F(\psi)\,\approx} \\ F(Z) & \xrightarrow{\iota_{F(Z)}} & F(Z) \coprod F(Z') \end{array}$$

because F preserves all finite coproducts in $\mathcal{C}$, where $F(\phi)$ is a surjective set map of finite sets in $\mathrm{Set}_{\mathrm{f}}$ because F preserves all strict epimorphisms in $\mathcal{C}$, $\iota_{F(Z)}$ is the inclusion of $F(Z)$ in $F(Z) \coprod F(Z')$, and $F(\psi)$ is a bijection of finite sets in $\mathrm{Set}_{\mathrm{f}}$. Since the composite of set maps of finite sets $F(\psi)^{-1}F(f) = \iota_{F(Z)}F(\phi)$ in $\mathrm{Set}_{\mathrm{f}}$ is injective, the surjective set map of finite sets $F(\phi)$ is also injective, thus a bijection of finite sets in $\mathrm{Set}_{\mathrm{f}}$. Because F reflects isomorphisms, this implies that ϕ is an isomorphism in $\mathcal{C}$, so f factors as follows in $\mathcal{C}$:

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle \phi\,\cong}\big\downarrow & & \big\uparrow{\scriptstyle \psi\,\cong} \\ Z & \xrightarrow{\iota_Z} & Z \coprod Z' \end{array}$$

At last, we conclude that $f = \psi \iota_Z \phi$ is a monomorphism in $\mathcal{C}$ because ι_Z is a monomorphism in $\mathcal{C}$ and both ϕ and ψ are isomorphisms in $\mathcal{C}$.

□

Corollary 7.2.9. *In a Galois category $\mathcal{C}$ with associated fiber functor F, a morphism f is an isomorphism in $\mathcal{C}$ if and only if it is both a monomorphism and a strict epimorphism in $\mathcal{C}$.*

Proof. Because F reflects isomorphisms, f is an isomorphism in $\mathcal{C}$ if and only if $F(f)$ is a bijection of finite sets in $\mathrm{Set}_{\mathrm{f}}$, which is equivalent to $F(f)$ being both an injective and a surjective set map of finite sets in $\mathrm{Set}_{\mathrm{f}}$. Because F also reflects all monomorphisms and all strict epimorphisms in $\mathcal{C}$ by lemma 7.2.8, this is, in turn, equivalent to f being both a monomorphism and a strict epimorphism in $\mathcal{C}$. This completes the proof. □

The reflection of monomorphisms in lemma 7.2.8 relieves us of some irksome confusion in our study:

Corollary 7.2.10. *In a Galois category $\mathcal{C}$ with associated fiber functor F, every finite coproduct morphism is a monomorphism in $\mathcal{C}$.*

Proof. If ι is a finite coproduct morphism in $\mathcal{C}$, then we apply the fiber functor F, which preserves all finite coproducts in $\mathcal{C}$, to ι to obtain the disjoint union inclusion map of finite sets $F(\iota)$, which is injective, thus a monomorphism in $\mathrm{Set}_{\mathrm{f}}$. Because the fiber functor F reflects monomorphisms by lemma 7.2.8, we conclude that ι is a monomorphism in $\mathcal{C}$, as required. □

What is more, the special factorization of morphisms in a Galois category is essentially unique:

Lemma 7.2.11. *In a Galois category $\mathcal{C}$ with associated fiber functor F, given two factorizations of a morphism $f : X \to Y$ in $\mathcal{C}$ as below:*

$$\begin{array}{ccc} W & \xrightarrow{\iota_W} & W \coprod W' \\ {\scriptstyle\kappa}\uparrow & & \downarrow{\scriptstyle\lambda\,\cong} \\ X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & \uparrow{\scriptstyle\psi\,\cong} \\ Z & \xrightarrow{\iota_Z} & Z \coprod Z' \end{array}$$

where ϕ and κ are strict epimorphisms in $\mathcal{C}$, ι_Z is the coproduct morphism from Z to the coproduct $Z \coprod Z'$ and a monomorphism in $\mathcal{C}$ and ι_W is the coproduct morphism from W to the coproduct $W \coprod W'$ and a monomorphism in $\mathcal{C}$, and ψ and λ are isomorphisms in $\mathcal{C}$, there exists an isomorphism $\omega : W \xrightarrow{\cong} Z$ in $\mathcal{C}$, not necessarily unique, such that the diagram below commutes in $\mathcal{C}$:

$$\begin{array}{ccccc} & & W & & \\ & {\scriptstyle\kappa}\nearrow & \vdots & \searrow{\scriptstyle j} & \\ X & & \xrightarrow{\quad f \quad} & & Y \\ & {\scriptstyle\phi}\searrow & \downarrow & \nearrow{\scriptstyle i} & \\ & & Z & & \end{array}$$

where $i := \psi\iota_Z$ is the composite of monomorphisms, thus a monomorphism, in $\mathcal{C}$ of the coproduct morphism ι_Z from Z to the coproduct $Z \coprod Z'$ post-composed with the isomorphism ψ in $\mathcal{C}$, and $j := \lambda\iota_W$ is the composite of monomorphisms, thus a monomorphism, in $\mathcal{C}$ of the coproduct morphism ι_W from W to the coproduct $W \coprod W'$ post-composed with the isomorphism λ in $\mathcal{C}$.

Proof. Since ϕ is a strict epimorphism in $\mathcal{C}$, we have the pullback square in $\mathcal{C}$ below:

$$\begin{array}{ccc} X \times_Z X & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle\phi} \\ X & \xrightarrow{\phi} & Z \end{array}$$

and ϕ is a co-equalizer of the pair of pullback maps $p : X \times_Z X \to X$ and $q : X \times_Z X \to X$. Similarly, since κ is a strict epimorphism in $\mathcal{C}$, we have the pullback square in $\mathcal{C}$ below:

$$\begin{array}{ccc} X \times_W X & \xrightarrow{q'} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle\kappa} \\ X & \xrightarrow{\kappa} & W \end{array}$$

and κ is a co-equalizer of the pair of pullback maps $p' : X \times_W X \to X$ and $q' : X \times_W X \to X$. We write down the commutative diagram in $\mathcal{C}$ below:

$$\begin{array}{ccccc} X \times_Z X & \overset{p}{\underset{q}{\rightrightarrows}} & X & \xrightarrow{\phi} & Z \\ & & & {\scriptstyle\kappa}\searrow & \\ & & & & W \end{array}$$

where the first row commutes as a co-equalizer in $\mathcal{C}$, and we compute in $\mathcal{C}$ that:

$$j\kappa p = fp = \underbrace{i\phi p = i\phi q}_{\phi p = \phi q} = fq = j\kappa q$$

which implies that $\kappa p = \kappa q$ because j is a monomorphism in $\mathcal{C}$. Then, the universal property of co-equalizers yields a unique morphism $\omega : W \to Z$ in $\mathcal{C}$ such that the diagram in $\mathcal{C}$ below commutes:

$$\begin{array}{ccccc} X \times_Z X & \overset{p}{\underset{q}{\rightrightarrows}} & X & \xrightarrow{\phi} & Z \\ & & & {\scriptstyle\kappa}\searrow & \downarrow{\scriptstyle\omega} \\ & & & & W \end{array}$$

Similarly, we write down the commutative diagram in $\mathcal{C}$ below:

$$\begin{array}{ccccc} X \times_W X & \overset{p'}{\underset{q'}{\rightrightarrows}} & X & \xrightarrow{\kappa} & W \\ & & & {\scriptstyle \phi}\searrow & \\ & & & & Z \end{array}$$

where the first row commutes as a co-equalizer in $\mathcal{C}$, and we compute in $\mathcal{C}$ that:

$$i\phi p' = fp' = \underbrace{j\kappa p' = j\kappa q'}_{\kappa p' = \kappa q'} = fq' = i\phi q'$$

which implies that $\phi p' = \phi q'$ because i is a monomorphism in $\mathcal{C}$. Then, the universal property of co-equalizers yields a unique morphism $\sigma : Z \to W$ in $\mathcal{C}$ such that the diagram in $\mathcal{C}$ below commutes:

$$\begin{array}{ccccc} X \times_W X & \overset{p'}{\underset{q'}{\rightrightarrows}} & X & \xrightarrow{\kappa} & W \\ & & & {\scriptstyle \phi}\searrow & \uparrow{\scriptstyle \sigma} \\ & & & & Z \end{array}$$

Then, we write down the commutative diagram in $\mathcal{C}$ below:

$$\begin{array}{ccccc} X \times_Z X & \overset{p}{\underset{q}{\rightrightarrows}} & X & \xrightarrow{\phi} & Z \\ & & & {\scriptstyle \phi}\searrow & \downarrow{\scriptstyle \sigma\omega} \\ & & & & Z \end{array}$$

where the first row commutes as a co-equalizer in $\mathcal{C}$, and we compute in $\mathcal{C}$ that $\sigma\omega\phi = \sigma\kappa = \phi$. However, we also have the commutative diagram in $\mathcal{C}$ below:

$$\begin{array}{ccccc} X \times_Z X & \overset{p}{\underset{q}{\rightrightarrows}} & X & \xrightarrow{\phi} & Z \\ & & & {\scriptstyle \phi}\searrow & \| \\ & & & & Z \end{array}$$

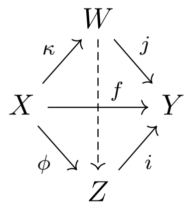
and the uniqueness in the universal property of co-equalizers forces $\sigma\omega = 1_Z$. Similarly, we write down the commutative diagram in $\mathcal{C}$ below:

$$\begin{array}{ccccc} X \times_W X & \overset{p'}{\underset{q'}{\rightrightarrows}} & X & \xrightarrow{\kappa} & W \\ & & & {\scriptstyle \kappa}\searrow & \downarrow{\scriptstyle \omega\sigma} \\ & & & & W \end{array}$$

where the first row commutes as a co-equalizer in $\mathcal{C}$, and we compute in $\mathcal{C}$ that $\omega\sigma\kappa = \omega\phi = \kappa$. However, we also have the commutative diagram in $\mathcal{C}$ below:

$$\begin{array}{ccccc} X \times_W X & \overset{p'}{\underset{q'}{\rightrightarrows}} & X & \xrightarrow{\kappa} & W \\ & & & {\scriptstyle \kappa}\searrow & \| \\ & & & & W \end{array}$$

and the uniqueness in the universal property of co-equalizers forces $\omega\sigma = 1_W$. The equations $\sigma\omega = 1_Z$ and $\omega\sigma = 1_W$ in $\mathcal{C}$ together inform us that $\omega : W \xrightarrow{\cong} Z$ is an isomorphism in $\mathcal{C}$ with unique two-sided inverse isomorphism $\sigma : Z \xrightarrow{\cong} W$ in $\mathcal{C}$. Finally, we verify that the diagram in $\mathcal{C}$ below commutes:



by recalling that $\omega\kappa = \phi$ in $\mathcal{C}$ and computing in $\mathcal{C}$ that $i\omega\kappa = i\phi = f = j\kappa$ to infer that $i\omega = j$ because κ, as a strict epimorphism in $\mathcal{C}$, is an epimorphism in $\mathcal{C}$. This completes the proof. □

Lastly, Galois categories inherit the following property from finite sets by means of their associated fiber functor:

Lemma 7.2.12. *Every Galois category $\mathcal{C}$ with associated fiber functor F is Artinian.*

Proof. For every descending chain of monomorphisms in $\mathcal{C}$:

$$\cdots \longrightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

we obtain a descending chain of injective set maps of finite sets in $\mathrm{Set_f}$ by applying the fiber functor F, which preserves monomorphisms:

$$\cdots \longrightarrow F(X_2) \xrightarrow{F(f_2)} F(X_1) \xrightarrow{F(f_1)} F(X_0)$$

and the above descending chain of injective set maps of finite sets in $\mathrm{Set_f}$ stabilizes, by cardinality considerations, at a natural number $n \in \mathbb{N}$, which depends on the given chain: for every natural number $m \geq n$, the injective set map of finite sets $F(f_m)$ is a bijection of finite sets in $\mathrm{Set_f}$. Because the fiber functor F reflects isomorphisms, this is equivalent to the given descending chain of monomorphisms in $\mathcal{C}$ stabilizing at n: for every natural number $m \geq n$, the monomorphism f_m is an isomorphism in $\mathcal{C}$. We conclude that $\mathcal{C}$ is Artinian, as required. □

7.3 Connected objects of Galois categories

We introduce (or recall, for the eagle-eyed reader of the attached appendix!) the categorical notion of a connected object:

Definition 7.3.1 (Connected object). A **connected object** in a category $\mathcal{C}$ with an initial object $\emptyset$ is an object X of $\mathcal{C}$ such that, if we have $X \cong Y \coprod Z$ in $\mathcal{C}$, then we must have (i) $X \cong Y$ and $Z \cong \emptyset$ in $\mathcal{C}$ or (ii) $X \cong Z$ and $Y \cong \emptyset$ in $\mathcal{C}$.

Example 7.3.2. Initial objects are connected.

Example 7.3.3. In the category Set of sets, the connected objects are precisely the empty set and all singleton sets.

Example 7.3.4. In the category Top of spaces, a connected object is precisely a connected space, justifying our terminology.

We proceed with proving some properties related to connected objects of Galois categories.

Lemma 7.3.5. *In a Galois category $\mathcal{C}$ with associated fiber functor F, a morphism $f : X \to Y$ of $\mathcal{C}$ is a monomorphism if and only if there exists an object X' of $\mathcal{C}$ such that $Y \cong X \coprod X'$ in $\mathcal{C}$ and $f : X \to Y$ is a coproduct morphism in $\mathcal{C}$ under this isomorphism in $\mathcal{C}$.*

Proof. If $f : X \to Y$ is a monomorphism in $\mathcal{C}$, then we factor f in $\mathcal{C}$ as:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & \uparrow{\scriptstyle\psi\,\cong} \\ Z & \xrightarrow{\iota_Z} & Z\coprod Z' \end{array}$$

where ϕ is a strict epimorphism in $\mathcal{C}$, ι_Z is the coproduct morphism from Z to the coproduct $Z\coprod Z'$ and a monomorphism in $\mathcal{C}$, and ψ is an isomorphism in $\mathcal{C}$. Then, $\psi^{-1}f = \iota_Z\phi$ is a monomorphism in $\mathcal{C}$ as a composite of monomorphisms in $\mathcal{C}$, so ϕ is not only a strict epimorphism in $\mathcal{C}$, but also a monomorphism in $\mathcal{C}$, and we infer that ϕ is an isomorphism in $\mathcal{C}$ by corollary 7.2.9. Conversely, if there exists an object X' of $\mathcal{C}$ such that $Y \cong X\coprod X'$ in $\mathcal{C}$ and $f : X \to Y$ is a coproduct morphism in $\mathcal{C}$ under this isomorphism in $\mathcal{C}$, then we apply the fiber functor F, which preserves all finite coproducts of $\mathcal{C}$, to f to obtain the inclusion $F(f) : F(X) \to F(X)\coprod F(X')$ of $F(X)$ in $F(X)\coprod F(X')$ in the category $\mathrm{Set_f}$ of finite sets, which is a monomorphism in $\mathrm{Set_f}$. Because F reflects all monomorphisms in $\mathcal{C}$ by lemma 7.2.8, we infer that $f : X \to Y$ is a monomorphism in $\mathcal{C}$, as required. □

Lemma 7.3.6. *If an object X of a Galois category $\mathcal{C}$ is disconnected, then there exist a connected and non-initial object X' of $\mathcal{C}$ and an object X'' of $\mathcal{C}$ such that $X \cong X'\coprod X''$ in $\mathcal{C}$.*

Proof. We apply the following algorithm. Firstly, since X is disconnected, we know that X is a non-initial object of $\mathcal{C}$ and that there exist non-initial objects X_1 and X_1' of $\mathcal{C}$ such that $X \cong X_1\coprod X_1'$. If X_1 is a connected object of $\mathcal{C}$, then we are done. Otherwise, X_1 is disconnected, so we know that X_1 is a non-initial object of $\mathcal{C}$ and that there exist non-initial objects X_2 and X_2' of $\mathcal{C}$ such that $X_1 \cong X_2\coprod X_2'$, and we repeat this algorithm until we obtain a connected object X_n of $\mathcal{C}$ for some $n \in \mathbb{N}$. The reason why our algorithm terminates in some finite time $n \in \mathbb{N}$ is because it yields a descending chain of coproduct monomorphisms, by corollary 7.2.10, in $\mathcal{C}$:

$$\cdots \longrightarrow X_3 \xrightarrow{\iota_3} X_2 \xrightarrow{\iota_2} X_1$$

which stabilizes at a natural number $n \in \mathbb{N}$, which depends on the chain, for the Galois category $\mathcal{C}$ is Artinian. □

Lemma 7.3.7. *Let $f : X \to Y$ be a morphism in a Galois category $\mathcal{C}$ with associated fiber functor F.*

1. *If X is a non-initial object of $\mathcal{C}$ and Y is a connected object of $\mathcal{C}$, then f is a strict epimorphism in $\mathcal{C}$.*

2. *If X is a connected object of $\mathcal{C}$ and f is a strict epimorphism in $\mathcal{C}$, then Y is a connected object of $\mathcal{C}$.*

Proof. We prove both claims in the order in which we stated them:

1. If X is a non-initial object of $\mathcal{C}$ and Y is a connected object of $\mathcal{C}$, then we factor f in $\mathcal{C}$ as:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\phi}\downarrow & & \uparrow{\scriptstyle\psi\,\cong} \\ Z & \xrightarrow{\iota_Z} & Z\coprod Z' \end{array}$$

   where ϕ is a strict epimorphism in $\mathcal{C}$, ι_Z is the coproduct morphism from Z to the coproduct $Z\coprod Z'$ and a monomorphism in $\mathcal{C}$, and ψ is an isomorphism in $\mathcal{C}$. Then, X being a non-initial object of $\mathcal{C}$ forces $Z' \cong \emptyset$ in $\mathcal{C}$: if Z' were a non-initial object of $\mathcal{C}$, then the connectedness of Y would force $Z' \cong Y$ and $Z \cong \emptyset$, and ϕ would be a strict epimorphism in $\mathcal{C}$ from a non-initial object of $\mathcal{C}$ to an initial object of $\mathcal{C}$, which is impossible. Thus, we have $Z' \cong \emptyset$ in $\mathcal{C}$, which implies $Z\coprod Z' \cong Z\coprod\emptyset \cong Z$ in $\mathcal{C}$, and the coproduct morphism ι_Z is the identity morphism of Z under said isomorphism in $\mathcal{C}$. At last, we conclude that $f = \psi\iota_Z\phi$ is a strict epimorphism in $\mathcal{C}$ because ϕ is a strict epimorphism in $\mathcal{C}$ and both ι_Z and ψ are isomorphisms in $\mathcal{C}$.

2. If X is a connected object of $\mathcal{C}$ and f is a strict epimorphism in $\mathcal{C}$, then we show that Y is a connected object of $\mathcal{C}$ by way of contradiction: we suppose that Y is a disconnected, thus non-initial, object of $\mathcal{C}$. Then, by lemma 7.3.6, there exist a connected and non-initial object Y' of $\mathcal{C}$ and an object Y'' of $\mathcal{C}$ such that $Y \cong Y' \coprod Y''$ in $\mathcal{C}$. We form the fiber product in $\mathcal{C}$ of the two morphisms:

$$\begin{array}{ccc} & & X \\ & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\iota_{Y'}} & Y \end{array}$$

where f is a strict epimorphism in $\mathcal{C}$ and $\iota_{Y'}$ is the coproduct morphism from Y' to the coproduct $Y \cong Y' \coprod Y''$ and a monomorphism in $\mathcal{C}$ by corollary 7.2.10, and said fiber product in $\mathcal{C}$ is:

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ {\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\iota_{Y'}} & Y \end{array}$$

where p is a monomorphism in $\mathcal{C}$ as a base change of the monomorphism $\iota_{Y'}$ in $\mathcal{C}$. We verify that X' is a non-initial object of $\mathcal{C}$ by gathering the data below:

(a) Because the fiber functor F reflects the initial object of $\mathcal{C}$ by lemma 7.2.8 and Y' is a non-initial object of $\mathcal{C}$, we know that $F(Y')$ is a non-empty finite set.

(b) Because the fiber functor F preserves strict epimorphisms in $\mathcal{C}$ and f is a strict epimorphism in $\mathcal{C}$, we know that $F(f)$ is a surjective set map of finite sets in $\mathrm{Set_f}$.

(c) Because the fiber functor F preserves all finite coproducts of $\mathcal{C}$ and $\iota_{Y'}$ is the coproduct morphism from Y' to the coproduct $Y \cong Y' \coprod Y''$ and a monomorphism in $\mathcal{C}$, we know that $F(\iota_{Y'})$ is the inclusion of $F(Y')$ in the disjoint union of finite sets $F(Y) \approx F(Y') \coprod F(Y'')$ in $\mathrm{Set_f}$.

We employ the data above to perform the computation in $\mathrm{Set_f}$ below:

$$\emptyset \neq F(Y') \subset F(Y) = (F(f))\,(F(X))$$

The above computation in $\mathrm{Set_f}$ implies that, as is the case for $F(Y')$, $F(X)$ is a non-empty finite set. Because the fiber functor F preserves all fiber products of $\mathcal{C}$, we infer that $F(X')$ is a non-empty finite set. As the fiber functor F preserves the initial object of $\mathcal{C}$, we conclude that X' is a non-initial object of $\mathcal{C}$.

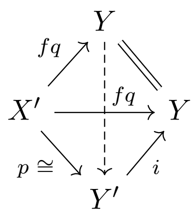
Knowing that X' is a non-initial object of $\mathcal{C}$, we use the previous claim in this lemma to infer that p is also a strict epimorphism in $\mathcal{C}$ because Y' is a connected object of $\mathcal{C}$, thus an isomorphism in $\mathcal{C}$ by corollary 7.2.9, as well as that q is a strict epimorphism in $\mathcal{C}$ because X is a connected object of $\mathcal{C}$. Thus, our Cartesian square at hand in $\mathcal{C}$ is:

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ {\scriptstyle p\,\cong}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\iota_{Y'}} & Y \end{array}$$

where p is an isomorphism in $\mathcal{C}$ and q is a strict epimorphism in $\mathcal{C}$. We have the two factorizations of the strict epimorphism $fq : X' \to Y$ in $\mathcal{C}$, the composite of the strict epimorphisms q and f in $\mathcal{C}$, below:

$$\begin{array}{ccc} Y & =\!= & Y \\ {\scriptstyle fq}\Big\uparrow & & \Big\| \\ X' & \xrightarrow{fq} & Y \\ {\scriptstyle p\,\cong}\Big\downarrow & & \Big\uparrow{\scriptstyle \psi\,\cong} \\ Y' & \xrightarrow{\iota_{Y'}} & Y' \coprod Y'' \end{array}$$

where the isomorphism p and fq both are strict epimorphisms in $\mathcal{C}$, $\iota_{Y'}$ is the coproduct morphism from Y' to the coproduct $Y' \coprod Y''$ and a monomorphism in $\mathcal{C}$, and $\psi : Y' \coprod Y'' \xrightarrow{\cong} Y$ is an isomorphism in $\mathcal{C}$. By lemma 7.2.11, there exists an isomorphism $\omega : Y \xrightarrow{\cong} Y'$ in $\mathcal{C}$, not necessarily unique, such that the diagram below commutes in $\mathcal{C}$:



where $i := \psi\iota_{Y'}$ is the composite of monomorphisms, thus a monomorphism, in $\mathcal{C}$ of the coproduct morphism $\iota_{Y'}$ from Y' to the coproduct $Y' \coprod Y''$ post-composed with the isomorphism ψ in $\mathcal{C}$. Since Y is assumed to be a disconnected object of $\mathcal{C}$ and Y' is a connected object of $\mathcal{C}$, the existence of an isomorphism $\omega : Y \xrightarrow{\cong} Y'$ in $\mathcal{C}$ constitutes our desired contradiction, thanks to which we safely conclude that Y is a connected object of $\mathcal{C}$, as required.

□

We are now ready to show that every object of a Galois category $\mathcal{C}$ admits an essentially unique decomposition as a finite coproduct of connected objects of $\mathcal{C}$:

Proposition 7.3.8. *Let X be a non-initial object of a Galois category $\mathcal{C}$ with associated fiber functor F.*

1. *There is a natural number $n \in \mathbb{N}$ and n connected objects $X_1, \ldots, X_n$ of $\mathcal{C}$ such that $X \cong \coprod_{i=1}^{n} X_i$ in $\mathcal{C}$.*

2. *Given an integer $m \in \mathbb{N}$ and m connected objects $X'_1, \ldots, X'_m$ of $\mathcal{C}$ such that $X \cong \coprod_{j=1}^{m} X'_j$ in $\mathcal{C}$, we must have $n = m$ and, after suitable re-ordering, $X_i \cong X'_i$ in $\mathcal{C}$ for every $i \in \{1, \ldots, n\}$.*

Proof. We prove the existence and the essential uniqueness of a decomposition of every object X of a Galois category $\mathcal{C}$ as a finite coproduct of connected objects of $\mathcal{C}$.

Existence: If X is already a connected object of $\mathcal{C}$, then we are done. Otherwise, lemma 7.3.6 guarantees that there exist a connected and non-initial object X_1 of $\mathcal{C}$ and an object X'_1 of $\mathcal{C}$ such that $X \cong X_1 \coprod X'_1$ in $\mathcal{C}$. If X'_1 is also a connected object of $\mathcal{C}$, then we are done. Otherwise, we again apply lemma 7.3.6 to X'_1 to obtain a connected and non-initial object X_2 of $\mathcal{C}$ and an object X'_2 of $\mathcal{C}$ such that $X'_1 \cong X_2 \coprod X'_2$ in $\mathcal{C}$, and we repeat this algorithm until we obtain a connected object X'_n of $\mathcal{C}$ for some $n \in \mathbb{N}$. The reason why our algorithm terminates in finite time is that it yields a descending chain of coproduct monomorphisms, by corollary 7.2.10:

$$\cdots \longrightarrow X'_2 \xrightarrow{\iota_3} X'_1 \xrightarrow{\iota_2} X$$

which stabilizes at a natural number $n \in \mathbb{N}$, which depends on the chain, for the Galois category $\mathcal{C}$ is Artinian.

Uniqueness: The existence claim of this lemma yields a natural number $n \in \mathbb{N}$ and n connected objects $X_1, \ldots, X_n$ of $\mathcal{C}$ such that $X \cong \coprod_{i=1}^{n} X_i$ in $\mathcal{C}$. Removing redundant initial objects of $\mathcal{C}$ without loss of generality if need be, we suppose that X_i is a non-initial connected object of $\mathcal{C}$ for every $i \in \{1, \ldots, n\}$. Let Y be a non-initial connected object of $\mathcal{C}$ and let $f : Y \to X$ be a monomorphism in $\mathcal{C}$. We gather the data below:

1. Because the fiber functor F reflects the initial object of $\mathcal{C}$ by lemma 7.2.8 and Y is a non-initial object of $\mathcal{C}$, we know that $F(Y)$ is a non-empty finite set.

2. Because the fiber functor F reflects the initial object of $\mathcal{C}$ by lemma 7.2.8 and X_i is a non-initial object of $\mathcal{C}$ for every $i \in \{1, \ldots, n\}$, we know that $F(X_i)$ is a non-empty finite set for every $i \in \{1, \ldots, n\}$.

3. Because the fiber functor F preserves monomorphisms in $\mathcal{C}$ and f is a monomorphism in $\mathcal{C}$, we know that $F(f)$ is an injective set map of finite sets in $\mathrm{Set_f}$.

4. Because the fiber functor F preserves all finite coproducts of $\mathcal{C}$ and $X \cong \coprod_{i=1}^n X_i$ in $\mathcal{C}$, we know that $F(X) \approx \coprod_{i=1}^n F(X_i)$ in $\mathrm{Set_f}$.

We also know that there exists some $j \in \{1, \ldots, n\}$ such that the intersection of finite sets $F(Y) \cap F(X_j)$ is a non-empty finite set. We form the fiber product in $\mathcal{C}$ of the two morphisms:

$$\begin{array}{ccc} & & X_j \\ & & \big\downarrow{\scriptstyle \iota_j} \\ Y & \xrightarrow{f} & X \end{array}$$

where f is a monomorphism in $\mathcal{C}$ and ι_j is the coproduct morphism from X_j to the coproduct $X \cong \coprod_{i=1}^n X_i$ and a monomorphism in $\mathcal{C}$ by corollary 7.2.10, and said fiber product in $\mathcal{C}$ is:

$$\begin{array}{ccc} Y' & \xrightarrow{q} & X_j \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \iota_j} \\ Y & \xrightarrow{f} & X \end{array}$$

where q is a monomorphism in $\mathcal{C}$ as a base change of the monomorphism f in $\mathcal{C}$ and p is a monomorphism in $\mathcal{C}$ as a base change of the coproduct monomorphism ι_j in $\mathcal{C}$. Because the fiber functor F preserves all fiber products of $\mathcal{C}$, we know that $F(Y') \approx F(Y) \cap F(X_j)$ is a non-empty finite set. Thus, because the fiber functor F preserves the initial object of $\mathcal{C}$, we infer that Y' is a non-initial object of $\mathcal{C}$, which we combine with the connectedness of the objects X_j and Y of $\mathcal{C}$ to infer, by the first claim in lemma 7.3.7, that the monomorphisms q and p in $\mathcal{C}$ are also strict epimorphisms in $\mathcal{C}$, thus isomorphisms in $\mathcal{C}$ by corollary 7.2.9. Thus, our Cartesian square at hand in $\mathcal{C}$ is:

$$\begin{array}{ccc} Y' & \xrightarrow{q\,\cong} & X_j \\ {\scriptstyle p\,\cong}\big\downarrow & & \big\downarrow{\scriptstyle \iota_j} \\ Y & \xrightarrow{f} & X \end{array}$$

where both q and p are isomorphisms in $\mathcal{C}$, which we compose to produce $qp^{-1} : Y \xrightarrow{\cong} X_j$ in $\mathcal{C}$. □

We conclude this section with the following property of connected objects of Galois categories:

Lemma 7.3.9. *If $n \in \mathbb{N}$ is a natural number and X and $Y_1, \ldots, Y_n$ are connected objects of a Galois category $\mathcal{C}$ with associated fiber functor F, then we have the bijection of Hom sets:*

$$\sum_{i=1}^{n} (\iota_i)_* : \coprod_{i=1}^{n} \mathrm{Hom}(X, Y_i) \xrightarrow{\approx} \mathrm{Hom}\left(X, \coprod_{i=1}^{n} Y_i\right)$$

where, for every $j \in \{1, \ldots, n\}$, ι_j is the coproduct morphism from Y_j to the finite coproduct $\coprod_{i=1}^n Y_i$ in $\mathcal{C}$.

Proof. Let $f : X \to \coprod_{i=1}^n Y_i$ be a morphism of $\mathcal{C}$. We factor f in $\mathcal{C}$ as:

$$\begin{array}{ccc} X & \xrightarrow{f} & \coprod_{i=1}^n Y_i \\ {\scriptstyle \phi}\big\downarrow & & \big\uparrow{\scriptstyle \psi\,\cong} \\ Z & \xrightarrow{\iota_Z} & Z \coprod Z' \end{array}$$

where ϕ is a strict epimorphism in $\mathcal{C}$, ι_Z is the coproduct morphism from Z to the coproduct $Z \coprod Z'$ and a monomorphism in $\mathcal{C}$, and ψ is an isomorphism in $\mathcal{C}$. Equivalently, we factor f in $\mathcal{C}$ as:

$$\begin{array}{ccc} X & \xrightarrow{f} & \coprod_{i=1}^{n} Y_i \\ & {\scriptstyle\phi}\searrow \quad \nearrow{\scriptstyle i} & \\ & Z & \end{array}$$

where ϕ is a strict epimorphism in $\mathcal{C}$ and $i := \psi\iota_Z$ is the composite morphism in $\mathcal{C}$ of the coproduct morphism ι_Z from Z to the coproduct $Z \coprod Z'$ post-composed with the isomorphism ψ in $\mathcal{C}$, so i is a monomorphism in $\mathcal{C}$ as a composite of monomorphisms in $\mathcal{C}$. Because X is a connected object of $\mathcal{C}$ and ϕ is a strict epimorphism in $\mathcal{C}$, claim 2 in lemma 7.3.7 informs us that Z is a connected object of $\mathcal{C}$. Moreover, by the essential uniqueness claim in proposition 7.3.8, the monomorphism i in $\mathcal{C}$ informs us that $Z \cong Y_k$ in $\mathcal{C}$ for a unique $k \in \{1, \ldots, n\}$, so, under this isomorphism in $\mathcal{C}$, the morphism $\phi : X \to Y_k$ of $\mathcal{C}$ is the unique pre-image of f under $\sum_{i=1}^{n} (\iota_i)_*$. □

7.4 Categories of elements of fiber functors

We proceed with defining the category of elements of a general set-valued functor:

Definition 7.4.1 (Category of elements). The **category of elements** $\mathrm{el}(F)$ of a set-valued functor $F : \mathcal{C} \to \mathrm{Set}$ from a category $\mathcal{C}$ has:

1. objects (X, ζ), where X is an object of $\mathcal{C}$ and ζ is an element of the set $F(X)$, and
2. morphisms $f : (X, \zeta) \to (Y, \eta)$, where f is a morphism of $\mathcal{C}$ such that $(F(f))(\zeta) = \eta \in F(Y)$.

A **connected element** (X, ζ) of $\mathrm{el}(F)$ is one such that X is a connected object of $\mathcal{C}$.

The **projection** associated with $\mathrm{el}(F)$ is the functor $P : \mathrm{el}(F) \to \mathcal{C}$ defined:

1. on the objects of $\mathrm{el}(F)$ by $P(X, \zeta) := X$, and
2. on the morphisms of $\mathrm{el}(F)$ by $P(f) := f$.

Remark 7.4.2. *We will be focusing on the category of elements* $\mathrm{el}(F)$ *of the fiber functor F associated with a Galois category $\mathcal{C}$, where we are technically enlarging the target category of F from the category* $\mathrm{Set_f}$ *of finite sets to the category Set of sets.*

Remark 7.4.3. *Because, by lemma 7.2.8, the fiber functor F associated with a Galois category $\mathcal{C}$ reflects the initial object of $\mathcal{C}$, the image of the projection functor $P : \mathrm{el}(F) \to \mathcal{C}$ on objects consists of all non-initial objects of $\mathcal{C}$.*

We also define the evaluation at an element of a general set-valued functor:

Definition 7.4.4 (Evaluation). The **evaluation** at an element (X, ζ) of a set-valued functor $F : \mathcal{C} \to \mathrm{Set}$ from a category $\mathcal{C}$ is the natural transformation $\mathrm{ev}_\zeta : \mathrm{Hom}(X, -) \to F$ of functors from the category $\mathcal{C}$ to the category Set of sets object-wise defined at an object Y of $\mathcal{C}$ by the set map:

$$\mathrm{ev}_\zeta(Y) : \mathrm{Hom}(X, Y) \to F(Y), \quad (\mathrm{ev}_\zeta(Y))(f) := (F(f))(\zeta) \in F(Y)$$

Remark 7.4.5. *Given a morphism $\phi : Y \to Z$, we verify that the naturality diagram of set maps below commutes:*

$$\begin{array}{ccc} \mathrm{Hom}(X, Y) & \xrightarrow{\phi_*} & \mathrm{Hom}(X, Z) \\ {\scriptstyle \mathrm{ev}_\zeta(Y)}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_\zeta(Z)} \\ F(Y) & \xrightarrow{F(\phi)} & F(Z) \end{array}$$

by computing, for every morphism $f : X \to Y$ of $\mathcal{C}$, that:

$$(\mathrm{ev}_\zeta(Z)\phi_*)(f) := (\mathrm{ev}_\zeta(Z))(\phi f) := (F(\phi f))(\zeta) = (F(\phi)F(f))(\zeta) =: (F(\phi)\mathrm{ev}_\zeta(Y))(f)$$

Remark 7.4.6. *We will be focusing on the evaluation at an element of the fiber functor F associated with a Galois category $\mathcal{C}$, where we are enlarging the target category of F from the category* $\mathrm{Set_f}$ *of finite sets to the category Set of sets also because the image of the covariant Hom functor* $\mathrm{Hom}(X,-)$ *associated with some object X of $\mathcal{C}$ may extend beyond the category of finite sets.*

We begin with showing that evaluation at a connected element of the fiber functor associated with a Galois category is always object-wise injective:

Lemma 7.4.7. *If (X,ζ) is a connected element of the fiber functor F of a Galois category $\mathcal{C}$, then the evaluation* $\mathrm{ev}_\zeta : \mathrm{Hom}(X,-) \to F$ *at (X,ζ) is object-wise injective.*

Proof. Let Y be an object of $\mathcal{C}$. If Y is an initial object of $\mathcal{C}$, then $\mathrm{ev}_\zeta(Y)$ is the identity map of the empty set, which is injective. Otherwise, if Y is a non-initial object of $\mathcal{C}$, then consider two morphisms $f : X \to Y$ and $g : X \to Y$ in $\mathcal{C}$ such that $(\mathrm{ev}_\zeta(Y))(f) = (\mathrm{ev}_\zeta(Y))(g)$, that is, $(F(f))(\zeta) = (F(g))(\zeta) \in F(Y)$. We form the equalizer in $\mathcal{C}$ of the two morphisms $f : X \to Y$ and $g : X \to Y$ in $\mathcal{C}$:

$$\mathrm{eq}(f,g) \xrightarrow{i} X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

where the equalizer morphism i is a monomorphism in $\mathcal{C}$. Because the fiber functor F preserves all equalizers of $\mathcal{C}$, applying F to the above equalizer in $\mathcal{C}$ yields the equalizer in the category $\mathrm{Set_f}$ of finite sets:

$$\mathrm{eq}\left(F(f),F(g)\right) \xrightarrow{j} F(X) \underset{F(g)}{\overset{F(f)}{\rightrightarrows}} F(Y)$$

where j is the inclusion of the equalizer $\mathrm{eq}\left(F(f),F(g)\right)$ in $F(X)$. We know that $\zeta \in \mathrm{eq}\left(F(f),F(g)\right) \subset F(X)$, so $\mathrm{eq}\left(F(f),F(g)\right)$ is a non-empty finite set, and so is its superset $F(X)$. Because the fiber functor F preserves the initial object of $\mathcal{C}$ and $F\left(\mathrm{eq}(f,g)\right) \approx \mathrm{eq}\left(F(f),F(g)\right)$ is a non-empty finite set, we know that $\mathrm{eq}(f,g)$ is a non-initial object of $\mathcal{C}$. Combining this with the connectedness of the object X of $\mathcal{C}$ and applying the first claim in lemma 7.3.7, we infer that the equalizer monomorphism $i : \mathrm{eq}(f,g) \to X$ of $\mathcal{C}$ is also a strict epimorphism of $\mathcal{C}$, thus an isomorphism $i : \mathrm{eq}(f,g) \xrightarrow{\cong} X$ of $\mathcal{C}$ by corollary 7.2.9, which is equivalent to $f = g$ in $\mathcal{C}$, as required. $\square$

Corollary 7.4.8. *If X is a connected object of a Galois category $\mathcal{C}$ with associated fiber functor F, then, for every object Y of $\mathcal{C}$, the Hom set* $\mathrm{Hom}(X,Y)$ *is a finite set. In particular, if X is a non-initial connected object of $\mathcal{C}$, then, for every object Y of $\mathcal{C}$, the Hom set* $\mathrm{Hom}(X,Y)$ *is a finite set of finite cardinality* $|\mathrm{Hom}(X,Y)| \leq |F(Y)|$.

Remark 7.4.9. *If X is an initial object of $\mathcal{C}$, then the Hom set* $\mathrm{Hom}(X,X)$ *of endomorphisms of X in $\mathcal{C}$ is a singleton set, whereas $F(X)$ is the empty set because the fiber functor F preserves the initial object of $\mathcal{C}$, so the finite cardinality inequality in the second claim in corollary 7.4.8 does not extend to any initial object X of $\mathcal{C}$.*

Henceforth, for two objects X and Y of a Galois category $\mathcal{C}$, we write $X \geq Y$ if $\mathrm{Hom}(X,Y)$ is a non-empty set. Similarly, for two elements (X,ζ) and (Y,η) of the fiber functor F associated with a Galois category $\mathcal{C}$, we write $(X,\zeta) \geq (Y,\eta)$ if $\mathrm{Hom}\left((X,\zeta),(Y,\eta)\right)$ is a non-empty set, that is, if there exists a morphism $f : X \to Y$ of $\mathcal{C}$ such that $(F(f))(\zeta) = \eta \in F(Y)$. Note that $(X,\zeta) \geq (Y,\eta)$ implies $X \geq Y$. Our notation alludes to a partial order: indeed, $\geq$ is reflexive and transitive, but it need not be even essentially antisymmetric - that is, antisymmetric up to isomorphism in $\mathcal{C}$ - so not dispelling this allusion will do more harm than good. Instead, we note that the relation $\geq$ makes a Galois category $\mathcal{C}$ directed. We view lemma 7.4.7 through our new lens:

Corollary 7.4.10. *If (X,ζ) is a connected element of the fiber functor F of a Galois category $\mathcal{C}$ and (Y,η) is an element of F such that $(X,\zeta) \geq (Y,\eta)$ in $\mathcal{C}$, then $(X,\zeta) \geq (Y,\eta)$ in $\mathcal{C}$ is witnessed by a unique morphism $f : X \to Y$ of $\mathcal{C}$: there exists a unique morphism $f : X \to Y$ of $\mathcal{C}$ such that $(F(f))(\zeta) = \eta \in F(Y)$.*

Proof. A morphism $f : X \to Y$ of $\mathcal{C}$ witnesses $(X,\zeta) \geq (Y,\eta)$ in $\mathcal{C}$ if and only if $(\mathrm{ev}_\zeta(Y))(f) = \eta$. $\square$

A different way to view lemma 7.4.7 through our new lens is:

Corollary 7.4.11. *If (X, ζ) is a connected element of the fiber functor F of a Galois category $\mathcal{C}$, then, for every element (Y, η) of F, the set* $\mathrm{Hom}((X, \zeta), (Y, \eta))$ *is either the empty set or a singleton set.*

We use our new perspective to state and prove some new results:

Lemma 7.4.12. *If $(Y_1, \eta_1), \ldots, (Y_n, \eta_n)$ are n connected elements of the fiber functor F of a Galois category $\mathcal{C}$, then there exists a connected element (X, ζ) of F such that $(X, \zeta) \geq (Y_i, \eta_i)$ in $\mathcal{C}$ for every $i \in \{1, \ldots, n\}$.*

Proof. We employ proposition 7.3.8 to essentially uniquely decompose the finite product $\prod_{r=1}^{n} Y_r$ in $\mathcal{C}$ as a finite coproduct of connected objects $\prod_{r=1}^{n} Y_r \cong \coprod_{j=1}^{m} X_j$ in $\mathcal{C}$. Because the fiber functor F preserves both finite products and finite coproducts in $\mathcal{C}$, we have $\prod_{r=1}^{n} F(Y_r) \approx \coprod_{j=1}^{m} F(X_j)$ in the category $\mathrm{Set_f}$ of finite sets, and we set:

$$\eta := (\eta_1, \ldots, \eta_n) \in \prod_{r=1}^{n} F(Y_r) \approx \coprod_{j=1}^{m} F(X_j)$$

Then, there exists a unique $k \in \{1, \ldots, m\}$ such that $\eta \in F(X_k)$, and, for every $i \in \{1, \ldots, n\}$, the composite morphism $p_i \iota_k$ in $\mathcal{C}$ of the coproduct morphism $\iota_k : X_k \to \coprod_{j=1}^{m} X_j$ post-composed with the product morphism $p_i : \prod_{r=1}^{n} Y_r \to Y_i$ witnesses $(X_k, \eta) \geq (Y_i, \eta_i)$ in $\mathcal{C}$, and actually does so uniquely by corollary 7.4.10. □

We extract the following useful consequence to conclude this section:

Corollary 7.4.13. *If Y is an object of a Galois category $\mathcal{C}$ with associated fiber functor F, then there exists a connected element (X, ζ) of F such that the evaluation set map $\mathrm{ev}_\zeta(Y) : \mathrm{Hom}(X, Y) \xrightarrow{\approx} F(Y)$ is a bijection of finite sets.*

Proof. By lemma 7.4.7, it suffices to produce a connected element (X, ζ) of F such that the evaluation set map $\mathrm{ev}_\zeta(Y) : \mathrm{Hom}(X, Y) \to F(Y)$ is surjective. If $F(Y)$ is the empty set - which is equivalent to Y being an initial object of $\mathcal{C}$, because the fiber functor F reflects the initial object of $\mathcal{C}$ by lemma 7.2.8 - then we are done. Otherwise:

$$F(Y) = \{\eta_1, \ldots, \eta_n\}$$

is a non-empty finite set of non-zero cardinality $n \in \mathbb{N}$. Lemma 7.4.12 provides us with a connected element (X, ζ) of F such that $(X, \zeta) \geq (Y, \eta_i)$ in $\mathcal{C}$ for every $i \in \{1, \ldots, n\}$. Equivalently, for every $i \in \{1, \ldots, n\}$, there exists a unique, by corollary 7.4.10, morphism $f_i : X \to Y$ such that $(\mathrm{ev}_\zeta(Y))(f_i) = \eta_i$. Hence, (X, ζ) is a connected element of F such that the evaluation set map $\mathrm{ev}_\zeta(Y) : \mathrm{Hom}(X, Y) \to F(Y)$ is surjective, as required. □

7.5 Galois objects of Galois categories

We begin with noting the observation below:

Lemma 7.5.1. *If X is a connected object of a Galois category $\mathcal{C}$ with associated fiber functor F, then every endomorphism of X in $\mathcal{C}$ is an automorphism of X in $\mathcal{C}$.*

Proof. Let $f : X \to X$ be an endomorphism of X in $\mathcal{C}$. By the first claim in lemma 7.3.7, the connectedness of X implies that f is a strict epimorphism in $\mathcal{C}$. Because the fiber functor F preserves strict epimorphisms in $\mathcal{C}$, we know that $F(f) : F(X) \to F(X)$ is a surjective self-map of the finite set $F(X)$, thus a permutation of the finite set $F(X)$ by the pigeonhole principle. At last, because the fiber functor F reflects isomorphisms, this implies that $f : X \xrightarrow{\cong} X$ is an automorphism of X in $\mathcal{C}$, as required. □

We restate lemma 7.5.1 as follows:

Corollary 7.5.2. *If X is a connected object of a Galois category $\mathcal{C}$ with associated fiber functor F, then* $\mathrm{End}(X) = \mathrm{Aut}(X)$*, that is, the set* $\mathrm{End}(X)$ *of endomorphisms of X in $\mathcal{C}$ is equal to, rather than a proper superset of, the group* $\mathrm{Aut}(X)$ *of automorphisms of X in $\mathcal{C}$.*

It is now time for us to define and study Galois objects of a Galois category:

Definition 7.5.3 (Galois object). A **Galois object** X of a Galois category $\mathcal{C}$ with associated fiber functor F is a connected object X of $\mathcal{C}$ such that:

1. X is an initial object of $\mathcal{C}$ or
2. there exists some $\zeta \in F(X)$ at which the evaluation set map:

$$\mathrm{ev}_\zeta(X) : \mathrm{Aut}(X) \xrightarrow{\approx} F(X)$$

   where $\mathrm{End}(X) = \mathrm{Aut}(X)$ by corollary 7.5.2, is a bijection of finite sets.

A **Galois element** (X, ζ) of F is one such that X is a Galois object of $\mathcal{C}$.

Remark 7.5.4. *Note that we logically need to separately postulate that all initial objects of a Galois category $\mathcal{C}$ be Galois: because the fiber functor F associated with a Galois category $\mathcal{C}$ preserves the initial object of $\mathcal{C}$, the existence condition fails for all initial objects of $\mathcal{C}$ as an existence condition placed on the empty set.*

We characterize non-initial Galois objects in many different ways:

Lemma 7.5.5. *Let X be a non-initial connected object of a Galois category $\mathcal{C}$ with associated fiber functor F. The following are equivalent:*

1. *X is a Galois object of $\mathcal{C}$.*
2. *The left group action of the group* $\mathrm{Aut}(X)$ *of automorphisms of X in $\mathcal{C}$ on the finite set $F(X)$ defined by $\omega \cdot \zeta := (F(\omega))(\zeta) \in F(X)$ is regular*[4]*, that is, both free and transitive.*
3. *The finite set $F(X)$ and the group* $\mathrm{Aut}(X)$ *of automorphisms of X in $\mathcal{C}$ have equal finite cardinality.*
4. *The group quotient object $X/\mathrm{Aut}(X)$ of $\mathcal{C}$ is terminal in $\mathcal{C}$.*

The second characterization of non-initial Galois objects in lemma 7.5.5 - in particular, the transitivity of the group action of the group $\mathrm{Aut}(X)$ of automorphisms of X in $\mathcal{C}$ on the finite set $F(X)$ - sheds light on:

Lemma 7.5.6. *If X is a Galois object of a Galois category $\mathcal{C}$ with associated fiber functor F, then, for every $\zeta \in F(X)$, the evaluation set map:*

$$\mathrm{ev}_\zeta(X) : \mathrm{Aut}(X) \xrightarrow{\approx} F(X)$$

where $\mathrm{End}(X) = \mathrm{Aut}(X)$ *by corollary 7.5.2, is a bijection.*

Remark 7.5.7. *Lemma 7.5.6 vacuously holds for all initial objects of $\mathcal{C}$: the fiber functor F preserves the initial object of $\mathcal{C}$, thus rendering the statement in lemma 7.5.6 a vacuously satisfied property of the empty set.*

[4] A **free** group action of a group G on a set S is one such that its associated group homomorphism $\sigma : G \to \mathrm{Sym}(S)$ from G to the symmetric group $\mathrm{Sym}(S)$ on S is injective. A **transitive** group action of a group G on a set S is one whose unique orbit is S itself. A **regular** group action of a group G on a set S is a free and transitive one.

Hence, if X is a connected object of a Galois category $\mathcal{C}$ with associated fiber functor F for which we find some element ζ of the finite set $F(X)$ satisfying the non-initial Galois object condition, then lemma 7.5.6 informs us that *every* element of the finite set $F(X)$ satisfies the non-initial Galois object condition.

We proceed with defining a Galois closure of an object of a Galois category:

Definition 7.5.8 (**Galois closure**). A **Galois closure** of an object X of a Galois category $\mathcal{C}$ with associated fiber functor F is a Galois object $\widehat{X}$ of $\mathcal{C}$ such that $\widehat{X} \geq X$ in $\mathcal{C}$.

Example 7.5.9. If X is already a Galois object of $\mathcal{C}$, then a sensible choice of a Galois closure of X is X itself.

The existence theorem below guarantees that every object of a Galois category has a Galois closure:

Theorem 7.5.10. *Every object X of a Galois category $\mathcal{C}$ with associated fiber functor F has a Galois closure $\widehat{X}$.*

Proof. If X is already a Galois object of $\mathcal{C}$, then a Galois closure of X is X itself. Otherwise, corollary 7.4.13 informs us that there still exists a connected element (Y, ζ) of F such that the evaluation set map $\mathrm{ev}_\zeta(X) : \mathrm{Hom}(Y, X) \xrightarrow{\approx} F(X)$ is a bijection of finite sets. If n is the common finite cardinality of the Hom set $\mathrm{Hom}(Y, X)$ and the finite set $F(X)$, we enumerate:

$$\mathrm{Hom}\,(Y, X) = \{f_1, \ldots, f_n\}$$

and:

$$F(X) = \{\zeta_1, \ldots, \zeta_n\}$$

so that, for every $i \in \{1, \ldots, n\}$, we have:

$$(\mathrm{ev}_\zeta(X))\,(f_i) := (F(f_i))\,(\zeta) = \zeta_i$$

Then, the morphism $f := (f_1, \ldots, f_n) : Y \to X^n := \prod_{i=1}^n X$ of $\mathcal{C}$ satisfies $(F(f))\,(\zeta) = (\zeta_1, \ldots, \zeta_n)$ and factors in $\mathcal{C}$ as:

$$\begin{array}{ccc} Y & \xrightarrow{f} & X^n \\ \phi\big\downarrow & & \psi \cong \big\uparrow \\ \widehat{X} & \xrightarrow{\iota_{\widehat{X}}} & \widehat{X} \coprod \widehat{X}' \end{array}$$

where ϕ is a strict epimorphism in $\mathcal{C}$, $\iota_{\widehat{X}}$ is the coproduct morphism from $\widehat{X}$ to the coproduct $\widehat{X} \coprod \widehat{X}'$ and a monomorphism in $\mathcal{C}$, and ψ is an isomorphism in $\mathcal{C}$. We note that:

1. by the second claim in lemma 7.3.7 $\widehat{X}$ is a connected object of $\mathcal{C}$ because ϕ is a strict epimorphism in $\mathcal{C}$ with connected source object Y of $\mathcal{C}$ as (Y, ζ) is a connected element of F, and

2. $\widehat{X} \geq X$ in $\mathcal{C}$, witnessed by the composite morphism $p_1 \psi \iota_{\widehat{X}} : \widehat{X} \to X$ in $\mathcal{C}$, where $p_1 : X^n \to X$ is the first product morphism in $\mathcal{C}$.

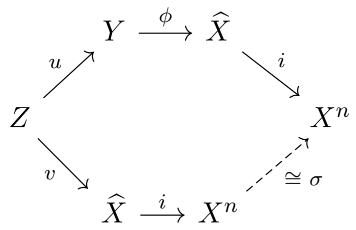
We justify our suggestive notation by showing that $\widehat{X}$ is a Galois object of $\mathcal{C}$. Knowing that $\widehat{X}$ is a connected object of $\mathcal{C}$, we are left with the task of finding an element $\widehat{\zeta}$ of the finite set $F(\widehat{X})$ at which the evaluation:

$$\mathrm{ev}_{\widehat{\zeta}}(\widehat{X}) : \mathrm{Aut}(\widehat{X}) \xrightarrow{\approx} F(\widehat{X})$$

where $\mathrm{End}(\widehat{X}) = \mathrm{Aut}(\widehat{X})$ by corollary 7.5.2, is a bijection of finite sets. We show that $\widehat{\zeta} := (F(\phi))\,(\zeta) \in F(\widehat{X})$ is such an element. By lemma 7.4.7, it suffices to show that the evaluation set map at $\widehat{\zeta}$ is surjective. Let η be an element of the finite set $F(\widehat{X})$. We find an element ω of the group $\mathrm{Aut}(\widehat{X})$ of automorphisms of $\widehat{X}$ in $\mathcal{C}$ such that:

$$\left(\mathrm{ev}_{\widehat{\zeta}}(\widehat{X})\right)(\omega) := F\omega\left(\widehat{\zeta}\right) = \eta \in F(\widehat{X})$$

By lemma 7.4.12, there exists a connected element (Z,θ) of F such that $(Z,\theta) \geq (\widehat{X},\eta)$ and $(Z,\theta) \geq (Y,\zeta)$ in $\mathcal{C}$, where Z is a non-initial connected object of $\mathcal{C}$. This is witnessed by two morphisms of elements $v : (Z,\theta) \to (\widehat{X},\eta)$ and $u : (Z,\theta) \to (Y,\zeta)$ whose underlying morphisms are strict epimorphisms in $\mathcal{C}$ by the first claim in lemma 7.3.7 because their common source object is the non-initial connected object Z of $\mathcal{C}$ and their targets $\widehat{X}$ and Y both are connected objects of $\mathcal{C}$. We shall need an automorphism $\sigma : X^n \xrightarrow{\cong} X^n$ of X^n in $\mathcal{C}$ making the diagram in $\mathcal{C}$ below commute:



where $i := \psi\iota_{\widehat{X}}$ is the composite of monomorphisms, thus a monomorphism, in $\mathcal{C}$ of the coproduct morphism $\iota_{\widehat{X}}$ from $\widehat{X}$ to the coproduct $\widehat{X} \coprod \widehat{X}'$ post-composed with the isomorphism ψ in $\mathcal{C}$. To that effect, we first show that, under the bijection of finite sets $F(X^n) \approx (F(X))^n$ because the fiber functor F preserves all finite products of $\mathcal{C}$, in $(F(i))(\eta) = (\zeta_{i_1}, \ldots, \zeta_{i_n}) \in (F(X))^n$, the elements $\zeta_{i_1}, \ldots, \zeta_{i_n}$ of the finite set $F(X)$ are pairwise distinct. We do so by equivalently showing that, if $p_j(F(i))(\eta) = p_k(F(i))(\eta)$ for two natural numbers j and k, where p_j is the j-th Cartesian product projection map and p_k is the k-th Cartesian product projection map, then we must have $j = k$. To do so, we first compute that:

$$\underbrace{p_j(F(i))(\eta) = (F(p_j i))(\eta)}_{F \text{ preserves all finite products in } \mathcal{C}} = (F(p_j iu))(\theta) =: (\mathrm{ev}_\theta(X))(p_j iu)$$

$$\underbrace{p_k(F(i))(\eta) = (F(p_k i))(\eta)}_{F \text{ preserves all finite products in } \mathcal{C}} = (F(p_k iu))(\theta) =: (\mathrm{ev}_\theta(X))(p_k iu)$$

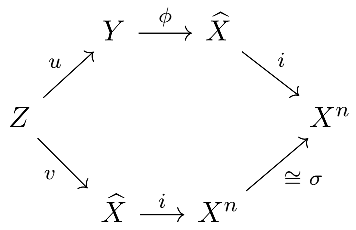
By lemma 7.4.7, the set map $\mathrm{ev}_\theta(X)$ is injective, so our two computations above together imply that $p_j iu = p_k iu$ in $\mathcal{C}$. Because u is a strict epimorphism in $\mathcal{C}$, we infer that $p_j i = p_k i$ in $\mathcal{C}$, and we obtain that:

$$f_j = \underbrace{p_j f = p_j i\phi}_{f = i\phi} = p_k i\phi = p_k f = f_k$$

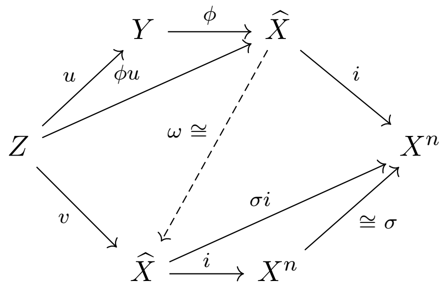
At last, $f_j = f_k$ forces $j = k$, so $(F(i))(\eta) = (\zeta_{i_1}, \ldots, \zeta_{i_n}) \in (FX)^n$ has pairwise distinct entries, as required. Because the symmetric group on n letters S_n acts transitively by permutations on said n letters, we safely choose $\sigma \in S_n$ such that $\sigma(F(i))(\eta) = (\zeta_1, \ldots, \zeta_n) \in (FX)^n$, and we show that its induced automorphism $\sigma : X^n \xrightarrow{\cong} X^n$ of X^n in $\mathcal{C}$ which permutes the n factors of the finite product $X^n := \prod_{i=1}^n X$ in $\mathcal{C}$ by the permutation $\sigma \in S_n$ is an automorphism of X^n in $\mathcal{C}$ which makes our previous diagram commute in $\mathcal{C}$, that is, we show that $i\phi u = \sigma iv$. To that effect, we compute that:

$$(\mathrm{ev}_\theta(X^n))(\sigma iv) := (F(\sigma iv))(\theta) = \sigma(F(i))(\eta) = (\zeta_1, \ldots, \zeta_n) =$$
$$= (F(f))(\zeta) = \underbrace{(F(fu))(\theta) = (F(i\phi u))(\theta)}_{f = i\phi} =: (\mathrm{ev}_\theta(X^n))(i\phi u)$$

By lemma 7.4.7, the set map $\mathrm{ev}_\theta(X^n)$ is injective, so our computation above implies that $i\phi u = \sigma iv$, and the diagram below commutes in $\mathcal{C}$:



We apply lemma 7.2.11 to $i\phi u = \sigma i v$ in $\mathcal{C}$ to obtain our desired automorphism $\omega : \widehat{X} \xrightarrow{\cong} \widehat{X}$ of $\widehat{X}$ in $\mathcal{C}$, which is induced by the factorizations of $i\phi u = \sigma i v$ in $\mathcal{C}$ in the commutative diagram in $\mathcal{C}$ below:



where both ϕu and v are strict epimorphisms in $\mathcal{C}$, the former as a composite of two strict epimorphisms in $\mathcal{C}$, and both $i := \psi \iota_{\widehat{X}}$ and σi are monomorphisms in $\mathcal{C}$ as composites of two monomorphisms in $\mathcal{C}$. At last, we compute that:

$$\left(\mathrm{ev}_{\widehat{\zeta}}(\widehat{X})\right)(\omega) := (F(\omega))\left(\widehat{\zeta}\right) = (F(\omega\phi))(\zeta) = \underbrace{(F(\omega\phi u))(\theta) = (F(v))(\theta)}_{\omega\phi u = v} = \eta \in F(\widehat{X})$$

The above computation proves that the evaluation set map at $\widehat{\zeta}$ is surjective, thus completing the proof. □

A useful consequence of the existence theorem 7.5.10 is the following:

Corollary 7.5.11. *If $X_1, \ldots, X_n$ are n objects of a Galois category $\mathcal{C}$ with associated fiber functor F, then there exists a common Galois closure $\widehat{X}$ of $X_1, \ldots, X_n$.*

Proof. We form the finite product $\prod_{j=1}^{n} X_j$ in the Galois category $\mathcal{C}$, which satisfies $\prod_{j=1}^{n} X_j \geq X_i$ for every $i \in \{1, \ldots, n\}$, witnessed by the i-th product morphism. Theorem 7.5.10 provides us with a Galois closure $\widehat{X}$ of $\prod_{j=1}^{n} X_j$, which is a common Galois closure of $X_1, \ldots, X_n$ by the transitivity of the $\geq$ relation. □

Another consequence of the existence theorem 7.5.10 in the language of the relation $\geq$ is the following:

Corollary 7.5.12. *Let $\mathcal{G}$ be the full subcategory[5] of the Galois objects of a Galois category $\mathcal{C}$ with associated fiber functor F, and let $F|_{\mathcal{G}}$ be the restriction[6] of F to $\mathcal{G}$. Then, with respect to the relation $\geq$, the category of elements $\mathrm{el}(F|_{\mathcal{G}})$ of $F|_{\mathcal{G}}$ is cofinal in the category of elements $\mathrm{el}(F)$ of F.*

We conclude this section with the following result:

Proposition 7.5.13. *Let (X, ζ) be a Galois element of the fiber functor F associated with a Galois category $\mathcal{C}$, and let $\mathcal{C}^X$ be the full subcategory of $\mathcal{C}$ of all objects Y of $\mathcal{C}$ such that $X \geq Y$ in $\mathcal{C}$. Then, the evaluation ev_ζ at ζ restricted to $\mathcal{C}^X$ is a natural isomorphism:*

$$\mathrm{ev}_\zeta^X : \mathrm{Hom}_{\mathcal{C}^X}(X, -) \xrightarrow{\cong} F|_{\mathcal{C}^X}$$

where, to be precise, the restriction ev_ζ^X is the natural transformation $\mathrm{ev}_\zeta \circ i^X$, where $i^X : \mathcal{C}^X \to \mathcal{C}$ denotes the full subcategory inclusion functor.

[5]A **full** subcategory $\mathcal{G}$ of a category $\mathcal{C}$ is one such that, for every pair of objects X and Y of $\mathcal{G}$, we have $\mathrm{Hom}_{\mathcal{G}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ rather than just $\mathrm{Hom}_{\mathcal{G}}(X, Y) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y)$ - in other words, such that the inclusion functor $i : \mathcal{G} \to \mathcal{C}$ is full, hence the choice of terminology. For example, the category Ab of abelian groups is a full subcategory of the category Grp of groups, but not a full subcategory of the category Set of sets: the constant self-map of $\mathbb{Z}$ with constant value 1, albeit a perfectly valid set map, is not an abelian group endomorphism of $\mathbb{Z}$ because it fails to respect its additive identity element 0.

[6]This restriction is the composite functor Fi of the full subcategory inclusion functor $i : \mathcal{G} \to \mathcal{C}$ post-composed with F.

Remark 7.5.14. *An object Y of $\mathcal{C}$ fails to also be an object of $\mathcal{C}^X$ if and only if* $\mathrm{Hom}\,(X,Y)$ *is the empty set. Combining this with the fact that the fiber functor F reflects the initial object of $\mathcal{C}$ by lemma 7.2.8, we realize that, for every non-initial object Y of $\mathcal{C}$ which fails to also be an object of $\mathcal{C}^X$, the set* $\mathrm{Hom}\,(X,Y)$ *is the empty set but the set $F(Y)$ is a non-empty finite set, so the claim in proposition 7.5.13 may fail to extend beyond the full subcategory $\mathcal{C}^X$ of a Galois category $\mathcal{C}$.*

Proof. Firstly, let Y be a connected object of $\mathcal{C}$ such that $X \geq Y$ in $\mathcal{C}$, and let $f : X \to Y$ be a morphism witnessing $Y \geq X$ in $\mathcal{C}$. By lemma 7.4.7, since (X,ζ) is assumed to be a Galois element, thus a connected element of the fiber functor F, it suffices to show that the evaluation set map $\mathrm{ev}_\zeta^X(Y) : \mathrm{Hom}_{\mathcal{C}^X}(X,Y) \xrightarrow{\cong} F(Y)$ at ζ is surjective. Because (X,ζ) is a Galois element of the fiber functor F, the naturality commutative diagram of set maps for $f : X \to Y$ is:

$$\begin{array}{ccc} \mathrm{Aut}\,(X) & \xrightarrow{f_*} & \mathrm{Hom}\,(X,Y) \\ {\scriptstyle \mathrm{ev}_\zeta(X)\,\approx}\downarrow & & {\scriptstyle \mathrm{ev}_\zeta(Y)}\downarrow \\ F(X) & \xrightarrow{F(f)} & F(Y) \end{array}$$

Moreover, (X,ζ) being an element of the fiber functor F forces X to be a non-initial object of $\mathcal{C}$, which we combine with the assumed connectedness of Y to infer that $f : X \to Y$ is a strict epimorphism in $\mathcal{C}$ by the first claim in lemma 7.3.7. Because the fiber functor F preserves all strict epimorphisms in $\mathcal{C}$, this implies that $F(f)$ is a surjective set map of finite sets. Thus, the set map $F(f)\mathrm{ev}_\zeta(X) = \mathrm{ev}_\zeta(Y)f_*$ is surjective as a composite of two surjective set maps, which implies that the evaluation set map $\mathrm{ev}_\zeta^X(Y)$ at ζ is surjective, as required.

Now, let Z be a disconnected, thus non-initial object of $\mathcal{C}$ such that $X \geq Z$ in $\mathcal{C}$. We employ proposition 7.3.8 to essentially uniquely decompose Z in $\mathcal{C}$ as a finite coproduct of connected objects $Z \cong \coprod_{i=1}^n Y_i$ in $\mathcal{C}$. Under this essentially unique decomposition of Z in $\mathcal{C}$, the evaluation set map $\mathrm{ev}_\zeta^X(Z)$ at ζ is $\sum_{i=1}^n \mathrm{ev}_\zeta^X(Y_i)$, which is a bijection because, for every $i \in \{1,\ldots,n\}$, the evaluation set map $\mathrm{ev}_\zeta^X(Y_i)$ is a bijection by our previous argument for connected objects of $\mathcal{C}$ because Y_i is a connected object of $\mathcal{C}$. This completes the proof. □

Corollary 7.5.15. *Let X be a Galois element of the fiber functor F associated with a Galois category $\mathcal{C}$, and let $\mathcal{C}^X$ be the full subcategory of $\mathcal{C}$ of all objects Y of $\mathcal{C}$ such that $X \geq Y$ in $\mathcal{C}$. Then, the restriction $F|_{\mathcal{C}^X}$ of F to $\mathcal{C}^X$ is a representable functor.*

Lastly, we combine the cofinality result in corollary 7.5.12 with proposition 7.5.13 to conclude that:

Corollary 7.5.16. *The fiber functor F of a Galois category $\mathcal{C}$ with full subcategory $\mathcal{G}$ of the Galois objects of $\mathcal{C}$ satisfies the natural isomorphisms:*

$$F \cong \varinjlim_{X\in\mathcal{G}} \mathrm{Hom}\,(X,-) \cong \varinjlim_{X\in\mathcal{C}} \mathrm{Hom}\,(X,-)$$

7.6 The fundamental theorem of Galois category theory

The fundamental theorem of Galois category theory uses the fundamental group of a Galois category:

Definition 7.6.1 (Fundamental group). The **fundamental group** $\pi_1(\mathcal{C},F)$ of a Galois category $\mathcal{C}$ relative to its associated fiber functor F is the automorphism group of F:

$$\pi_1(\mathcal{C},F) := \mathrm{Aut}(F) := \left\{ t : F \xrightarrow{\cong} F \right\}$$

that is, $\pi_1(\mathcal{C},F)$ is the group of natural automorphisms of the fiber functor F.

We shall make reference to the following central result about the fundamental group of a Galois category:

Proposition 7.6.2. *The fundamental group $\pi_1(\mathcal{C}, F)$ of a Galois category $\mathcal{C}$ relative to its associated fiber functor F is a profinite group acting continuously on the finite set $F(X)$ for every object X of $\mathcal{C}$. In particular, denoting by $\mathcal{G}$ the full subcategory of $\mathcal{C}$ of its Galois objects, we have that $\pi_1(\mathcal{C}, F)$ is a profinite group as the inverse limit:*

$$\pi_1(\mathcal{C}, F) \cong \lim_{X \in \mathcal{G}} \operatorname{Aut}(X)$$

of the finite, by corollary 7.4.8, and discrete groups of automorphisms in $\mathcal{C}$ of the Galois objects of $\mathcal{C}$.

Remark 7.6.3. *The proof of proposition 7.6.2 makes use of the essential smallness in the definition of a Galois category, thus demystifying this requirement for a category to be Galois.*

We also state and prove the auxiliary lemma below:

Lemma 7.6.4. *Let Z be a non-initial Galois object, let X be a connected object, and let $f : Z \to X$ be a morphism in a Galois category $\mathcal{C}$ with associated fiber functor F. Under the right group action by pre-composition of the finite, by corollary 7.4.8, group $\operatorname{Aut}(Z)$ of automorphisms of Z in $\mathcal{C}$ on the finite, by corollary 7.4.8, Hom set $\operatorname{Hom}(Z, X)$ defined by $g \cdot \omega := g\omega$, let H denote the stabilizer of f. Then, we have $X \cong Z/H$ in $\mathcal{C}$.*

Remark 7.6.5. *The subgroup H of the finite, by corollary 7.4.8, group $\operatorname{Aut}(Z)$ of automorphisms of Z in $\mathcal{C}$ acts on Z by the symmetries of Z in $\mathcal{C}$, and the quotient Z/H by this finite group action exists in the Galois category $\mathcal{C}$.*

Proof. Because the finite group H is the stabilizer of f, there exists a unique morphism $\widetilde{f} : Z/H \to X$ in $\mathcal{C}$ such that the diagram in $\mathcal{C}$ below commutes:

$$\begin{array}{ccc} Z & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle q}\searrow \quad \nearrow{\scriptstyle \widetilde{f}} & \\ & Z/H & \end{array}$$

where $q : Z \to Z/H$ is the quotient morphism. By the first claim in lemma 7.3.7, because Z is a non-initial object of $\mathcal{C}$ and X is a connected object of $\mathcal{C}$, the morphism $f : Z \to X$ is a strict epimorphism in $\mathcal{C}$. Similarly, the uniquely induced morphism $\widetilde{f} : Z/H \to X$ is also a strict epimorphism in $\mathcal{C}$. Therefore, by corollary 7.2.9, it suffices to show that $\widetilde{f} : Z/H \to X$ is a monomorphism in $\mathcal{C}$, which is equivalent to showing that the set map of finite sets $F(\widetilde{f})$ is injective because the fiber functor F reflects all monomorphisms in $\mathcal{C}$ by lemma 7.2.8. Because the fiber functor F preserves all quotients by finite groups in $\mathcal{C}$, the image under F of the previous commutative diagram in $\mathcal{C}$ is the commutative diagram of set maps of finite sets below:

$$\begin{array}{ccc} F(Z) & \xrightarrow{\quad F(f) \quad} & F(X) \\ & {\scriptstyle p}\searrow \quad \nearrow{\scriptstyle F(\widetilde{f})} & \\ & F(Z)/H & \end{array}$$

where $p : F(Z) \to F(Z)/H$ is the quotient map, and both $F(f)$ and $F(\widetilde{f})$ are surjective because the fiber functor F preserves all strict epimorphisms in $\mathcal{C}$. Suppose that two elements ζ and ζ' of the finite set $F(Z)$ satisfy the equation $\left(F(\widetilde{f})\right)([\zeta]) = \left(F(\widetilde{f})\right)([\zeta'])$ in $F(X)$. Applying the second characterization of non-initial Galois objects in lemma 7.5.5 to Z, we find an automorphism ω of Z in $\mathcal{C}$ such that $\omega \cdot \zeta := (F(\omega))(\zeta) = \zeta'$ in $F(Z)$, and we compute that:

$$(\mathrm{ev}_\zeta(X))(f\omega) := (F(f\omega))(\zeta) = (F(f)F(\omega))(\zeta) = (F(f))(\zeta') = \left(F(\widetilde{f})p\right)(\zeta') =$$
$$= \left(F(\widetilde{f})\right)([\zeta']) = \left(F(\widetilde{f})\right)([\zeta]) = \left(F(\widetilde{f})p\right)(\zeta) = (F(f))(\zeta) =: (\mathrm{ev}_\zeta(X))(f)$$

Combining our above computation with the injectivity of the evaluation set map $\mathrm{ev}_\zeta(X)$ guaranteed to us by an application of lemma 7.4.7 to the connected object (Z, ζ) of the fiber functor F, we infer that $f\omega = f$, so the automorphism ω of Z in $\mathcal{C}$ is an element of the stabilizer H of f. At last, this implies $[\zeta] = [\omega \cdot \zeta] = [\zeta']$. □

We are now ready to state and prove the fundamental theorem of Galois category theory:

Theorem 7.6.6 (**Fundamental theorem of Galois category theory**). *If $\mathcal{C}$ is a Galois category with associated fiber functor F and full subcategory $\mathcal{G}$ of its Galois objects, then F induces an equivalence of categories $\widetilde{F} : \mathcal{C} \xrightarrow{\simeq} \pi_1(\mathcal{C}, F) - \mathrm{Set_f}$ from $\mathcal{C}$ to the Galois category $\pi_1(\mathcal{C}, F) - \mathrm{Set_f}$ of finite and discrete $\pi_1(\mathcal{C}, F)$-spaces[7] such that the diagram of functors below commutes:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad F \quad} & \mathrm{Set_f} \\ & {\scriptstyle \widetilde{F}\,\simeq}\searrow \qquad \nearrow{\scriptstyle \Lambda} & \\ & \pi_1(\mathcal{C}, F) - \mathrm{Set_f} & \end{array}$$

where the forgetful functor Λ is the fiber functor associated with the Galois category $\pi_1(\mathcal{C}, F) - \mathrm{Set_f}$.

Remark 7.6.7. *The fundamental theorem of Galois category theory states that the example of the category of finite and discrete G-spaces and G-equivariant continuous maps for a profinite group G, with which we associate the forgetful functor to the category $\mathrm{Set_f}$ of finite sets, is the essentially universal example of a Galois category.*

Proof. It is equivalent[8] to prove that the fiber functor F induces a functor $\widetilde{F} : \mathcal{C} \to \pi_1(\mathcal{C}, F) - \mathrm{Set_f}$ from $\mathcal{C}$ to the Galois category $\pi_1(\mathcal{C}, F) - \mathrm{Set_f}$ of finite and discrete $\pi_1(\mathcal{C}, F)$-spaces such that the diagram of functors below commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\quad F \quad} & \mathrm{Set_f} \\ & {\scriptstyle \widetilde{F}}\searrow \qquad \nearrow{\scriptstyle \Lambda} & \\ & \pi_1(\mathcal{C}, F) - \mathrm{Set_f} & \end{array}$$

where the forgetful functor Λ is the fiber functor associated with the Galois category $\pi_1(\mathcal{C}, F) - \mathrm{Set_f}$ and $\widetilde{F}$ is both essentially surjective[9] and fully faithful[10]. The commutativity of the above diagram of functors combined with the fact that Λ is a forgetful functor forces $\widetilde{F}$ to be defined as the fiber functor F is, and we prove that this functor $\widetilde{F} : \mathcal{C} \to \pi_1(\mathcal{C}, F) - \mathrm{Set_f}$ induced by F is essentially surjective, faithful, and full, in that order.

Essentially surjective: By the preservation of the initial object and all finite coproducts of $\mathcal{C}$ by F combined with proposition 7.3.8, our work reduces to considering a non-empty, connected, finite $\pi_1(\mathcal{C}, F)$-set E, where $\phi : \pi_1(\mathcal{C}, F) \to \mathrm{Sym}(E)$ is the group homomorphism from $\pi_1(\mathcal{C}, F)$ to the symmetric group $\mathrm{Sym}(E)$ of the finite set E defining the group action of the profinite group $\pi_1(\mathcal{C}, F)$ on E. We know that the kernel $\ker(\phi)$ of the group homomorphism ϕ is an open normal subgroup of the profinite group $\pi_1(\mathcal{C}, F)$, and that ϕ uniquely induces a group homomorphism $\widetilde{\phi} : \pi_1(\mathcal{C}, F) / \ker(\phi) \to \mathrm{Sym}(E)$ such that the diagram of group homomorphisms below commutes:

$$\begin{array}{ccc} \pi_1(\mathcal{C}, F) & \xrightarrow{\quad \phi \quad} & \mathrm{Sym}(E) \\ & {\scriptstyle q}\searrow \qquad \nearrow{\scriptstyle \widetilde{\phi}} & \\ & \pi_1(\mathcal{C}, F) / \ker(\phi) & \end{array}$$

[7] Recall that, by proposition 7.6.2, the fundamental group $\pi_1(\mathcal{C}, F)$ of $\mathcal{C}$ relative to F is profinite.

[8] A reference for the theorem in category theory we are employing here, which states that a functor is an equivalence of categories if and only if it is both essentially surjective and fully faithful, is [8], p. 31, theorem 1.5.9.

[9] An **essentially surjective functor** is a functor such that every object of its target category is isomorphic to an object in its image. For example, the forgetful functor from finite groups to non-empty finite sets, albeit not surjective on objects, is essentially surjective by virtue of finite cyclic groups alone, but the forgetful functor from finite groups to finite sets fails to be even essentially surjective due to the empty set.

[10] A **full functor** is a functor that is surjective on morphisms. A **faithful functor** is a functor that is injective on morphisms. A **fully faithful functor** is a functor that is both full and faithful, that is, a functor that is bijective on morphisms.

where q is the quotient group homomorphism and $\widetilde{\phi}$ is a group monomorphism. Since, by proposition 7.6.2, the fundamental group $\pi_1(\mathcal{C}, F)$ of $\mathcal{C}$ relative to F is profinite as the inverse limit:

$$\pi_1(\mathcal{C}, F) \cong \lim_{X \in \mathcal{G}} \mathrm{Aut}(X)$$

of the finite, by corollary 7.4.8, and discrete groups of automorphisms in $\mathcal{C}$ of the Galois objects of $\mathcal{C}$, we infer that there exists a Galois object Y of $\mathcal{C}$ such that:

$$\pi_1(\mathcal{C}, F)/\ker(\phi) \cong \underbrace{\mathrm{Aut}(Y)/\ker(\phi) \cong F(Y)/\ker(\phi)}_{\text{lemma 7.5.5, characterization 3}} \cong \underbrace{F(Y/\ker(\phi)) \cong \mathrm{Aut}(Y/\ker(\phi))}_{\text{lemma 7.5.5, characterization 3}}$$

At last, applying the orbit-stabilizer theorem for the transitive group action of $\pi_1(\mathcal{C}, F)/\ker(\phi) \cong \mathrm{Aut}(Y/\ker(\phi))$ on the non-empty finite $\pi_1(\mathcal{C}, F)$-set E to an element e of E, we conclude that:

$$E \approx \underbrace{\mathrm{Aut}(Y/\ker(\phi))/\mathrm{stab}(e) \approx F(Y/\ker(\phi))/\mathrm{stab}(e)}_{\text{lemma 7.5.5, characterization 3}} \approx F((Y/\ker(\phi))/\mathrm{stab}(e))$$

where $\mathrm{Aut}(Y/\ker(\phi))/\mathrm{stab}(e)$ denotes the finite set of left cosets of the subgroup $\mathrm{stab}(e)$ of $\mathrm{Aut}(Y/\ker(\phi))$, which need not be normal in $\mathrm{Aut}(Y/\ker(\phi))$.

Faithful: By proposition 7.3.8, our work reduces to considering two morphisms $f : X \to Y$ and $g : X \to Y$ in $\mathcal{C}$ with non-initial and connected source object X of $\mathcal{C}$ such that $F(f) = F(g)$. Because the fiber functor F reflects the initial object of $\mathcal{C}$ by lemma 7.2.8, we know there exists an element ζ of $F(X)$, so (X, ζ) is a connected element of F. We evaluate at ζ to compute that:

$$(\mathrm{ev}_\zeta(Y))(f) := \underbrace{(F(f))(\zeta) = (F(g))(\zeta)}_{F(f)=F(g)} =: (\mathrm{ev}_\zeta(Y))(g)$$

We combine the above computation with lemma 7.4.7 to conclude that $f = g$, as required.

Full: By proposition 7.3.8, our work reduces to considering a $\pi_1(\mathcal{C}, F)$-equivariant map of finite $\pi_1(\mathcal{C}, F)$-sets $\widetilde{u} : F(X) \to F(Y)$ where X and Y are non-initial and connected objects of $\mathcal{C}$. Because the fiber functor F reflects the initial object of $\mathcal{C}$ by lemma 7.2.8, we know that both $F(X)$ and $F(Y)$ are non-empty finite sets. By proposition 7.5.13, we may choose a Galois element (Z, ζ) of the fiber functor F such that $Z \geq X$ in $\mathcal{C}$ and the evaluation set map $\mathrm{ev}_\zeta(Y) : \mathrm{Hom}(Z, Y) \xrightarrow{\approx} F(Y)$ at ζ is a bijection of finite sets. Let $f : Z \to X$ be a morphism in $\mathcal{C}$ witnessing $Z \geq X$ in $\mathcal{C}$. By the first claim in lemma 7.3.7, f is a strict epimorphism in $\mathcal{C}$, so $F(f)$ is a surjective set map of finite sets because the fiber functor F preserves all strict epimorphisms in $\mathcal{C}$. We observe that $\widetilde{u}(F(f))(\zeta)$ is an element of the non-empty finite set $F(Y)$, and the evaluation set map $\mathrm{ev}_\zeta(Y)$ being a bijection of finite sets yields a unique morphism $f' : Z \to Y$ in $\mathcal{C}$ such that:

$$(\mathrm{ev}_\zeta(Y))(f') := \big(F(f')\big)(\zeta) = \widetilde{u}(F(f))(\zeta) \in F(Y)$$

Moreover, by lemma 7.6.4 and using the notation in the statement of lemma 7.6.4, we have $X \cong Z/H$ in $\mathcal{C}$. Because the fiber functor F preserves all quotients by finite groups in $\mathcal{C}$ and by the transitivity in characterization 3 in lemma 7.5.5 applied to the Galois object Z of $\mathcal{C}$, we obtain the commutative diagram of set maps of finite sets:

$$\begin{array}{ccc} F(Z) & \xrightarrow{F(f)} & F(Z)/H \\ & {\scriptstyle F(f')} \searrow & \downarrow {\scriptstyle \widetilde{u}} \\ & & F(Y) \end{array}$$

Then, for every element ω of the finite stabilizer $H := \mathrm{stab}(f)$ of f, we compute that:

$$F(f'\omega) = F(f')F(\omega) = \widetilde{u}F(f)F(\omega) = \underbrace{\widetilde{u}F(f\omega) = \widetilde{u}F(f)}_{\omega \in H := \mathrm{stab}(f)} = F(f')$$

Having established faithfulness, we infer that $f'\omega = f'$ for every element ω of H, so there exists a unique morphism $u : Z/H \to Y$ in $\mathcal{C}$ such that the diagram below commutes in $\mathcal{C}$:

$$\begin{array}{ccc} Z & \xrightarrow{f} & Z/H \\ & \searrow^{f'} & \downarrow{\scriptstyle u} \\ & & Y \end{array}$$

We apply the fiber functor F to the above commutative diagram in $\mathcal{C}$ to obtain the commutative diagram of set maps of finite sets below:

$$\begin{array}{ccc} F(Z) & \xrightarrow{F(f)} & F(Z)/H \\ & \searrow^{F(f')} & \downarrow{\scriptstyle F(u)} \\ & & F(Y) \end{array}$$

and we compute that $\widetilde{u}F(f) = F(f') = F(u)F(f)$. At last, because $F(f)$ is a surjective set map of finite sets, we conclude that $\widetilde{u} = F(u)$, as required. This completes the proof. $\square$

7.7 Example: Finite-sheeted covering spaces

We conclude our study of Galois category theory with expanding our discussion of the following example of a Galois category from section 1. If S is a path-connected, locally path-connected, and semi-locally simply connected pointed space - that is, a pointed space satisfying the hypotheses in the Galois correspondence theorem for covering spaces - with basepoint s_0, then the category $\mathrm{Cov}_{\mathrm{f}}^{S}$ of finite-sheeted covering spaces of S and finite-sheeted covering space maps over S is a Galois category whose associated fiber functor $F : \mathrm{Cov}_{\mathrm{f}}^{S} \to \mathrm{Set}_{\mathrm{f}}$ sends each finite-sheeted covering space of S to its finite fiber at s_0 and each finite-sheeted covering space map over S to its induced set map of finite fibers at s_0.

Given a finite-sheeted covering space $p_X : X \to S$ of S, we categorically describe the finite fiber $F(p_X) := p_X^{-1}(s_0)$ of p_X at s_0 by the Cartesian square below:

$$\begin{array}{ccc} F(p_X) & \xrightarrow{i} & X \\ {\scriptstyle g_{F(p_X)}}\downarrow & & \downarrow{\scriptstyle p_X} \\ * & \xrightarrow{s_0} & S \end{array}$$

where $g_{F(p_X)}$ is the terminal map of the finite fiber $F(p_X)$ and i is the inclusion of said finite fiber $F(p_X)$ in X. What is more, every finite-sheeted covering space map over S is locally determined near s_0 by its induced set map of finite fibers at s_0: given two finite-sheeted covering spaces $p_X : X \to S$ and $p_Y : Y \to S$ of S, a finite-sheeted covering space map $f : X \to Y$ over S from p_X to p_Y, and an open neighborhood U of S that contains s_0 and is evenly covered both by p_X and by p_Y, we have the commutative diagram of local triviality in U below:

$$\begin{array}{ccc} U \times F(p_X) & \xrightarrow{1_U \times F(f)} & U \times F(p_Y) \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle p_1'} & \\ & U & \end{array}$$

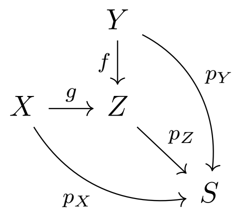
where p_1 and p_1' are the respective Cartesian product projections to the first coordinate, $F(p_X) := p_X^{-1}(s_0)$ is the finite fiber of p_X at s_0, $F(p_Y) := p_Y^{-1}(s_0)$ is the finite fiber of p_Y at s_0, and the set map $F(f)$ of finite fibers at s_0 induced by the finite-sheeted covering space map f over S locally determines f near s_0 in the precise sense of the above commutative diagram of continuous maps.

We gather some remarks on this example of a Galois category under the umbrella of the proposition below:

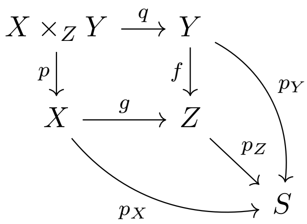
Proposition 7.7.1. *The category* $\mathrm{Cov}_{\mathrm{f}}^{S}$ *of finite-sheeted covering spaces of and finite-sheeted covering space maps over a path-connected, locally path-connected, and semi-locally simply connected pointed space* S *with basepoint* s_0 *is a Galois category whose associated fiber functor* $F : \mathrm{Cov}_{\mathrm{f}}^{S} \to \mathrm{Set}_{\mathrm{f}}$ *sends each finite-sheeted covering space of* S *to its finite fiber at* s_0 *and each finite-sheeted covering space map over* S *to its induced set map of finite fibers at* s_0*.*

Proof. After noting that $\mathrm{Cov}_{\mathrm{f}}^{S}$ is essentially small by an application of the Galois correspondence theorem that S is rigged by its assumed topological properties to satisfy, we gather some remarks on the satisfaction of each enumerated axiom in the definition of a Galois category with its associated fiber functor:

1. The terminal object of $\mathrm{Cov}_{\mathrm{f}}^{S}$ is the identity map $1_S : S \xrightarrow{=} S$ of S, a single-sheeted covering space of S. As for the construction of fiber products in $\mathrm{Cov}_{\mathrm{f}}^{S}$, given two finite-sheeted covering space maps over S with the same target:



   we form their fiber product in the continuous category Top of topological spaces:



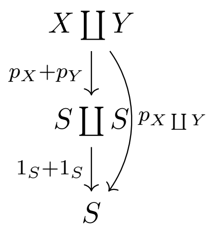
   and $p_{X \times_Z Y} := p_X p = p_Y q : X \times_Z Y \to S$ is another finite-sheeted covering space of S: given an open neighborhood U of S that is evenly covered by p_X, p_Y, and p_Z, we compute that:

$$p_{X\times_Z Y}^{-1}(U) \cong \underbrace{p_X^{-1}(U) \times_{p_Z^{-1}(U)} p_Y^{-1}(U) \cong (U \times F(p_X)) \times_{(U\times F(p_Z))} (U \times F(p_Y))}_{U \text{ is evenly covered}} \cong$$

$$\cong U \times \big(F(p_X) \times_{F(p_Z)} F(p_Y)\big)$$

   where the fiber product $F(p_X) \times_{F(p_Z)} F(p_Y)$ is a finite set because $F(p_X)$, $F(p_Y)$, and $F(p_Z)$ all are.

2. The initial object of $\mathrm{Cov}_{\mathrm{f}}^{S}$ is the initial map $f_S : \emptyset \to S$ of S, the unique zero-sheeted covering space of S. As for the construction of finite coproducts in $\mathrm{Cov}_{\mathrm{f}}^{S}$, given two finite-sheeted covering spaces $p_X : X \to S$ and $p_Y : Y \to S$ of S, we define $p_{X\coprod Y} : X \coprod Y \to S$ to be the composite of continuous maps below:



   where $1_S + 1_S$ is the folding map of S. This is another finite-sheeted covering space of S: given an open neighborhood U of S that is evenly covered both by p_X and by p_Y, we compute that:

$$p_{X\coprod Y}^{-1}(U) = \underbrace{p_X^{-1}(U) \coprod p_X^{-1}(U) \cong (U \times F(p_X)) \coprod (U \times F(p_Y))}_{U \text{ is evenly covered}} \cong U \times \Big(F(p_X) \coprod F(p_Y)\Big)$$

where the disjoint union $F(p_X) \coprod F(p_Y)$ is a finite set as a disjoint union of two finite sets. Lastly, as for the construction of quotients by finite groups in $\mathrm{Cov}_{\mathrm{f}}^{S}$, given a finite group G acting by deck transformations on a finite-sheeted covering space $p_X : X \to S$ of S, there exists a unique continuous map $p_{X/G} : X/G \to X$ such that the diagram of continuous maps below commutes:

$$\begin{array}{ccc} X & \xrightarrow{\ p_X\ } & S \\ & {\scriptstyle q}\searrow \quad \nearrow {\scriptstyle p_{X/G}} & \\ & X/G & \end{array}$$

where q is the orbit space map. This is another finite-sheeted covering space of S: given an open neighborhood U of S that is evenly covered by p_X, we compute that:

$$p_{X/G}^{-1}(U) = \underbrace{p_X^{-1}(U)/G \cong (U \times F(p_X))/G}_{U \text{ is evenly covered}} \cong U \times (F(p_X)/G)$$

where the quotient set $F(p_X)/G$ is a finite set as a quotient of a finite set.

3. A morphism in $\mathrm{Cov}_{\mathrm{f}}^{S}$ - that is, finite-sheeted covering space map over S - is a strict epimorphism in $\mathrm{Cov}_{\mathrm{f}}^{S}$ if and only if it is an epimorphism in $\mathrm{Cov}_{\mathrm{f}}^{S}$. Moreover, all finite coproduct morphisms in $\mathrm{Cov}_{\mathrm{f}}^{S}$ are finite disjoint union inclusions, thus injective, thus monomorphisms (left-cancellable) in $\mathrm{Cov}_{\mathrm{f}}^{S}$.

4. The fiber functor F preserves the terminal object of $\mathrm{Cov}_{\mathrm{f}}^{S}$ because $F(1_S) = \{s_0\}$, and it preserves all fiber products of $\mathrm{Cov}_{\mathrm{f}}^{S}$ by the following computation in our remark on axiom 1:

$$p_{X \times_Z Y}^{-1}(U) \cong U \times \left(F(p_X) \times_{F(p_Z)} F(p_Y)\right)$$

which implies $F(p_{X \times_Z Y}) \approx F(p_X) \times_{F(p_Z)} F(p_Y)$ in the category $\mathrm{Set}_{\mathrm{f}}$ of finite sets.

5. The fiber functor F preserves the initial object of $\mathrm{Cov}_{\mathrm{f}}^{S}$ because $F(f_S) = \emptyset$, it preserves all finite coproducts of $\mathrm{Cov}_{\mathrm{f}}^{S}$ by the following computation in our remark on axiom 2:

$$p_{X \coprod Y}^{-1}(U) \cong U \times \left(F(p_X) \coprod F(p_Y)\right)$$

which implies $F\left(p_{X \coprod Y}\right) \approx F(p_X) \coprod F(p_Y)$ in the category $\mathrm{Set}_{\mathrm{f}}$ of finite sets, and it preserves all quotients by finite groups in $\mathrm{Cov}_{\mathrm{f}}^{S}$ by the following computation in our remark on axiom 2:

$$p_{X/G}^{-1}(U) \cong U \times (F(p_X)/G)$$

which implies $F\left(p_{X/G}\right) \approx F(p_X)/G$ in the category $\mathrm{Set}_{\mathrm{f}}$ of finite sets.

6. Every (strict) epimorphism f in $\mathrm{Cov}_{\mathrm{f}}^{S}$ induces a surjective set map $F(f)$ of finite fibers at s_0.

7. If a finite-sheeted covering space map f over S induces a bijection $F(f)$ of finite fibers at s_0, then f is forced to be a homeomorphism over S, that is, an isomorphism in $\mathrm{Cov}_{\mathrm{f}}^{S}$. The converse implication holds in the greatest generality: every functor preserves all isomorphisms.

□

If S is a path-connected, locally path-connected, and semi-locally simply connected pointed space with basepoint s_0, then we know that S admits a universal cover $\widetilde{p} : \widetilde{S} \to S$, where 'universality' means that $\widetilde{S}$ is a simply connected space. Moreover, the Galois correspondence theorem applied to S informs us that the group $\mathrm{Aut}(\widetilde{p})$ of deck transformations of the universal cover $\widetilde{p}$ of S is isomorphic to the fundamental group $\pi_1(S)$ of the pointed space S at its basepoint s_0: $\mathrm{Aut}(\widetilde{p}) \cong \pi_1(S)$. We can say a bit more:

Proposition 7.7.2. *If S is a path-connected, locally path-connected, and semi-locally simply connected pointed space with basepoint s_0 and universal cover $\widetilde{p} : \widetilde{S} \to S$, and $\mathrm{Cov}_{\mathrm{f}}^{S}$ is the Galois category of finite-sheeted covering spaces of and finite-sheeted covering space maps over S, with associated fiber functor $F : \mathrm{Cov}_{\mathrm{f}}^{S} \to \mathrm{Set}_{\mathrm{f}}$ sending each finite-sheeted covering space of S to its finite fiber at s_0 and each finite-sheeted covering space map over S to its induced set map of finite fibers at s_0, then, for every finite-sheeted covering space p of S, we have the natural bijection of finite sets:*

$$F(p) \approx \mathrm{Hom}\left(\widetilde{p}, p\right)$$

in other words, we have the natural isomorphism $F \cong \mathrm{Hom}\left(\widetilde{p}, -\right)$ of set-valued functors from $\mathrm{Cov}_{\mathrm{f}}^{S}$, and the fiber functor F is representable with representing object[11] *$\widetilde{p}$ in $\mathrm{Cov}_{\mathrm{f}}^{S}$.*

An application of proposition 7.6.2 enables us to also describe the fundamental group $\pi_1\left(\mathrm{Cov}_{\mathrm{f}}^{S}, F\right)$ of the Galois category $\mathrm{Cov}_{\mathrm{f}}^{S}$ relative to its associated fiber functor F:

Proposition 7.7.3. *If S is a path-connected, locally path-connected, and semi-locally simply connected pointed space with basepoint s_0, and $\mathrm{Cov}_{\mathrm{f}}^{S}$ is the Galois category of finite-sheeted covering spaces of and finite-sheeted covering space maps over S, with associated fiber functor $F : \mathrm{Cov}_{\mathrm{f}}^{S} \to \mathrm{Set}_{\mathrm{f}}$ sending each finite-sheeted covering space of S to its finite fiber at s_0 and each finite-sheeted covering space map over S to its induced set map of finite fibers at s_0, then the profinite fundamental group $\pi_1\left(\mathrm{Cov}_{\mathrm{f}}^{S}, F\right)$ of $\mathrm{Cov}_{\mathrm{f}}^{S}$ relative to F is isomorphic to the profinite completion*[12] *$\widehat{\pi_1(S)}$ of the (discrete) fundamental group $\pi_1(S)$ of S at s_0:*

$$\pi_1\left(\mathrm{Cov}_{\mathrm{f}}^{S}, F\right) \cong \widehat{\pi_1(S)}$$

We combine proposition 7.7.3 with the fundamental theorem of Galois category theory to infer that:

Corollary 7.7.4. *If S is a path-connected, locally path-connected, and semi-locally simply connected pointed space with basepoint s_0, and $\mathrm{Cov}_{\mathrm{f}}^{S}$ is the Galois category of finite-sheeted covering spaces of and finite-sheeted covering space maps over S, with associated fiber functor $F : \mathrm{Cov}_{\mathrm{f}}^{S} \to \mathrm{Set}_{\mathrm{f}}$ sending each finite-sheeted covering space of S to its finite fiber at s_0 and each finite-sheeted covering space map over S to its induced set map of finite fibers at s_0, then F induces an equivalence of categories $\widetilde{F} : \mathrm{Cov}_{\mathrm{f}}^{S} \xrightarrow{\simeq} \widehat{\pi_1(S)} - \mathrm{Set}_{\mathrm{f}}$ from $\mathrm{Cov}_{\mathrm{f}}^{S}$ to the Galois category $\widehat{\pi_1(S)} - \mathrm{Set}_{\mathrm{f}}$ of finite and discrete $\widehat{\pi_1(S)}$-spaces such that the diagram of functors below commutes:*

$$\begin{array}{ccc}
\mathrm{Cov}_{\mathrm{f}}^{S} & \xrightarrow{\quad F \quad} & \mathrm{Set}_{\mathrm{f}} \\
 & {\scriptstyle \widetilde{F}\,\simeq}\searrow \qquad \nearrow{\scriptstyle \Lambda} & \\
 & \widehat{\pi_1(S)} - \mathrm{Set}_{\mathrm{f}} &
\end{array}$$

where the forgetful functor Λ is the fiber functor associated with the Galois category $\widehat{\pi_1(S)} - \mathrm{Set}_{\mathrm{f}}$.

Corollary 7.7.5. *If S is a path-connected, locally path-connected, and semi-locally simply connected pointed space with basepoint s_0, and $\mathrm{Cov}_{\mathrm{f}}^{S}$ is the Galois category of finite-sheeted covering spaces of and finite-sheeted covering space maps over S, with associated fiber functor $F : \mathrm{Cov}_{\mathrm{f}}^{S} \to \mathrm{Set}_{\mathrm{f}}$ sending each finite-sheeted covering space of S to its finite fiber at s_0 and each finite-sheeted covering space map over S to its induced set map of finite fibers at s_0, then, for every pair p_X and p_Y of finite-sheeted covering spaces of S, F induces a bijection of Hom sets:*

$$\widetilde{F}_{X,Y} : \mathrm{Hom}\left(p_X, p_Y\right) \xrightarrow{\approx} \mathrm{Hom}_{\widehat{\pi_1(S)}}\left(F(p_X), F(p_Y)\right)$$

[11] It is a consequence of the Yoneda lemma ([8], p. 57, theorem 2.2.4) that the representing object of a representable functor is unique up to isomorphism.

[12] The **profinite completion** of a discrete group G is the profinite group $\widehat{G}$ defined to be the inverse limit $\widehat{G} := \lim_{[G:N]<\infty} G/N$ over all finite and discrete quotients of G. By the universal property of inverse limits, there is always a canonical continuous group homomorphism $\tau_G : G \to \widehat{G}$ from a discrete group G to its profinite completion $\widehat{G}$. For example, the profinite completion $\widehat{\mathbb{Z}}$ of the discrete group $\mathbb{Z}$ is $\widehat{\mathbb{Z}} := \lim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z} \cong \prod_p \mathbb{Z}_p$, the product of the profinite group $\mathbb{Z}_p$ of the p-adic integers over all primes p.

where $\mathrm{Hom}_{\widehat{\pi_1(S)}}(F(p_X), F(p_Y))$ *is the Hom set of* $\widehat{\pi_1(S)}$*-equivariant continuous maps from the finite and discrete fiber* $F(p_X)$ *to the finite and discrete fiber* $F(p_Y)$*. In particular, for every finite-sheeted covering space* p_X *of* S*,* F *induces an isomorphism of endomorphism monoids:*

$$\widetilde{F}_{X,X} : \mathrm{End}\,(p_X) \xrightarrow{\cong} \mathrm{End}_{\widehat{\pi_1(S)}}\,(F(p_X))$$

We conclude this section with studying two different types of actions that arise in the Galois category $\mathrm{Cov}_{\mathrm{f}}^{S}$. If $p_X : X \to S$ is a finite-sheeted covering space of a path-connected, locally path-connected, and semi-locally simply connected pointed space S with basepoint s_0, then the group $\mathrm{Aut}\,(p_X)$ of deck transformations of p_X is a subgroup of the group $\mathrm{Homeo}\,(X)$ of self-homeomorphisms of X, thus continuously acting on the space X from the left by self-homeomorphisms of X. What is more, the fundamental group $\pi_1(S)$ of S acts on X from the right as follows. Let $\widetilde{x}$ be a point in X, write $x := p_X(\widetilde{x})$ in S, and let $[\gamma]$ be an element of the fundamental group $\pi_1(S)$ of S at x represented by a loop $\gamma : [0,1] \to S$ with $\gamma(0) = \gamma(1) = x$ in S. We equip the closed unit interval $[0,1]$ with the basepoint 0 and we write down the diagram of pointed continuous maps below:

$$\begin{array}{ccc} \{0\} & \xrightarrow{\widetilde{x}} & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p_X} \\ [0,1] & \xrightarrow{\gamma} & S \end{array}$$

where i denotes the pointed subspace inclusion. By the unique path lifting property of the finite-sheeted covering space p_X of S, we infer that the path γ admits a unique lift $\widetilde{\gamma} : [0,1] \to X$, which need not be a loop in X, such that $\widetilde{\gamma}(0) = \widetilde{x}$ in X in the precise sense of the commutative diagram of pointed continuous maps below:

$$\begin{array}{ccc} \{0\} & \xrightarrow{\widetilde{x}} & X \\ {\scriptstyle i}\downarrow & \overset{\widetilde{\gamma}}{\nearrow} & \downarrow{\scriptstyle p_X} \\ [0,1] & \xrightarrow{\gamma} & S \end{array}$$

At last, we describe a well-defined right action of the fundamental group $\pi_1(S)$ of S on X by:

$$\widetilde{x} \cdot [\gamma] := \widetilde{\gamma}(1) \in X$$

We verify that the left action of the group $\mathrm{Aut}\,(p_X)$ of deck transformations of p_X on X by self-homeomorphisms of X and the above right action of the fundamental group $\pi_1(S)$ of S on X commute. Given a deck transformation $\phi : X \xrightarrow{\cong} X$ of p_X, a point $\widetilde{x}$ of X with $x := p_X(\widetilde{x})$ in S, and an element $[\gamma]$ of the fundamental group $\pi_1(S)$ of S at x represented by a loop $\gamma : [0,1] \to S$ with $\gamma(0) = \gamma(1) = x$ in S, we write down the commutative diagram of pointed continuous maps below:

$$\begin{array}{ccccc} \{0\} & \xrightarrow{\widetilde{x}} & X & \xrightarrow{\phi\ \cong} & X \\ {\scriptstyle i}\downarrow & \overset{\widetilde{\gamma}}{\nearrow} & \downarrow{\scriptstyle p_X} & \swarrow{\scriptstyle p_X} & \\ [0,1] & \xrightarrow{\gamma} & S & & \end{array}$$

and we employ the uniqueness in the unique path lifting property of the finite-sheeted covering space p_X of S to compute in X that:

$$(\phi \cdot \widetilde{x}) \cdot [\gamma] := \underbrace{\phi\,(\widetilde{x}) \cdot [\gamma] := \phi\widetilde{\gamma}(1)}_{\text{uniqueness of the lift}} =: \phi \cdot (\widetilde{x} \cdot [\gamma])$$

Thus, in the precise sense of the above computation in X, the left action of the group $\mathrm{Aut}\,(p_X)$ of deck transformations of p_X on X by self-homeomorphisms of X and the right action of the fundamental group $\pi_1(S)$ of S on X commute. Lastly, note that:

1. the left action of the group $\mathrm{Aut}\,(p_X)$ of deck transformations of p_X on X by self-homeomorphisms of X restricts to a left action of the group $\mathrm{Aut}\,(p_X)$ of deck transformations of p_X on X on the finite fiber $F\,(p_X)$ of p_X at s_0, and

2. the right action of the fundamental group $\pi_1(S)$ of S on X also restricts to a right action of the fundamental group $\pi_1(S)$ of S at its basepoint s_0 on the finite fiber $F\,(p_X)$ of p_X at s_0.

7.8 Appendix on category theory

We explain some notions from category theory which arise in these notes, redirecting the reader interested in their comprehensive treatment or our presupposed background in category theory to the textbooks [7], [9], and [8] and the crash courses in category theory in chapter 1 of [10] and in section 2 of [5].

Although categorical definitions precede concrete examples in our exposition to render definitions easy and quick to find, the reader should keep in mind that, historically, categorical definitions formalize and generalize concrete examples in the world of sets, topological spaces, (abelian) groups, vector spaces and modules, and other ubiquitous mathematical structures whose study historically predates the advent of category theory, the brainchild of Samuel Eilenberg and Saunders Mac Lane, in the 1940s.

7.8.1 Initial, terminal, and zero objects

We begin with studying initial, terminal, and zero objects and various examples of such:

Definition 7.8.1 (**Initial object**)**.** An **initial object** of a category $\mathcal{C}$ is an object $\emptyset$ of $\mathcal{C}$ satisfying the universal property that, for every object X of $\mathcal{C}$, there exists a unique morphism $f_X : \emptyset \to X$ in $\mathcal{C}$.

Remark 7.8.2. *A category $\mathcal{C}$ need not have an initial object, and if it does, then it is unique up to unique isomorphism in $\mathcal{C}$, thus allowing us to speak of 'the' initial object of $\mathcal{C}$.*

Remark 7.8.3. *If $\emptyset$ is an initial object of $\mathcal{C}$ and $\mathcal{D}$ is a full subcategory[13] of $\mathcal{C}$ with $\emptyset$, then $\emptyset$ is an initial object of $\mathcal{D}$.*

Example 7.8.4. As our notation deliberately suggests, the empty set $\emptyset$ is the initial object of the category Set of sets, as well as its full subcategory Set_f of finite sets (unique not only up to set bijection, but actually on the nose in these special cases). However, its full subcategory $\text{Set}_{\geq 1}$ of non-empty sets is an example of a category with no initial object, reminding us that a category need not have one.

Example 7.8.5. The empty space $\emptyset$ is the initial object of the category Top of topological spaces, as well is every full subcategory of Top containing it, such as its full subcategory $\mathcal{T}_2$ of Hausdorff spaces[14], its full subcategory MS of metric spaces, or its full subcategory $\text{Top}_{\geq 2}$ of simply connected spaces (again, unique not only up to homeomorphism, but actually on the nose in these special cases).

Example 7.8.6. The singletons are precisely the initial objects of the category Top_* of pointed spaces and pointed continuous maps between them. Pointedness excludes $\emptyset$ from the objects of the category Top_*.

Example 7.8.7. The zero group 0 is the initial object of the category Grp of groups, as well as in its full subcategory Ab of abelian groups, its full subcategory Grp_f of finite groups, and its full subcategory Div of divisible abelian groups. However, its full subcategory $\text{Grp}_{\neq 0}$ of non-zero groups is an example of a category with no initial object.

Example 7.8.8. The zero module 0 is the initial object of the category R-Mod of left modules over a unital ring R, with the case $R = \mathbb{Z}$ recovering the above example in the category Ab of abelian groups.

[13] A **full** subcategory $\mathcal{G}$ of a category $\mathcal{C}$ is one such that, for every pair of objects X and Y of $\mathcal{G}$, we have $\text{Hom}_\mathcal{G}(X, Y) = \text{Hom}_\mathcal{C}(X, Y)$ rather than just $\text{Hom}_\mathcal{G}(X, Y) \subset \text{Hom}_\mathcal{C}(X, Y)$ - in other words, such that the inclusion functor $i : \mathcal{G} \to \mathcal{C}$ is full, hence the choice of terminology. For example, the category Ab of abelian groups is a full subcategory of the category Grp of groups, but not a full subcategory of the category Set of sets: the constant self-map of $\mathbb{Z}$ with constant value 1, albeit a perfectly valid set map, is not an abelian group endomorphism of $\mathbb{Z}$ because it fails to respect its additive identity element 0.

[14] The notation $\mathcal{T}_2$ follows the Kolmogorov classification of increasingly, index-wise, properly stricter separation axioms for topological spaces, where $\mathcal{T}$ stands for the German word 'Trennungsaxiom' literally translating to English to 'separation axiom'.

Example 7.8.9. The ring of integers $\mathbb{Z}$ is the initial object of the category CRing of commutative, unital rings and ring homomorphisms respecting the multiplicative unit.

Example 7.8.10. The initial object, if it exists, of the poset category associated with a poset is the minimal element of said poset. For example, the poset category $\mathbb{N}$ associated with the poset $(\mathbb{N}, \leq)$ with the usual partial order $\leq$ has (unique, on the nose) initial object 1, whereas the poset category $\mathbb{Z}$ associated with the poset $(\mathbb{Z}, \leq)$ with the usual partial order $\leq$ has no initial object because the poset $(\mathbb{Z}, \leq)$ has no minimal element. Moreover, for every set X, the poset category $\mathcal{P}(X)$ associated with the power set poset $(\mathcal{P}(X), \subset)$ with the subset partial order $\subset$ has (unique, on the nose) initial object $\emptyset$.

The dual notion to that of an initial object is that of a terminal object:

Definition 7.8.11 (Terminal object). A **terminal object** of a category $\mathcal{C}$ is an object $*$ of $\mathcal{C}$ satisfying the universal property that, for every object X of $\mathcal{C}$, there exists a unique morphism $g_X : X \to *$ in $\mathcal{C}$.

Remark 7.8.12. *A category $\mathcal{C}$ need not have a terminal object, and if it does, then it is unique up to unique isomorphism in $\mathcal{C}$, thus allowing us to speak of 'the' terminal object of $\mathcal{C}$. In particular, a category $\mathcal{C}$ having an initial object need not have a terminal object, and vice versa. Moreover, if $\mathcal{C}$ has both an initial and a terminal object, they need not be isomorphic in $\mathcal{C}$.*

Remark 7.8.13. *If $*$ is a terminal object of $\mathcal{C}$ and $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ with $*$, then $*$ is a terminal object of $\mathcal{D}$.*

Remark 7.8.14. *'Duality' precisely translates to the fact that, for a category $\mathcal{C}$:*

1. *$*$ being a terminal object of $\mathcal{C}$ is equivalent to $*$ being an initial object of the opposite category $\mathcal{C}^{\mathrm{op}}$ and*

2. *$\emptyset$ being an initial object of $\mathcal{C}$ is equivalent to $\emptyset$ being a terminal object of the opposite category $\mathcal{C}^{\mathrm{op}}$.*

Example 7.8.15. As our notation deliberately suggests, the singletons are precisely the terminal objects of the category Set of sets, as well as its full subcategory $\mathrm{Set}_{\mathrm{f}}$ of finite sets and its full subcategory $\mathrm{Set}_{\geq 1}$ of non-empty sets, the latter having no initial object. However, its full subcategory $\mathrm{Set}_{\neq 1}$ of non-singleton sets has the unique initial object $\emptyset$ but no terminal object.

Example 7.8.16. The singleton spaces are precisely the terminal objects of the category Top of topological spaces, as well as its full subcategory $\mathcal{T}_2$ of Hausdorff spaces, its full subcategory MS of metric spaces, and its full subcategory $\mathrm{Top}_{\geq 2}$ of simply connected spaces.

Example 7.8.17. The singleton spaces are precisely the terminal objects of the category Top_* of pointed spaces, coinciding with the initial objects in Top_*.

Example 7.8.18. The zero group 0 is the terminal object of the category Grp of groups, as well as its full subcategory Ab of abelian groups, its full subcategory $\mathrm{Grp}_{\mathrm{f}}$ of finite groups, and its full subcategory Div of divisible abelian groups, coinciding with the initial object in all the aforementioned categories. However, its full subcategory $\mathrm{Grp}_{\neq 0}$ of non-zero groups is an example of a category with no terminal or initial object.

Example 7.8.19. The zero module 0 is the terminal object of the category R-Mod of left modules over a unital ring R, coinciding with its initial object, with the case $R = \mathbb{Z}$ recovering the above example in the category Ab of abelian groups.

Example 7.8.20. The zero ring 0 is the terminal object of the category CRing of commutative, unital rings and ring homomorphisms respecting the multiplicative unit.

Example 7.8.21. The terminal object, if it exists, of the poset category associated with a poset is the maximal element of said poset. For example, neither the poset category $\mathbb{N}$ associated with the poset $(\mathbb{N}, \leq)$ with the usual partial order $\leq$ nor the poset category $\mathbb{Z}$ associated with the poset $(\mathbb{Z}, \leq)$ with the usual partial order $\leq$ has a terminal object because neither the poset $(\mathbb{N}, \leq)$ nor the poset $(\mathbb{Z}, \leq)$ has a maximal element. Moreover, for every set X, the poset category $\mathcal{P}(X)$ associated with the power set poset $(\mathcal{P}(X), \subset)$ with the subset partial order $\subset$ has (unique, on the nose) terminal object X.

The occasional coincidence of the initial and terminal object gives rise to the notion of a zero object:

Definition 7.8.22 (Zero object). A **zero object** of a category $\mathcal{C}$ is an object 0 that is both initial and terminal.

Remark 7.8.23. *A category $\mathcal{C}$ need not have a zero object, and if it does, then it is unique up to unique isomorphism in $\mathcal{C}$, thus allowing us to speak of 'the' zero object of $\mathcal{C}$. In particular, a category $\mathcal{C}$ having an initial object and a terminal object need not have a zero object, for the initial and the terminal object of $\mathcal{C}$ need not be isomorphic.*

Remark 7.8.24. *If 0 is a zero object of $\mathcal{C}$ and $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ with 0, then 0 is a zero object of $\mathcal{D}$.*

Remark 7.8.25. *A category $\mathcal{C}$ has a zero object if and only if $\mathcal{C}$ has both an initial and a terminal object and the two are isomorphic in $\mathcal{C}$.*

Example 7.8.26. As our notation deliberately suggests, the zero group 0 is the zero object of the category Grp of groups, as well as its full subcategory Ab of abelian groups, its full subcategory $\mathrm{Grp}_{\mathrm{f}}$ of finite groups, and its full subcategory Div of divisible abelian groups.

Example 7.8.27. The zero module 0 is the zero object of the category R-Mod of left modules over a unital ring R, with the case $R = \mathbb{Z}$ recovering the above example in the category Ab of abelian groups.

Example 7.8.28. The category Set of sets has the empty set as its (unique, on the nose) initial object and the singletons as its terminal objects, but Set has no zero object because no singleton can be in bijection with the empty set. Similarly, the category Top of topological spaces has the empty space as its (unique, on the nose) initial object and the singleton spaces as its terminal objects, but Top has no zero object because no singleton space can be in bijection with, let alone homeomorphic to the empty space.

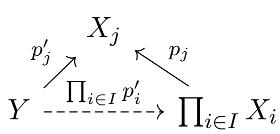
Example 7.8.29. The category CRing of commutative, unital rings has the ring of integers $\mathbb{Z}$ as its initial object and the zero ring 0 as its terminal object, but CRing has no zero object because the ring of integers $\mathbb{Z}$ is not even in bijection with, let alone isomorphic to the zero ring 0.

Example 7.8.30. The zero object, if it exists, of the poset category associated with a poset is simultaneously both the minimal and the maximal element of said poset. This forces our poset to be a 1-element poset, such as the power set of the empty set with the subset partial order, whose associated poset category consists of 1 object and its identity morphism.

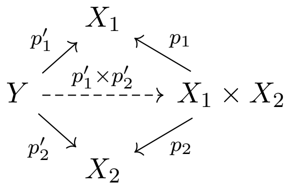
7.8.2 Products, coproducts, and connected objects

We proceed with studying products and coproducts and various examples of such:

Definition 7.8.31 (Product). A **product** of given objects $\{X_i\}_{i\in I}$ of a category $\mathcal{C}$, where I is an indexing set, is an object $\prod_{i\in I} X_i$ of $\mathcal{C}$ together with, for each $j \in I$, a morphism $p_j : \prod_{i\in I} X_i \to X_j$ satisfying the universal property that, for every object Y of $\mathcal{C}$ together with, for each $j \in I$, morphisms $p'_j : Y \to X_j$, there exists a unique morphism $\prod_{i\in I} p'_i : Y \to \prod_{i\in I} X_i$ such that, for all $j \in I$, the diagram below commutes in $\mathcal{C}$:



In the finite case $I = \{1, 2\}$, we obtain that a binary product of two given objects X_1 and X_2 of a category $\mathcal{C}$ is an object $X_1 \times X_2$ of $\mathcal{C}$ and two morphisms $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$ satisfying the universal property that, for every object Y of $\mathcal{C}$ together with two morphisms $p'_1 : Y \to X_1$ and $p'_2 : Y \to X_2$, there exists a unique morphism $p'_1 \times p'_2 : Y \to X_1 \times X_2$ such that the diagram below commutes in $\mathcal{C}$:



Remark 7.8.32. *A product of a given set, or even a pair, of objects of a category $\mathcal{C}$ need not exist in $\mathcal{C}$, and if it does, then it is unique up to canonical isomorphism in $\mathcal{C}$ provided by its universal property, thus allowing us to speak of 'the' product of said set, or pair, of objects.*

Remark 7.8.33. *In the degenerate case $I = \emptyset$, the definition of a product reduces to the definition of the terminal object of the category at hand $\mathcal{C}$.*

Remark 7.8.34. *In the degenerate case $I = \{1\}$, a product of a given object X is sensibly chosen to be X itself with its identity morphism $1_X : X \xrightarrow{=} X$.*

Remark 7.8.35. *By an induction argument, the existence of arbitrary finite products in a category $\mathcal{C}$ is equivalent to that of arbitrary binary products, but the latter does not generally imply the existence even of arbitrary countably infinite products, as we will discuss in the examples below.*

Remark 7.8.36. *The universal property of products extends to the categorical setting the following natural associativity and commutativity isomorphisms whenever both sides are defined in the category at hand:*

1. *Associativity:* $(X_1 \times X_2) \times X_3 \cong X_1 \times (X_2 \times X_3)$
2. *Commutativity:* $X_1 \times X_2 \cong X_2 \times X_1$

Moreover, if the category at hand possesses a terminal object $*$, then the universal property of products extends to the categorical setting the following natural unitality isomorphism whenever both sides are defined in the category at hand:

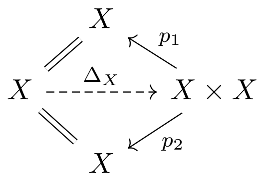
3. *Unitality:* $X \times * \cong X$

Combined with the commutativity isomorphism of products, the above isomorphism implies that $* \times X \cong X$ whenever both sides are defined in the category at hand.

Example 7.8.37. As our notation deliberately suggests, a product of given sets $\{X_i\}_{i \in I}$, where I is an indexing set, is their Cartesian product $\prod_{i \in I} X_i$ together with, for each $j \in I$, the j-th projection map $p_j : \prod_{i \in I} X_i \to X_j$. In the finite case $I = \{1, 2\}$, a binary product of given sets X_1 and X_2 is their Cartesian product $X_1 \times X_2$ together with the two projection maps $p_1 : X_1 \times X_2 \to X_1$ and $p_2 : X_1 \times X_2 \to X_2$. This construction justifies the notation used to denote products and works verbatim in the category Top of topological spaces and its full subcategories $\mathcal{T}_2$ of Hausdorff spaces, MS of metric spaces, and $\mathrm{Top}_{\geq 2}$ of simply connected spaces - by the fundamental group isomorphism $\pi_1 (X \times Y) \cong \pi_1 (X) \times \pi_1 (Y)$ - as well as in the category Top_* of pointed spaces, the category Grp of groups, the category Ab of abelian groups, the category R-Mod of left modules over a unital ring R, and the category CRing of commutative, unital rings, so all these categories have all products.

Example 7.8.38 (Diagonal morphisms). For every object X of a category $\mathcal{C}$ in which the binary product $X \times X$ exists, the universal property of products yields the **diagonal morphism** $\Delta_X :=$

$1_X \times 1_X : X \to X \times X$ as below:



When the binary product $X \times X$ is constructed by means of the Cartesian product construction in the previous example, the diagonal morphism $\Delta_X : X \to X \times X$ is the diagonal set map defined by $\Delta_X(x) := (x, x) \in X \times X$, justifying our terminology and notation for general diagonal morphisms.

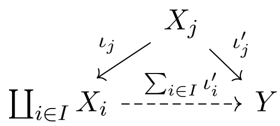
Example 7.8.39. If $\{f_i\}_{i\in I}$ are isomorphisms in a category $\mathcal{C}$, so is $\prod_{i\in I} f_i$, with $\left(\prod_{i\in I} f_i\right)^{-1} = \prod_{i\in I} f_i^{-1}$.

Example 7.8.40. The category $\mathrm{Set_f}$ of finite sets, albeit closed under all finite products by the Cartesian product construction, is not closed even under countably infinite products, for the countably infinite Cartesian product $\prod_{i\in\mathbb{N}} \{0,1\} =: 2^{\mathbb{N}} \approx \mathcal{P}(\mathbb{N})$, the set of binary sequences or, equivalently, the power set of the set of all natural numbers $\mathbb{N}$, lies outside $\mathrm{Set_f}$ because it is (countably) infinite. Similarly, the category $\mathbb{C}-\mathrm{Vect_{fd}}$ of finite-dimensional complex vector spaces, albeit closed under all finite products by the Cartesian product construction, is not closed even under countably infinite products, for the countably infinite Cartesian product $\prod_{i\in\mathbb{N}} \mathbb{C} =: \mathbb{C}^{\mathbb{N}}$ of complex sequences lies outside $\mathbb{C}-\mathrm{Vect_{fd}}$ as it is (countably) infinite-dimensional.

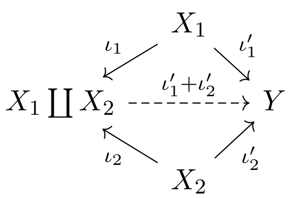
Example 7.8.41. The product, if it exists, of given objects $\{x_i\}_{i\in I}$, where I is an indexing set, in the poset category associated with a poset is the meet $\bigwedge_{i\in I} x_i$ of the elements $\{x_i\}_{i\in I}$ of said poset. For example, for every set X, the poset category $\mathcal{P}(X)$ associated with the power set poset $(\mathcal{P}(X), \subset)$ with the subset partial order $\subset$ has all products because the poset $(\mathcal{P}(X), \subset)$ has all meets, which are intersections. However, the poset category $\mathbb{Z}$ associated with the poset $(\mathbb{Z}, \leq)$ with the usual partial order $\leq$, even though it has all finite products because it has all finite meets, does not have the (countably) infinite product $\prod_{n\in\mathbb{Z}} n$ because the poset $(\mathbb{Z}, \leq)$ has no minimal element.

The dual notion to that of a product is that of a coproduct:

Definition 7.8.42 (Coproduct). A **coproduct** of given objects $\{X_i\}_{i\in I}$ of a category $\mathcal{C}$, where I is an indexing set, is an object $\coprod_{i\in I} X_i$ of $\mathcal{C}$ together with, for each $j \in I$, a morphism $\iota_j : X_j \to \coprod_{i\in I} X_i$ satisfying the universal property that, for every object Y of $\mathcal{C}$ together with, for each $j \in I$, morphisms $\iota'_j : X_j \to Y$, there exists a unique morphism $\sum_{i\in I} \iota'_i : \coprod_{i\in I} X_i \to Y$ such that, for all $j \in I$, the diagram below commutes:



In the finite case $I = \{1, 2\}$, we obtain that a binary coproduct of two given objects X_1 and X_2 of a category $\mathcal{C}$ is an object $X_1 \coprod X_2$ of $\mathcal{C}$ and two morphisms $\iota_1 : X_1 \to X_1 \coprod X_2$ and $\iota_2 : X_2 \to X_1 \coprod X_2$ satisfying the universal property that, for every object Y of $\mathcal{C}$ together with two morphisms $\iota'_1 : X_1 \to Y$ and $\iota'_2 : X_2 \to Y$, there exists a unique morphism $\iota'_1 + \iota'_2 : X_1 \coprod X_2 \to Y$ such that the diagram below commutes in $\mathcal{C}$:



Remark 7.8.43. *A coproduct of a given set, or even a pair, of objects of a category $\mathcal{C}$ need not exist in $\mathcal{C}$, and if it does, then it is unique up to canonical isomorphism in $\mathcal{C}$ provided by its universal property, thus allowing us to speak of 'the' coproduct of said set, or pair, of objects.*

Remark 7.8.44. *In the degenerate case $I = \emptyset$, the definition of a coproduct reduces to the definition of the initial object of the category at hand $\mathcal{C}$.*

Remark 7.8.45. *In the degenerate case $I = \{1\}$, a coproduct of a given object X is sensibly chosen to be X itself with its identity morphism $1_X : X \xrightarrow{=} X$.*

Remark 7.8.46. *By an induction argument, the existence of arbitrary finite coproducts in a category $\mathcal{C}$ is equivalent to that of arbitrary binary coproducts, but the latter does not generally imply the existence even of arbitrary countably infinite coproducts, as we will discuss in the examples below.*

Remark 7.8.47. *The universal property of coproducts extends to the categorical setting the following natural associativity and commutativity isomorphisms whenever both sides are defined in the category at hand:*

1. *Associativity:* $(X_1 \coprod X_2) \coprod X_3 \cong X_1 \coprod (X_2 \coprod X_3)$
2. *Commutativity:* $X_1 \coprod X_2 \cong X_2 \coprod X_1$

Moreover, if the category at hand possesses an initial object $\emptyset$, then the universal property of coproducts extends to the categorical setting the following natural unitality isomorphism whenever both sides are defined in the category at hand:

3. *Unitality:* $X \coprod \emptyset \cong X$

Combined with the commutativity isomorphism of coproducts, the above isomorphism implies that $\emptyset \coprod X \cong X$ whenever both sides are defined in the category at hand.

Remark 7.8.48. *'Duality' precisely translates to the fact that:*

1. *a product in a category $\mathcal{C}$ is equivalent to a coproduct in the opposite category $\mathcal{C}^{\mathrm{op}}$ and*
2. *a coproduct in a category $\mathcal{C}$ is equivalent to a product in $\mathcal{C}^{\mathrm{op}}$.*

Example 7.8.49. As our notation deliberately suggests, a coproduct of given sets $\{X_i\}_{i \in I}$, where I is an indexing set, is their disjoint union $\coprod_{i \in I} X_i$ together with, for each $j \in I$, the j-th inclusion map $\iota_j : X_j \to \coprod_{i \in I} X_i$. In the finite case $I = \{1, 2\}$, a binary coproduct of given sets X_1 and X_2 is their disjoint union $X_1 \coprod X_2$ together with the two inclusion maps $\iota_1 : X_1 \to X_1 \coprod X_2$ and $\iota_2 : X_2 \to X_1 \coprod X_2$. This construction justifies the notation used to denote coproducts and works verbatim in the category Top of topological spaces.

Example 7.8.50. In the category Top_* of pointed spaces, a coproduct of given pointed spaces $\{X_i\}_{i \in I}$, where I is an indexing set, is their wedge sum $\bigvee_{i \in I} X_i$ together with, for each $j \in I$, the j-th inclusion map $\iota_j : X_j \to \bigvee_{i \in I} X_i$. In the finite case $I = \{1, 2\}$, a binary coproduct of given pointed spaces X_1 and X_2 is their disjoint union $X_1 \vee X_2$ together with the two inclusion maps $\iota_1 : X_1 \to X_1 \vee X_2$ and $\iota_2 : X_2 \to X_1 \vee X_2$. The discrepancy between the construction of coproducts in the category Top and that in the category Top_* is due to the fact that spaces in Top_* are equipped with a designated basepoint each, and continuous maps in Top_* all are also required to preserve said basepoints.

Example 7.8.51. In the category Ab of abelian groups, a coproduct of given abelian groups $\{X_i\}_{i \in I}$, where I is an indexing set, is their direct sum $\bigoplus_{i \in I} X_i$ together with, for each $j \in I$, the j-th inclusion map $\iota_j : X_j \to \bigoplus_{i \in I} X_i$. In the finite case $I = \{1, 2\}$, a binary coproduct of given abelian groups X_1 and X_2 is their direct sum $X_1 \oplus X_2$ together with the two inclusion maps $\iota_1 : X_1 \to X_1 \oplus X_2$ and $\iota_2 : X_2 \to X_1 \oplus X_2$. This construction generalizes verbatim to the category R-Mod of left modules over a unital ring R.

Example 7.8.52. In the category Grp of groups, a coproduct of given groups $\{X_i\}_{i\in I}$, where I is an indexing set, is their free product $*_{i\in I}X_i$ together with, for each $j \in I$, the j-th inclusion map $\iota_j : X_j \to *_{i\in I}X_i$. In the finite case $I = \{1, 2\}$, a binary coproduct of given groups X_1 and X_2 is their free product $X_1 * X_2$ together with the two inclusion maps $\iota_1 : X_1 \to X_1 * X_2$ and $\iota_2 : X_2 \to X_1 * X_2$.

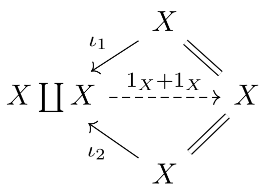
Note that the binary coproduct of $\mathbb{Z}/2\mathbb{Z}$ with itself as an abelian group is the Klein 4-group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, whereas the binary coproduct of $\mathbb{Z}/2\mathbb{Z}$ with itself as a group is the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$, and:

$$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$

as groups because the Klein 4-group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is abelian and has 4 elements, whereas the free product $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is non-abelian and infinite!

Example 7.8.53. In the category CRing of commutative, unital rings, a coproduct of given commutative rings groups $\{X_i\}_{i\in I}$, where I is an indexing set, is their tensor product $\bigotimes_{i\in I} X_i$ together with, for each $j \in I$, the j-th structure map $\iota_j : X_j \to \bigotimes_{i\in I} X_i$. In the finite case $I = \{1, 2\}$, a binary coproduct of given commutative rings X_1 and X_2 is their tensor product $X_1 \otimes X_2$ together with the two structure maps $\iota_1 : X_1 \to X_1 \otimes X_2$ and $\iota_2 : X_2 \to X_1 \otimes X_2$.

Example 7.8.54 (Folding morphisms). For every object X of a category $\mathcal{C}$ in which the binary coproduct $X \coprod X$ exists, the universal property of coproducts yields the **folding morphism** $1_X + 1_X : X \coprod X \to X$ as below:



In the category Top of topological spaces, the folding morphism $1_X + 1_X : X \coprod X \to X$ of a space X is the folding map which folds both disjoint copies of X to X, justifying our terminology for general folding morphisms.

Example 7.8.55. If $\{f_i\}_{i\in I}$ are isomorphisms in a category $\mathcal{C}$, so is $\sum_{i\in I} f_i$, with $\left(\sum_{i\in I} f_i\right)^{-1} = \sum_{i\in I} f_i^{-1}$.

Example 7.8.56 (Connected objects). A **connected object** in a category $\mathcal{C}$ with an initial object $\emptyset$ is an object X of $\mathcal{C}$ such that, if we have $X \cong Y \coprod Z$ in $\mathcal{C}$, then we must have (i) $X \cong Y$ and $Z \cong \emptyset$ in $\mathcal{C}$ or (ii) $X \cong Z$ and $Y \cong \emptyset$ in $\mathcal{C}$.

Initial objects are connected. In the category Set of sets, the connected objects are precisely the empty set and all singleton sets. In the category Top of spaces, a connected object is precisely a connected space, justifying our terminology.

Example 7.8.57. The category $\mathrm{Set_f}$ of finite sets, albeit closed under all finite coproducts by the disjoint union construction, is not closed even under countably infinite coproducts, for the countably infinite disjoint union $\coprod_{i\in\mathbb{N}} \{0, 1\}$ lies outside $\mathrm{Set_f}$ because it is (countably) infinite. Similarly, the category $\mathbb{C}-\mathrm{Vect_{fd}}$ of finite-dimensional complex vector spaces, albeit closed under all finite coproducts by the direct sum construction, is not closed even under countably infinite coproducts, for the countably infinite direct sum $\bigoplus_{i\in\mathbb{N}} \mathbb{C} =: \mathbb{C}^{\oplus\mathbb{N}}$ of complex sequences with finitely many non-zero entries, which is a proper complex subspace of $\mathbb{C}^{\mathbb{N}}$, lies outside $\mathbb{C}-\mathrm{Vect_{fd}}$ as it is (countably) infinite-dimensional.

Example 7.8.58. The coproduct, if it exists, of given objects $\{x_i\}_{i\in I}$, where I is an indexing set, in the poset category associated with a poset is the join $\bigvee_{i\in I} x_i$ of the elements $\{x_i\}_{i\in I}$ of said poset. For example, for every set X, the poset category $\mathcal{P}(X)$ associated with the power set poset $(\mathcal{P}(X), \subset)$ with the subset partial order $\subset$ has all coproducts because the poset $(\mathcal{P}(X), \subset)$ has all joins, which are unions. However, the poset category $\mathbb{Z}$ associated with the poset $(\mathbb{Z}, \leq)$ with the usual partial order $\leq$, even though it has all finite coproducts because it has all finite joins, does not have the (countably) infinite coproduct $\coprod_{n\in\mathbb{Z}} n$ because the poset $(\mathbb{Z}, \leq)$ has no maximal element.

7.8.3 Equalizers and co-equalizers

We proceed with studying equalizers and co-equalizers and various examples of such:

Definition 7.8.59 (**Equalizer**)**.** An **equalizer** of a pair of morphisms with the same source and target $h : X \to Y$ and $h' : X \to Y$ in a category $\mathcal{C}$ is a morphism $f : \mathrm{eq}(h, h') \to X$ in $\mathcal{C}$ satisfying the universal property that:

1. the diagram below commutes in $\mathcal{C}$:

$$\mathrm{eq}(h, h') \xrightarrow{f} X \underset{h'}{\overset{h}{\rightrightarrows}} Y$$

   that is, $hf = h'f$, and

2. if $g : Q \to X$ is a morphism in $\mathcal{C}$ such that the diagram below commutes in $\mathcal{C}$:

$$\begin{array}{ccccc} Q & \xrightarrow{g} & X & \underset{h'}{\overset{h}{\rightrightarrows}} & Y \\ & & \nearrow_{f} & & \\ \mathrm{eq}(h, h') & & & & \end{array}$$

   that is, such that $hg = h'g$, then there exists a unique morphism $\phi : Q \to \mathrm{eq}(h, h')$ in $\mathcal{C}$ such that the diagram below commutes in $\mathcal{C}$:

$$\begin{array}{ccccc} Q & \xrightarrow{g} & X & \underset{h'}{\overset{h}{\rightrightarrows}} & Y \\ {\scriptstyle \phi}\downarrow & \nearrow_{f} & & & \\ \mathrm{eq}(h, h') & & & & \end{array}$$

Remark 7.8.60. *An equalizer of a pair of morphisms with the same source and target in $\mathcal{C}$ need not exist in $\mathcal{C}$, and if it does, then it is unique up to canonical isomorphism in $\mathcal{C}$ provided by its universal property, thus allowing us to speak of 'the' equalizer of said pair of morphisms.*

Remark 7.8.61. *In the degenerate case $h = h'$, an equalizer of $h = h'$ is sensibly chosen to be the identity morphism $1_X : X \xrightarrow{=} X$. For example, only such degenerate cases occur in a poset category associated with a poset.*

Example 7.8.62. In the category Set of sets, given an arbitrary pair of set maps with the same source and target $h : X \to Y$ and $h' : X \to Y$, we construct their equalizer $\mathrm{eq}(h, h')$ by defining:

$$\mathrm{eq}(h, h') := \big\{x \in X : h(x) = h'(x) \in Y\big\} \subset X$$

a subset of X, and $f : \mathrm{eq}(h, h') \to X$ to be the subset inclusion. This construction works verbatim in the category $\mathrm{Set}_{\mathrm{f}}$ of finite sets, the category Top of topological spaces, the category Grp of groups, the category Ab of abelian groups, the category R-Mod of left modules over a unital ring R, and the category CRing of commutative, unital rings, so all the aforementioned categories have all equalizers.

Example 7.8.63. In the category Set of sets, given a pair of set maps with the same source and target $h : X \to Y$ and $h' : X \to Y$ where Y is non-empty and $h'(x) := y \in Y$ for all $x \in X$, by the above explicit construction, their equalizer is the fiber $h^{-1}(y)$ of $y \in Y$. This example works verbatim in the category $\mathrm{Set}_{\mathrm{f}}$ of finite sets, the category Top of topological spaces, and the category Top_* of pointed spaces, but only works to categorically describe kernels in Grp, Ab, and R-Mod for every unital ring R as group homomorphisms and (left) R-linear maps both must preserve identity elements.

Example 7.8.64. In the category Set of sets, given a self-map $h : X \to X$, by the above explicit construction, the equalizer of h and the identity map 1_X is the fixed-point subset F_h of X. This example works verbatim in the category $\mathrm{Set_f}$ of finite sets, the category Top of topological spaces, the category Top_* of pointed spaces, the category Grp of groups, the category Ab of abelian groups, the category R-Mod of left modules over a unital ring R, and the category CRing of commutative, unital rings.

The dual notion to that of an equalizer is that of a co-equalizer:

Definition 7.8.65 (Co-equalizer). A **co-equalizer** of a pair of morphisms with the same source and target $h : X \to Y$ and $h' : X \to Y$ in a category $\mathcal{C}$ is a morphism $f : Y \to \mathrm{coeq}(h, h')$ in $\mathcal{C}$ satisfying the universal property that:

1. the diagram below commutes in $\mathcal{C}$:

$$X \overset{h}{\underset{h'}{\rightrightarrows}} Y \xrightarrow{f} \mathrm{coeq}(h, h')$$

   that is, $fh = fh'$, and

2. if $g : Y \to Q$ is a morphism in $\mathcal{C}$ such that the diagram below commutes in $\mathcal{C}$:

$$\begin{array}{ccccc} X & \overset{h}{\underset{h'}{\rightrightarrows}} & Y & \xrightarrow{f} & \mathrm{coeq}(h, h') \\ & & & {\scriptstyle g}\searrow & \\ & & & & Q \end{array}$$

   that is, such that $gh = gh'$, then there exists a unique morphism $\phi : \mathrm{coeq}(h, h') \to Q$ in $\mathcal{C}$ such that the diagram below commutes in $\mathcal{C}$:

$$\begin{array}{ccccc} X & \overset{h}{\underset{h'}{\rightrightarrows}} & Y & \xrightarrow{f} & \mathrm{coeq}(h, h') \\ & & & {\scriptstyle g}\searrow & \Big\downarrow{\scriptstyle \phi} \\ & & & & Q \end{array}$$

Remark 7.8.66. *A co-equalizer of a pair of morphisms with the same source and target in $\mathcal{C}$ need not exist in $\mathcal{C}$, and if it does, then it is unique up to canonical isomorphism in $\mathcal{C}$ provided by its universal property, thus allowing us to speak of 'the' co-equalizer of said pair of morphisms.*

Remark 7.8.67. *In the degenerate case $h = h'$, a co-equalizer of $h = h'$ is sensibly chosen to be the identity morphism $1_Y : Y \xrightarrow{=} Y$. For example, only such degenerate cases occur in a poset category associated with a poset.*

Remark 7.8.68. *'Duality' precisely translates to the fact that:*

1. *an equalizer in a category $\mathcal{C}$ is equivalent to a co-equalizer in the opposite category $\mathcal{C}^{\mathrm{op}}$ and*

2. *a co-equalizer in a category $\mathcal{C}$ is equivalent to an equalizer in the opposite category $\mathcal{C}^{\mathrm{op}}$.*

Example 7.8.69. In the category Set of sets, given an arbitrary pair of set maps with the same source and target $h : X \to Y$ and $h' : X \to Y$, we construct their co-equalizer $\mathrm{coeq}(h, h')$ by defining:

$$\mathrm{coeq}(h, h') := \frac{Y}{h(x) \sim h'(x),\ x \in X}$$

a quotient of Y, and $f : Y \to \mathrm{coeq}(h, h')$ to be the quotient map. This construction works verbatim in the category $\mathrm{Set_f}$ of finite sets and the category Top of topological spaces. Mutatis mutandis, it also works in the category Grp of groups by defining the group:

$$\mathrm{coeq}(h, h') := \frac{Y}{\langle\langle h(x)\,(h'(x))^{-1} : x \in X \rangle\rangle}$$

quotienting by the suitable normal closure, as well as in the category Ab of abelian groups and, more generally, the category R-Mod of left modules over a unital ring R by defining:

$$\mathrm{coeq}(h, h') := \frac{Y}{\langle h(x) - h'(x) : x \in X \rangle}$$

quotienting by the suitable subspace, and the category CRing of commutative, unital rings by defining:

$$\mathrm{coeq}(h, h') := \frac{Y}{\langle h(x) - h'(x) : x \in X \rangle}$$

quotienting by the suitable ideal. Thus, Set, $\mathrm{Set_f}$, Top, Grp, Ab, R-Mod, and CRing all have all co-equalizers.

7.8.4 Fiber products (pullbacks) and pushouts

We proceed with studying fiber products (pullbacks) and pushouts and various examples of such:

Definition 7.8.70 (Fiber product (Pullback)). A **fiber product** or **pullback** of a pair of morphisms in a category $\mathcal{C}$ with the same target:

$$\begin{array}{ccc} & & A \\ & & \Big\downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & B \end{array}$$

is an object $A \times_B C$ of $\mathcal{C}$ together with two morphisms $q : A \times_B C \to A$ (**base change** of g) and $p : A \times_B C \to C$ (**base change** of f) in $\mathcal{C}$ satisfying the universal property that:

1. the **pullback square**, or **Cartesian square**, below commutes in $\mathcal{C}$:

$$\begin{array}{ccc} A \times_B C & \xrightarrow{q} & A \\ {\scriptstyle p}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & B \end{array}$$

2. and, for every commutative diagram in $\mathcal{C}$ as below:

$$\begin{array}{ccc} D & \xrightarrow{q'} & A \\ {\scriptstyle p'}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & B \end{array}$$

there exists a unique morphism $\omega : D \to A \times_B C$ such that the diagram below commutes in $\mathcal{C}$:

$$\begin{array}{cccc} D & & & \\ & \overset{\omega}{\searrow} & \overset{q'}{\longrightarrow} & \\ & A \times_B C & \xrightarrow{p} & A \\ {\scriptstyle p'}\searrow & {\scriptstyle q}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ & C & \xrightarrow{g} & B \end{array}$$

Remark 7.8.71. *A fiber product of a pair of morphisms in $\mathcal{C}$ with the same target need not exist in $\mathcal{C}$, and if it does, then it is unique up to canonical isomorphism in $\mathcal{C}$ provided by its universal property, thus allowing us to speak of 'the' fiber product of said pair of morphisms.*

Remark 7.8.72. *Although the 'pullback' term is more prevalent in topology, the 'fiber product' term is more prevalent in algebraic geometry, so we mostly follow the latter terminology for the purposes of the seminar.*

Example 7.8.73. In the category Set of sets, given an arbitrary pair of set maps with the same target:

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & B \end{array}$$

we construct the fiber product $A \times_B C$ by defining:

$$A \times_B C := \{(a, c) \in A \times C : f(a) = g(c) \in B\} = \bigcup_{b \in B} f^{-1}(b) \times g^{-1}(b) \subset A \times C$$

and taking $q : A \times_B C \to A$ and $p : A \times_B C \to A$ to be the restrictions to the fiber product $A \times_B C$ of the respective product maps. This construction justifies the notation and terminology for fiber products and works verbatim in the category $\mathrm{Set_f}$ of finite sets, the category Top of topological spaces, the category Top_* of pointed spaces, the category Grp of groups, the category Ab of abelian groups, the category R-Mod of left modules over a unital ring R, and the category CRing of commutative, unital rings, all of which have all fiber products.

Example 7.8.74. In a category $\mathcal{C}$ with a terminal object $*$, the fiber product of the pair of terminal object morphisms:

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle g_A} \\ B & \xrightarrow{g_B} & * \end{array}$$

if it exists in $\mathcal{C}$, is precisely the binary product $A \times B$ of A and B. This applies to the category Set of sets, the category $\mathrm{Set_f}$ of finite sets, the category Top of topological spaces, the category Top_* of pointed spaces, the category Grp of groups, the category Ab of abelian groups, the category R-Mod of left modules over a unital ring R, and the category CRing of commutative, unital rings.

Example 7.8.75. In a category $\mathcal{C}$ having two morphisms $h : X \to Y$ and $h' : X \to Y$ and the binary product $Y \times Y$, the fiber product:

$$\begin{array}{ccc} & & X \\ & & \downarrow{\scriptstyle h \times h'} \\ Y & \xrightarrow{\Delta_Y} & Y \times Y \end{array}$$

if it exists in $\mathcal{C}$, is precisely the equalizer of the pair h and h'. This applies to the category Set of sets, the category $\mathrm{Set_f}$ of finite sets, the category Top of topological spaces, the category Top_* of pointed spaces, the category Grp of groups, the category Ab of abelian groups, the category R-Mod of left modules over a unital ring R, and the category CRing of commutative, unital rings.

Example 7.8.76. In the category Set of sets, the fiber product of the pair of set maps with the same target:

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle f} \\ * & \xrightarrow{g} & B \end{array}$$

where B is non-empty and $g(*) := b \in B$, is, by the above explicit construction, (sensibly identified with) the fiber $f^{-1}(b)$ of $b \in B$. This example works verbatim in the category $\mathrm{Set_f}$ of finite sets, the category Top of topological spaces, and the category Top_* of pointed spaces, but only works to categorically describe kernels in Grp, Ab, and R-Mod for every unital ring R as group homomorphisms and (left) R-linear maps both must preserve identity elements.

Example 7.8.77. In the category Set of sets, the fiber product of the pair of subset inclusions:

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ C & \longrightarrow & B \end{array}$$

is, by the above explicit construction, (sensibly identified with) the intersection $A \cap C$. This example works verbatim in the category $\mathrm{Set}_{\mathrm{f}}$ of finite sets, the category Top of topological spaces, the category Top_* of pointed spaces, the category Grp of groups, the category Ab of abelian groups, the category R-Mod of left modules over a unital ring R, and the category CRing of commutative, unital rings.

Example 7.8.78. In the poset category associated with a poset, the fiber product of:

$$\begin{array}{ccc} & & a \\ & & \downarrow \\ c & \longrightarrow & b \end{array}$$

is the meet $a \wedge c$ of the elements a and c of said poset. For example, for every set X, the poset category $\mathcal{P}(X)$ associated with the power set poset $(\mathcal{P}(X), \subset)$ with the subset partial order $\subset$ has all fiber products because the poset $(\mathcal{P}(X), \subset)$ has all binary meets, which are binary intersections. Moreover, the poset category $\mathbb{Z}$ associated with the poset $(\mathbb{Z}, \leq)$ with the usual partial order $\leq$ has all fiber products because the poset $(\mathbb{Z}, \leq)$ has all binary meets, as does the poset category $\mathbb{N}$ associated with the poset $(\mathbb{N}, \leq)$ with the partial order $\leq$.

The dual notion to that of a fiber product is that of a pushout:

Definition 7.8.79 (Pushout). A **pushout** of a pair of morphisms in a category $\mathcal{C}$ with the same source:

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

is an object $B \cup_A C$ of $\mathcal{C}$ together with two morphisms $j : B \to B \cup_A C$ (**co-base change** of g) and $i : C \to B \cup_A C$ in $\mathcal{C}$ (**co-base change** of f) satisfying the universal property that:

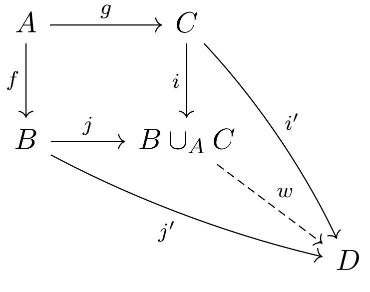
1. the **pushout square**, or **co-Cartesian square**, below commutes in $\mathcal{C}$:

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} \\ B & \xrightarrow{j} & B \cup_A C \end{array}$$

2. and, for every commutative diagram in $\mathcal{C}$ as below:

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i'} \\ B & \xrightarrow{j'} & D \end{array}$$

there exists a unique morphism $w : B \cup_A C \to D$ such that the diagram below commutes in $\mathcal{C}$:



Remark 7.8.80. *A pushout of a pair of morphisms in $\mathcal{C}$ with the same source need not exist in $\mathcal{C}$, and if it does, then it is unique up to canonical isomorphism in $\mathcal{C}$ provided by its universal property, thus allowing us to speak of 'the' pushout of said pair of morphisms.*

Remark 7.8.81. *'Duality' precisely translates to the fact that:*

1. *a fiber product in a category $\mathcal{C}$ is equivalent to a pushout in the opposite category $\mathcal{C}^{\mathrm{op}}$ and*
2. *a pushout in a category $\mathcal{C}$ is equivalent to a fiber product in the opposite category $\mathcal{C}^{\mathrm{op}}$.*

Example 7.8.82. In the category Set of sets, given an arbitrary pair of set maps with the same source:

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

we construct the pushout $B \cup_A C$ by defining:

$$B \cup_A C := \frac{B \coprod C}{f(a) \sim g(a),\ a \in A}$$

and taking $j : B \to B \cup_A C$ and $i : C \to B \cup_A C$ in $\mathcal{C}$ to be the respective coproduct maps followed by the quotient map. This gluing construction works verbatim in the category $\mathrm{Set_f}$ of finite sets, as well as in the category Top of topological spaces, where it lies at the heart of the definition and study of CW complexes because it formally describes the geometric procedure of gluing cells (often uncountably many of them, via involved attaching maps, or inductively ad infinitum, or all at once). Mutatis mutandis, it also works in the category Grp of groups by defining the free product with amalgamation:

$$B *_A C := \frac{B * C}{\langle\langle f(a)\,(g(a))^{-1} : a \in A\rangle\rangle}$$

that is, forming the free product of B and C - the group-theoretic coproduct of B and C - and quotienting by the suitable normal closure, as well as in the category Ab of abelian groups and, more generally, the category R-Mod of left modules over a unital ring R by defining:

$$B \oplus_A C := \frac{B \oplus C}{\langle (f(a), -g(a)) : a \in A\rangle}$$

that is, forming the direct sum of B and C - the module-theoretic coproduct of B and C - and quotienting by the suitable subspace, and the category CRing of commutative, unital rings by defining:

$$B \otimes_A C := \frac{B \otimes C}{\langle (f(a) \otimes 1) - (1 \otimes g(a)) : a \in A\rangle}$$

that is, forming the tensor product of B and C - the ring-theoretic coproduct of B and C - and quotienting by the suitable ideal. Thus, Set, $\mathrm{Set_f}$, Top, Grp, Ab, R-Mod over a unital ring R, and CRing all have all pushouts.

Example 7.8.83. In a category $\mathcal{C}$ with an initial object $\emptyset$, the pushout of the pair of initial object morphisms:

$$\begin{array}{ccc} \emptyset & \xrightarrow{f_A} & A \\ {\scriptstyle f_B}\downarrow & & \\ B & & \end{array}$$

if it exists in $\mathcal{C}$, is precisely the binary coproduct $A \coprod B$ of A and B. This applies to the category Set of sets, the category $\mathrm{Set_f}$ of finite sets, the category Top of topological spaces, the category Top_* of pointed spaces, the category Grp of groups, the category Ab of abelian groups, the category R-Mod of left modules over a unital ring R, and the category CRing of commutative, unital rings.

Example 7.8.84. In a category $\mathcal{C}$ having two morphisms $h : X \to Y$ and $h' : X \to Y$ and the binary coproduct $X \coprod X$, the pushout:

$$\begin{array}{ccc} X \coprod X & \xrightarrow{1_X + 1_X} & X \\ {\scriptstyle h+h'} \big\downarrow & & \\ Y & & \end{array}$$

if it exists in $\mathcal{C}$, is precisely the co-equalizer of the pair h and h'. This applies to the category Set of sets, the category Set_{f} of finite sets, the category Top of topological spaces, the category Top_* of pointed spaces, the category Grp of groups, the category Ab of abelian groups, the category R-Mod of left modules over a unital ring R, and the category CRing of commutative, unital rings.

Example 7.8.85. In the poset category associated with a poset, the pushout of:

$$\begin{array}{ccc} a & \longrightarrow & c \\ \big\downarrow & & \\ b & & \end{array}$$

is the join $b \vee c$ of the elements b and c of said poset. For example, for every set X, the poset category $\mathcal{P}(X)$ associated with the power set poset $(\mathcal{P}(X), \subset)$ with the subset partial order $\subset$ has all pushouts because the poset $(\mathcal{P}(X), \subset)$ has all binary joins, which are binary unions. Moreover, the poset category $\mathbb{Z}$ associated with the poset $(\mathbb{Z}, \leq)$ with the usual partial order $\leq$ has all pushouts because the poset $(\mathbb{Z}, \leq)$ has all binary joins. The same is true for the poset category $\mathbb{N}$ associated with the poset $(\mathbb{N}, \leq)$ with the usual partial order $\leq$.

7.8.5 Limits and colimits and their preservation

We follow section 2 of [5] in our study of general limits and colimits, which unifies all our previous discussions, at the expected cost of higher abstraction. We shall need the following functor:

Definition 7.8.86 (Constant functor). Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. The **constant functor** $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$ from $\mathcal{C}$ to the functor category $\mathcal{C}^{\mathcal{D}}$ of functors from $\mathcal{D}$ to $\mathcal{C}$ is defined:

1. on the objects of $\mathcal{C}$ by assigning to X the constant functor $\Delta(X) : \mathcal{D} \to \mathcal{C}$ sending all objects of $\mathcal{D}$ to X and all morphisms of $\mathcal{D}$ to 1_X, the identity morphism of X in $\mathcal{C}$, and
2. on the morphisms of $\mathcal{C}$ by assigning to $f : X \to Y$ the natural transformation of constant functors $\Delta(f) : \Delta(X) \to \Delta(Y)$ defined object-wise to be $f : X \to Y$.

We first define general limits using the language of constant functors:

Definition 7.8.87 (Limit). A **limit** of a functor $F : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is a small[15] category, is an object $\lim F$ of $\mathcal{C}$ together with a natural transformation $t : \Delta(\lim F) \to F$ satisfying the universal property that, for every other object X of $\mathcal{C}$ together with a natural transformation $t' : \Delta(X) \to F$, there exists a unique morphism $f : X \to \lim F$ of $\mathcal{C}$ such that the diagram below commutes in the functor category $\mathcal{C}^{\mathcal{D}}$:

$$\begin{array}{ccc} \Delta(X) & \xrightarrow{t'} & F \\ {\scriptstyle \Delta(f)} \big\downarrow & \nearrow_{t} & \\ \Delta(\lim F) & & \end{array}$$

[15]A category is **small** if its objects form a set, rather than a proper class. Moreover, especially in our current setting, a functor $F : \mathcal{D} \to \mathcal{C}$ is often called a $\mathcal{D}$**-shaped diagram**, for it formalizes a specific type of diagram in $\mathcal{C}$, as illustrated in our examples.

Remark 7.8.88. *A limit of a functor $F : \mathcal{D} \to \mathcal{C}$ out of a small category $\mathcal{D}$ need not exist, and if it does, then it is unique up to canonical isomorphism in $\mathcal{C}$ provided by its universal property, thus allowing us to speak of 'the' limit of said functor. If every functor $F : \mathcal{D} \to \mathcal{C}$ out of every small (resp. finite[16]) category $\mathcal{D}$ - that is, every small (resp. finite) diagram in $\mathcal{C}$ - has a limit in $\mathcal{C}$, then we say that $\mathcal{C}$* ***has all small (resp. finite) limits****. A category with all small limits is also often called 'complete' or 'small-complete'.*

If every functor $F : \mathcal{D} \to \mathcal{C}$ out of a specific small category $\mathcal{D}$ has a limit in $\mathcal{C}$, then taking limits of functors out of $\mathcal{D}$ assembles to a limit functor $\lim : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$, which is right adjoint to the constant functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$.

Remark 7.8.89. *Limits are often referred to as 'inverse limits' or 'projective limits' to distinguish them from colimits.*

We describe how general limits unify our aforementioned categorical constructions:

Example 7.8.90 (Terminal objects). When $\mathcal{D}$ is the empty category and $F : \mathcal{D} \to \mathcal{C}$ is the unique functor out of the empty category to $\mathcal{C}$, the limit $\lim F$, if it exists, is the same as the terminal object $*$ of $\mathcal{C}$.

Example 7.8.91 (Products). When $\mathcal{D}$ is the discrete small category consisting of a set of objects I and their identity morphisms, a functor $F : \mathcal{D} \to \mathcal{C}$ is precisely a selection $\{X_i\}_{i \in I}$ of objects of $\mathcal{C}$ indexed by I, and the limit $\lim F$, if it exists, is precisely the same as the product $\prod_{i \in I} X_i$ in $\mathcal{C}$. In the finite case $I = \{1, 2\}$, we recover the binary product $X_1 \times X_2$ in $\mathcal{C}$, if it exists.

Example 7.8.92 (Equalizers). When $\mathcal{D}$ is a 2-object small category with 2 non-identity morphisms:

$$\bullet \rightrightarrows \bullet$$

where the 2 identity morphisms are suppressed, a functor $F : \mathcal{D} \to \mathcal{C}$ is a selection of a pair of morphisms with the same source and target in $\mathcal{C}$, and the limit $\lim F$, if it exists, is the same as its equalizer in $\mathcal{C}$.

Example 7.8.93 (Fiber products (Pullbacks)). When $\mathcal{D}$ is a 3-object small category with 2 non-identity morphisms:

$$\begin{array}{ccc} & & \bullet \\ & & \downarrow \\ \bullet & \longrightarrow & \bullet \end{array}$$

where the 3 identity morphisms are suppressed, a functor $F : \mathcal{D} \to \mathcal{C}$ is a diagram of the following type in $\mathcal{C}$:

$$\begin{array}{ccc} & & A \\ & & \downarrow{\scriptstyle f} \\ C & \xrightarrow{g} & B \end{array}$$

and the limit $\lim F$, if it exists, is precisely the same as the fiber product (pullback) $A \times_C B$ in $\mathcal{C}$.

Example 7.8.94 (Projective limits and the ring $\mathbb{Z}_p$ of p-adic integers). The poset category $\mathbb{N}$ associated with the poset $(\mathbb{N}, \leq)$ with the usual partial order $\leq$ has non-identity morphisms:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n \longrightarrow n+1 \longrightarrow \cdots$$

where all identity morphisms are suppressed, so its opposite category $\mathbb{N}^{\mathrm{op}}$ has non-identity morphisms:

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \cdots \longleftarrow n \longleftarrow n+1 \longleftarrow \cdots$$

[16] A category is **finite** if its objects form a finite set.

and a functor $F : \mathbb{N}^{\text{op}} \to \mathcal{C}$ is a countably infinite string of composable morphisms in $\mathcal{C}$ as below:

$$X_1 \xleftarrow{f_2} X_2 \xleftarrow{f_3} X_3 \longleftarrow \cdots \longleftarrow X_n \xleftarrow{f_{n+1}} X_{n+1} \longleftarrow \cdots$$

When $\mathcal{C}$ is the category CRing of commutative, unital rings, p is a prime number, and the diagram below in CRing consists of the countably infinite string of quotient projections below:

$$\mathbb{Z}/p\mathbb{Z} \xleftarrow{q_2} \mathbb{Z}/p^2\mathbb{Z} \xleftarrow{q_3} \mathbb{Z}/p^3\mathbb{Z} \longleftarrow \cdots \longleftarrow \mathbb{Z}/p^n\mathbb{Z} \xleftarrow{q_{n+1}} \mathbb{Z}/p^{n+1}\mathbb{Z} \longleftarrow \cdots$$

the projective limit exists in CRing and it is called **the ring $\mathbb{Z}_p$ of p-adic integers**. An alternative, non-categorical definition of $\mathbb{Z}_p$ is that of the closed unit disk in the p-adic numbers $\mathbb{Q}_p$ with the p-adic norm.

Example 7.8.95. As explained in [5], pp. 9-10, the category Set of sets, the category Top of topological spaces, and the category R-Mod of left modules over a unital ring R all have all small limits, vastly generalizing the existence of a terminal object and all products and fiber products in all the aforementioned 3 categories. However, the category Set_{f} of finite sets is not closed under infinite products, thus does not have all small limits, and the category $\text{Set}_{\neq 1}$ of non-singleton sets lacks a terminal object, thus does not have all finite limits.

The dual notion to that of a limit is that of a colimit:

Definition 7.8.96 (Colimit). A **colimit** of a functor $F : \mathcal{D} \to \mathcal{C}$, where $\mathcal{D}$ is a small category, is an object $\text{colim}F$ of $\mathcal{C}$ together with a natural transformation $s : F \to \Delta(\text{colim}F)$ satisfying the universal property that, for every other object X of $\mathcal{C}$ together with a natural transformation $s' : F \to \Delta(X)$, there exists a unique morphism $f : \text{colim}F \to X$ of $\mathcal{C}$ such that the diagram below commutes in the functor category $\mathcal{C}^{\mathcal{D}}$:

$$\begin{array}{ccc} \Delta(\text{colim}F) & \xrightarrow{s} & F \\ {\scriptstyle \Delta(f)}\big\downarrow & \nearrow_{s'} & \\ \Delta(X) & & \end{array}$$

Remark 7.8.97. *A colimit of a functor $F : \mathcal{D} \to \mathcal{C}$ out of a small category $\mathcal{D}$ need not exist, and if it does, then it is unique up to canonical isomorphism in $\mathcal{C}$ provided by its universal property, thus allowing us to speak of 'the' colimit of said functor. If every functor $F : \mathcal{D} \to \mathcal{C}$ out of every small (resp. finite) category $\mathcal{D}$ - that is, every small (resp. finite) diagram in $\mathcal{C}$ - has a colimit in $\mathcal{C}$, then we say that $\mathcal{C}$* ***has all small (resp. finite) colimits****. A category with all small colimits is also often called 'co-complete' or 'small-co-complete'.*

If every functor $F : \mathcal{D} \to \mathcal{C}$ out of a specific small category $\mathcal{D}$ has a colimit in $\mathcal{C}$, then taking colimits of functors out of $\mathcal{D}$ assembles to a functor colim $: \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$*, which is left adjoint to the constant functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$.*

Remark 7.8.98. *Colimits are often referred to as 'direct limits' or 'inductive limits' to distinguish them from limits.*

Remark 7.8.99. *'Duality' precisely translates to the fact that:*

1. *a limit in a category $\mathcal{C}$ is equivalent to a colimit in the opposite category $\mathcal{C}^{\text{op}}$ and*
2. *a colimit in a category $\mathcal{C}$ is equivalent to a limit in the opposite category $\mathcal{C}^{\text{op}}$.*

A category $\mathcal{C}$ has all small (finite) limits if and only if its opposite category $\mathcal{C}^{\text{op}}$ has all small (finite) colimits.

We describe how general colimits unify our aforementioned categorical constructions:

Example 7.8.100 (Initial objects). When $\mathcal{D}$ is the empty category and $F : \mathcal{D} \to \mathcal{C}$ is the unique functor out of the empty category to $\mathcal{C}$, the colimit $\mathrm{colim}F$, if it exists, is the same as the initial object $\emptyset$ of $\mathcal{C}$.

Example 7.8.101 (Coproducts). When $\mathcal{D}$ is the discrete small category consisting of a set of objects I and their identity morphisms, a functor $F : \mathcal{D} \to \mathcal{C}$ is precisely a selection $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$ indexed by I, and the colimit $\mathrm{colim}F$, if it exists, is precisely the same as the coproduct $\coprod_{i\in I} X_i$ in $\mathcal{C}$. In the finite case $I = \{1, 2\}$, we recover the binary coproduct $X_1 \coprod X_2$ in $\mathcal{C}$, if it exists.

Example 7.8.102 (Co-equalizers). When $\mathcal{D}$ is a 2-object small category with 2 non-identity morphisms:

$$\bullet \rightrightarrows \bullet$$

where the 2 identity morphisms are suppressed, a functor $F : \mathcal{D} \to \mathcal{C}$ is a selection of a pair of morphisms with the same source and target in $\mathcal{C}$, and the colimit $\mathrm{colim}F$, if it exists, is the same as its co-equalizer in $\mathcal{C}$.

Example 7.8.103 (Pushouts). When $\mathcal{D}$ is a 3-object small category with 2 non-identity morphisms:

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array}$$

where the 3 identity morphisms are suppressed, a functor $F : \mathcal{D} \to \mathcal{C}$ is a diagram of the following type in $\mathcal{C}$:

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

and the colimit $\mathrm{colim}F$, if it exists, is precisely the same as the pushout $B \cup_A C$ in $\mathcal{C}$.

Example 7.8.104 (Inductive limits and the CW topology). The poset category $\mathbb{N}$ associated with the poset $(\mathbb{N}, \leq)$ with the usual partial order $\leq$ has non-identity morphisms:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n \longrightarrow n+1 \longrightarrow \cdots$$

where all identity morphisms are suppressed, so a functor $F : \mathbb{N} \to \mathcal{C}$ is a countably infinite string of composable morphisms in $\mathcal{C}$ as below:

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{f_n} X_{n+1} \longrightarrow \cdots$$

When $\mathcal{C}$ is the category Top of topological spaces and the diagram below in Top consists of the countably infinite string of subspace inclusions below:

$$X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} X_3 \longrightarrow \cdots \longrightarrow X_n \xrightarrow{i_n} X_{n+1} \longrightarrow \cdots$$

the inductive limit exists in Top and it is, sensibly, the ascending union $\bigcup_{j=1}^{\infty} X_j$ equipped with the colimit topology: a subset V of $\bigcup_{j=1}^{\infty} X_j$ is closed if and only if, for every $j \in \mathbb{N}$, the subset $V \cap X_j$ of X_j is closed in X_j. This topology is also called the **CW topology** because, in the context of CW complexes, X_j is the j-th skeleton of the CW complex one is constructing, and the resulting CW complex $\bigcup_{j=1}^{\infty} X_j$ assembled by gluing cells is endowed, by the definition of a CW complex, with the aforementioned colimit topology.

Example 7.8.105. As explained in [5], pp. 7-8, the category Set of sets, the category Top of topological spaces, and the category R-Mod of left modules over a unital ring R all have all small colimits, vastly generalizing the existence of an initial object and all coproducts and pushouts in all the aforementioned 3 categories. However, the category $\mathrm{Set_f}$ of finite sets is not closed under infinite coproducts, thus does not have all small colimits, and the category $\mathrm{Set}_{\geq 1}$ of non-empty sets lacks an initial object, thus does not even have all finite colimits.

We state a theorem, and its dual result, on the existence of limits and colimits in functor categories, such as categories of presheaves in algebraic geometry and the category of simplicial sets - or other simplicial objects, such as simplicial abelian groups - in (simplicial) homotopy theory:

Theorem 7.8.106 ([7], p. 116, corollary)**.** *If a category $\mathcal{C}$ has all small (resp. finite) limits, then so does every functor category $\mathcal{C}^{\mathcal{K}}$ of functors mapping to $\mathcal{C}$ from $\mathcal{K}$.*

Remark 7.8.107 ([5], p. 11)**.** *Under the hypothesis of theorem 7.8.106, limits in a functor category $\mathcal{C}^{\mathcal{K}}$ of functors mapping to $\mathcal{C}$ from $\mathcal{K}$ are evaluated point-wise: if $X : \mathcal{D} \to \mathcal{C}^{\mathcal{K}}$ is a functor out of a small category $\mathcal{D}$ and we wish to evaluate the limit functor $\lim X$ in $\mathcal{C}^{\mathcal{K}}$ at an object a of K, then we study the associated functor $X_a : \mathcal{D} \to \mathcal{C}$ defined on objects by $X_a(d) := (X(d))(a)$ and on morphisms by $X_a(f) := (X(f))(a)$, whose limit exists by the hypothesis of theorem 7.8.106, and we employ the following natural isomorphism in $\mathcal{C}$:*

$$(\lim X)(a) \cong \lim X_a$$

Corollary 7.8.108. *If a category $\mathcal{C}$ has all small (resp. finite) colimits, then so does every functor category $\mathcal{C}^{\mathcal{K}}$ of functors mapping to $\mathcal{C}$ from $\mathcal{K}$.*

Remark 7.8.109. *Under the hypothesis of corollary 7.8.108, colimits in a functor category $\mathcal{C}^{\mathcal{K}}$ of functors mapping to $\mathcal{C}$ from $\mathcal{K}$ are evaluated point-wise: if $X : \mathcal{D} \to \mathcal{C}^{\mathcal{K}}$ is a functor out of a small category $\mathcal{D}$ and we wish to evaluate the colimit functor $\mathrm{colim} X$ in $\mathcal{C}^{\mathcal{K}}$ at an object a of K, then we study the associated functor $X_a : \mathcal{D} \to \mathcal{C}$ defined on objects by $X_a(d) := (X(d))(a)$ and on morphisms by $X_a(f) := (X(f))(a)$, whose colimit exists by the hypothesis of corollary 7.8.108, and we employ the following natural isomorphism in $\mathcal{C}$:*

$$(\mathrm{colim} X)(a) \cong \mathrm{colim} X_a$$

Lastly, we state without proof two powerful theorems, and their dual results, which significantly reduce one's work when they are studying whether a specific category has all small, or at least all finite, limits or colimits:

Theorem 7.8.110 ([7], p. 113, corollary 1)**.** *A category has all finite limits if and only if it has a terminal object, all equalizers, and all binary products - equivalently, if and only if it has all finite products and all equalizers.*

Remark 7.8.111. *Recall that the empty product is the terminal object.*

Corollary 7.8.112. *A category has all finite colimits if and only if it has an initial object, all co-equalizers, and all binary coproducts - equivalently, if and only if it has all finite coproducts and all co-equalizers.*

Remark 7.8.113. *Recall that the empty coproduct is the initial object.*

Theorem 7.8.114 ([7], p. 113, corollary 2)**.** *A category has all small limits if and only if it has all equalizers and all products (including the empty product, which is the terminal object).*

Corollary 7.8.115. *A category has all small colimits if and only if it has all co-equalizers and all coproducts (including the empty coproduct, which is the initial object).*

The category $\mathrm{Set}_{\mathrm{f}}$ of finite sets has all finite limits and colimits, but neither all small limits nor all small colimits.

We proceed with defining and studying the preservation of limits by functors:

Definition 7.8.116 (Preservation of limits). A functor $G : \mathcal{C} \to \mathcal{C}'$ **preserves all small (resp. finite) limits** if, for every functor $F : \mathcal{D} \to \mathcal{C}$ out of a small (resp. finite) category $\mathcal{D}$ such that $\lim F$ exists in $\mathcal{C}$:

1. $\lim GF$ exists in $\mathcal{C}'$ and
2. we have the isomorphism $\lim GF \cong G(\lim F)$ in $\mathcal{C}'$.

Remark 7.8.117. *Alluding to calculus, some authors refer to functors preserving all small limits as 'continuous' functors.*

Example 7.8.118. All identity functors and all covariant Hom functors ([7], p. 116, theorem 1), thus all covariant representable functors[17], preserve all small limits, and the composite of two functors preserving all small (resp. finite) limits also preserves all small (resp. finite) limits.

Example 7.8.119 (Terminal objects). A functor $G : \mathcal{C} \to \mathcal{C}'$ vacuously preserves the terminal object of $\mathcal{C}$ if $\mathcal{C}$ has no terminal object. Otherwise, G preserves the terminal object of $\mathcal{C}$ if:

1. $\mathcal{C}'$ also has a terminal object and
2. G sends the terminal object of $\mathcal{C}$ to that of $\mathcal{C}'$.

The forgetful functor from groups to sets preserves the terminal object, as does the inclusion functor from non-singleton sets to sets, albeit vacuously in the latter example. The inclusion functor from the terminal[18] category $*$ consisting of 1 object and its identity morphism to the poset category $\{1, 2\}$ associated with the poset $(\{1, 2\}, \leq)$ with the usual partial order $\leq$ which sends the unique object of $*$ to 1 fails to preserve the terminal object: the unique, on the nose, terminal object of $*$ is its unique object, and the unique, again on the nose, terminal object of $\{1, 2\}$ is its unique maximal element 2 with the usual partial order $\leq$, but the inclusion functor at hand sends the unique terminal object of $*$ to 1, and $1 \ncong 2$ in the poset category $\{1, 2\}$ because $1 \neq 2$ in the poset $(\{1, 2\}, \leq)$.

Example 7.8.120 (Products). A functor $G : \mathcal{C} \to \mathcal{C}'$ vacuously preserves the product of a given set $\{X_i\}_{i \in I}$ of objects of $\mathcal{C}$, where I is an indexing set, if said product does not exist in $\mathcal{C}$. Otherwise, G preserves the product $\prod_{i \in I} X_i$ in $\mathcal{C}$ if:

1. the product $\prod_{i \in I} G(X_i)$ also exists in $\mathcal{C}'$ and
2. we have the isomorphism $\prod_{i \in I} G(X_i) \cong G\left(\prod_{i \in I} X_i\right)$ in $\mathcal{C}'$.

In the finite case $I = \{1, 2\}$, a functor $G : \mathcal{C} \to \mathcal{C}'$ vacuously preserves the product of two objects X_1 and X_2 of $\mathcal{C}$ if said product does not exist in $\mathcal{C}$. Otherwise, G preserves the product $X_1 \times X_2$ in $\mathcal{C}$ if:

1. the product $G(X_1) \times G(X_2)$ also exists in $\mathcal{C}'$ and
2. we have the isomorphism $G(X_1) \times G(X_2) \cong G(X_1 \times X_2)$ in $\mathcal{C}'$.

The inclusion functor from finite sets to sets preserves all products: finite products are essentially sent to themselves, only in a larger category, and most infinite products are preserved vacuously because they do not exist in the category of finite sets. The fundamental group functor π_1 from pointed spaces to groups also preserves all products, none of them vacuously, as does the forgetful functor from spaces to sets.

Example 7.8.121 (Equalizers). A functor $G : \mathcal{C} \to \mathcal{C}'$ vacuously preserves the equalizer of a pair of morphisms with the same source and target $h : X \to Y$ and $h' : X \to Y$ in $\mathcal{C}$ if said equalizer does not exist in $\mathcal{C}$. Otherwise, G preserves its equalizer in $\mathcal{C}$ if:

1. the equalizer of the pair of morphisms with the same source and target $G(h) : G(X) \to G(Y)$ and $G(h') : G(X) \to G(Y)$ also exists in $\mathcal{C}'$ and

[17] A covariant **representable functor** is a functor that is naturally isomorphic to a covariant Hom functor.

[18] In the category Cat of small categories and functors between them, the terminal category $*$ is the terminal object of Cat.

2. G sends the equalizer of h and h' to that of $G(h)$ and $G(h')$.

For example, the inclusion functor from finite sets to sets preserves all equalizers, none of them vacuously.

Example 7.8.122 (Fiber products (Pullbacks)). A functor $G : \mathcal{C} \to \mathcal{C}'$ vacuously preserves the fiber product (pullback) of a given diagram in $\mathcal{C}$:

$$\begin{array}{ccc} & & A \\ & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

if said fiber product does not exist in $\mathcal{C}$. Otherwise, G preserves the fiber product $A \times_C B$ in $\mathcal{C}$ if:

1. the fiber product $G(A) \times_{G(C)} G(B)$ of the diagram in $\mathcal{C}'$ below:

$$\begin{array}{ccc} & & G(A) \\ & & \downarrow G(f) \\ G(C) & \xrightarrow{G(g)} & G(B) \end{array}$$

   also exists in $\mathcal{C}'$ and

2. we have the isomorphism $G\left(A \times_C B\right) \cong G(A) \times_{G(C)} G(B)$ in $\mathcal{C}'$.

The inclusion functor from finite sets to sets preserves all fiber products, none of them vacuously, as does the forgetful functor from spaces to sets. The fundamental group functor π_1 from pointed spaces to groups does not generally preserve all pullbacks, but it preserves pullback under the sensible hypotheses of the Seifert-van Kampen theorem. Not even analogues of this decomposition theorem hold for higher homotopy groups - that is, for the higher homotopy group functors π_n from pointed spaces to abelian groups for $n \geq 2$ - adding to the notorious difficulty of their computation.

We dually define and study the preservation of colimits by functors:

Definition 7.8.123 (Preservation of colimits). A functor $G : \mathcal{C} \to \mathcal{C}'$ **preserves all small (resp. finite) colimits** if, for every functor $F : \mathcal{D} \to \mathcal{C}$ out of a small (resp. finite) category $\mathcal{D}$ such that $\text{colim}F$ exists in $\mathcal{C}$:

1. $\text{colim}GF$ exists in $\mathcal{C}'$ and

2. we have the isomorphism $\text{colim}GF \cong G\left(\text{colim}F\right)$ in $\mathcal{C}'$.

Remark 7.8.124. *Alluding to calculus, some authors refer to functors preserving all small colimits as 'co-continuous'.*

Example 7.8.125. All identity functors preserve all small colimits, and the composite of two functors preserving all small (resp. finite) colimits also preserves all small (resp. finite) colimits.

Example 7.8.126 (Initial objects). A functor $G : \mathcal{C} \to \mathcal{C}'$ vacuously preserves the initial object of $\mathcal{C}$ if $\mathcal{C}$ has no initial object. Otherwise, G preserves the initial object of $\mathcal{C}$ if:

1. $\mathcal{C}'$ also has an initial object and

2. G sends the initial object of $\mathcal{C}$ to that of $\mathcal{C}'$.

The inclusion functor from finite sets to sets preserves the initial object, as does the inclusion functor from non-empty sets to sets, albeit vacuously in the latter example. The forgetful functor from abelian groups to groups also preserves the initial object, as does the forgetful functor from left modules over a unital ring R to groups. However, the forgetful functor from groups to sets does not preserve the initial object, for it sends the zero group to its ambient singleton set, which is non-empty. The same applies to the forgetful functor from abelian groups to sets and, more generally, the forgetful functor from left modules over a unital ring R to sets.

Example 7.8.127 (Coproducts). A functor $G : \mathcal{C} \to \mathcal{C}'$ vacuously preserves the coproduct of a given set $\{X_i\}_{i\in I}$ of objects of $\mathcal{C}$, where I is an indexing set, if said coproduct does not exist in $\mathcal{C}$. Otherwise, G preserves the coproduct $\coprod_{i\in I} X_i$ in $\mathcal{C}$ if:

1. the coproduct $\coprod_{i\in I} G(X_i)$ also exists in $\mathcal{C}'$ and
2. we have the isomorphism $\coprod_{i\in I} G(X_i) \cong G\left(\coprod_{i\in I} X_i\right)$ in $\mathcal{C}'$.

In the finite case $I = \{1, 2\}$, a functor $G : \mathcal{C} \to \mathcal{C}'$ vacuously preserves the coproduct of two objects X_1 and X_2 of $\mathcal{C}$ if said coproduct does not exist in $\mathcal{C}$. Otherwise, G preserves the coproduct $X_1 \coprod X_2$ in $\mathcal{C}$ if:

1. the coproduct $G(X_1) \coprod G(X_2)$ also exists in $\mathcal{C}'$ and
2. we have the isomorphism $G(X_1) \coprod G(X_2) \cong G(X_1 \coprod X_2)$ in $\mathcal{C}'$.

The inclusion functor from finite sets to sets preserves all coproducts: finite coproducts are essentially sent to themselves, only in a larger category, and most infinite coproducts are preserved vacuously because they do not exist in the category of finite sets. The abelianization functor from groups to abelian groups also preserves all coproducts, as does the forgetful functor from spaces to sets. However, the forgetful functor from abelian groups to groups does not preserve even binary coproducts, for the groups:

$$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

are not isomorphic: the free product of groups $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ is non-abelian and infinite, whereas the group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is abelian with 4 elements. Moreover, the forgetful functor from groups to sets does not preserve even binary coproducts, for the sets:

$$\mathbb{Z}/2\mathbb{Z} \coprod \mathbb{Z}/2\mathbb{Z} \not\approx \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$

are not in bijection: the disjoint union of sets $\mathbb{Z}/2\mathbb{Z} \coprod \mathbb{Z}/2\mathbb{Z}$ is finite with $2 + 2 = 4$ elements, whereas the ambient set of the free product $\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}$ is infinite. Similarly, the forgetful functor from abelian groups to sets does not preserve even binary coproducts, for the sets:

$$\mathbb{Z}/2\mathbb{Z} \coprod \mathbb{Z}/3\mathbb{Z} \not\approx \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$$

are not in bijection: the disjoint union of sets $\mathbb{Z}/2\mathbb{Z} \coprod \mathbb{Z}/3\mathbb{Z}$ has $2 + 3 = 5$ elements, whereas the ambient set of the direct sum $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ has $2 \cdot 3 = 6$ elements. Lastly, the forgetful functor from pointed spaces to spaces does not preserve even binary coproducts, for the spaces:

$$S^1 \vee S^1 \ncong S^1 \coprod S^1$$

are not homeomorphic: the wedge sum of two circles $S^1 \vee S^1$ is path-connected, whereas the disjoint union of two circles $S^1 \coprod S^1$ consists of 2 path components.

Example 7.8.128 (Co-equalizers). A functor $G : \mathcal{C} \to \mathcal{C}'$ vacuously preserves the co-equalizer of a pair of morphisms with the same source and target $h : X \to Y$ and $h' : X \to Y$ in $\mathcal{C}$ if said co-equalizer does not exist in $\mathcal{C}$. Otherwise, G preserves its co-equalizer in $\mathcal{C}$ if:

1. the co-equalizer of the pair of morphisms with the same source and target $G(h) : G(X) \to G(Y)$ and $G(h') : G(X) \to G(Y)$ also exists in $\mathcal{C}'$ and
2. G sends the co-equalizer of h and h' to that of $G(h)$ and $G(h')$.

For example, the inclusion functor from finite sets to sets preserves all co-equalizers, none of them vacuously. The abelianization functor from groups to abelian groups also preserves all co-equalizers, none of them vacuously.

Example 7.8.129 (Pushouts). A functor $G : \mathcal{C} \to \mathcal{C}'$ vacuously preserves the pushout of a given diagram in $\mathcal{C}$:

$$\begin{array}{ccc} A & \xrightarrow{g} & C \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

if said pushout does not exist in $\mathcal{C}$. Otherwise, G preserves the pushout $B \cup_A C$ in $\mathcal{C}$ if:

1. the pushout $G(B) \cup_{G(A)} G(C)$ of the diagram in $\mathcal{C}'$ below:

$$\begin{array}{ccc} G(A) & \xrightarrow{G(g)} & G(C) \\ {\scriptstyle G(f)}\downarrow & & \\ G(B) & & \end{array}$$

   also exists in $\mathcal{C}'$ and

2. we have the isomorphism $G(B \cup_A C) \cong G(B) \cup_{G(A)} G(C)$ in $\mathcal{C}'$.

The inclusion functor from finite sets to sets preserves all pushouts, none of them vacuously. The abelianization functor from groups to abelian groups also preserves all pushouts. Since binary coproducts of sets, groups, and abelian groups are pushouts of pairs of initial object maps, the failure of the forgetful functor from groups to sets and the forgetful functor from abelian groups to sets to preserve binary coproducts implies the failure of the forgetful functor from groups to sets and the forgetful functor from abelian groups to sets to preserve pushouts.

Theorem 7.8.110 has the following consequences, the latter being the categorical dual of the former:

Theorem 7.8.130. *A functor preserves all finite limits if and only if it preserves the terminal object, all equalizers, and all binary products - equivalently, if and only if it preserves all finite products and all equalizers.*

Remark 7.8.131. *Recall that the empty product is the terminal object.*

Corollary 7.8.132. *A functor preserves all finite colimits if and only if it preserves the initial object, all co-equalizers, and all binary coproducts - equivalently, if and only if it preserves all finite coproducts and all co-equalizers.*

Remark 7.8.133. *Recall that the empty coproduct is the initial object.*

Similarly, theorem 7.8.114 has the following consequences, the latter being the categorical dual of the former:

Theorem 7.8.134. *A functor preserves all small limits if and only if it preserves all equalizers and all products (including the empty product, which is the terminal object).*

Corollary 7.8.135. *A functor preserves all small colimits if and only if it preserves all co-equalizers and all coproducts (including the empty coproduct, which is the initial object).*

We also state a theorem, and its dual result, on the preservation of limits and colimits under adjunctions:

Theorem 7.8.136 ([8], p. 136, theorem 4.5.2)**.** *Right adjoint functors preserve all small limits.*

Corollary 7.8.137 ([8], p. 138, theorem 4.5.3)**.** *Left adjoint functors preserve all small colimits.*

Theorem 7.8.136 and corollary 7.8.137 both are most powerful given the ubiquity of adjunctions across mathematics: we infer from free-forget adjunctions that the free construction at hand preserves all small colimits and the forgetful functor at hand preserves all small limits, and we also combine said results with tensor-Hom adjunctions. Moreover, a functor that has both a left adjoint and a right adjoint is then known to preserve all small limits and colimits. Such a functor is the forgetful functor from spaces to sets, whose left adjoint is the discrete topology functor and whose right adjoint is the indiscrete topology functor.

Lastly, the contrapositives of theorem 7.8.136 and corollary 7.8.137 imply that a functor which fails to preserve a small limit cannot possibly have a left adjoint functor and, dually, that a functor which fails to preserve a small colimit cannot possibly have a right adjoint functor. For instance, the forgetful functor from pointed spaces to spaces and the forgetful functor from groups to sets both cannot possibly have a right adjoint functor, for they both fail to preserve even binary coproducts. Another application is that the Cartesian product endofunctor[19] of the category Set of sets $X \times -$ associated with a set X has a left adjoint functor if and only if X is a singleton set: if X is a singleton set, then $X \times -$ is naturally isomorphic to the identity endofunctor of Set, whose left and right adjoint functor is itself; otherwise, $X \times -$ fails to preserve the terminal object of Set by cardinality considerations, so it does not have a left adjoint functor by the contrapositive of theorem 7.8.136. On the same note, for every set X, the Cartesian product endofunctor of the category Set of sets $X \times -$ associated with X is left adjoint to the Hom endofunctor $\mathrm{Hom}(X, -)$ of Set associated with X, so $X \times -$ preserves all small colimits by corollary 7.8.137, and theorem 7.8.136 recovers that the Hom functor $\mathrm{Hom}(X, -)$ preserves all small limits, as all covariant Hom functors ([7], p. 116, theorem 1), thus all covariant representable functors do.

7.8.6 G-objects and their group quotient objects and fixed-point objects

We apply our theory on general limits and colimits by studying G-objects, where G is a group, and their associated group quotient objects and fixed-point objects. The key idea is to view a group G as a category:

Definition 7.8.138 (Group category). The **group category** G associated with a group G is the category with unique object $*$ with endomorphism monoid[20] $\mathrm{End}(*) := G$ with morphism composition.

Remark 7.8.139. *All morphisms in G are automorphisms of its unique object $*$.*

Remark 7.8.140. *A functor between group categories is equivalent to a group homomorphism at the level of morphisms. In fact, we have a full embedding functor[21] $F : \mathrm{Grp} \to \mathrm{Cat}$ of the category Grp of groups in the category Cat of small categories sending each group G to its associated group category and each group homomorphism to the functor between group categories that it uniquely defines at the level of morphisms. Under this full embedding, we can, and often do, think of Grp as a full subcategory of the category Cat of small categories.*

Example 7.8.141. The group category associated with the zero group is the terminal category $*$ consisting of a unique object and its identity morphism.

Viewing a group G as a category allows us to define the general notion of a G-object:

Definition 7.8.142 (G-object). A **G-object** X of a category $\mathcal{C}$ is a functor $X : G \to \mathcal{C}$ sending $*$ to X at the level of objects and being defined at the level of morphisms by a group homomorphism $X : G \to \mathrm{Aut}(X)$ from G to the automorphism group $\mathrm{Aut}(X)$ of X in $\mathcal{C}$.

[19] An **endofunctor** is a functor whose source category and whose target category are the same.

[20] A **monoid** is a pair $(M, \circ)$ of a set M equipped with a binary operation $\circ$ on M which is associative and unital, though an element of M need not have a two-sided inverse with respect to the binary operation $\circ$. Thus, every group is a monoid, but $(\mathbb{Z}, \cdot)$ is an abelian monoid which is not an abelian group, for only 1 and -1 are integers whose two-sided multiplicative inverse exists and is also an integer. Similarly, $(\mathbb{N}, \cdot)$ is also an abelian monoid which is not an abelian group, for only 1 is a natural number whose two-sided multiplicative inverse exists and is also a natural number.

[21] A **full functor** is a functor that is surjective on morphisms. A **faithful functor** is a functor that is injective on morphisms. An **embedding functor** is a functor that is both faithful and injective on objects.

Remark 7.8.143. *A functor $X : G \to \mathcal{C}$ sending $*$ to X at the level of objects is a priori defined at the level of morphisms by a monoid homomorphism*[22] $X : G \to \mathrm{End}(X)$ *from G to the endomorphism monoid* $\mathrm{End}(X)$ *of X in $\mathcal{C}$, which is then jazzed up by the group structure of G to be a group homomorphism* $X : G \to \mathrm{Aut}(X)$ *from G to the automorphism group* $\mathrm{Aut}(X)$ *of X in $\mathcal{C}$. This group homomorphism defines how G acts on the object X by its symmetries in $\mathcal{C}$ and renders X a G-object in the standard sense.*

Example 7.8.144. A G-object of the category Set of sets is precisely a G-set.

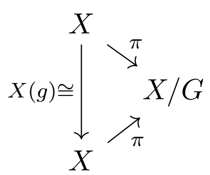
Example 7.8.145. A G-object of the category Top of topological spaces is precisely a G-space.

Example 7.8.146. If X is an object of a category $\mathcal{C}$ and G is a subgroup of the automorphism group $\mathrm{Aut}(X)$ of X in $\mathcal{C}$, then G acts on X by the symmetries of X in $\mathcal{C}$, and X is a G-object of $\mathcal{C}$.

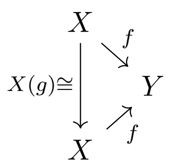
With a G-object of a category $\mathcal{C}$, we may associate a group quotient object of $\mathcal{C}$:

Definition 7.8.147 (Group quotient object). A **group quotient object** of a G-object X in a category $\mathcal{C}$ is an object X/G of $\mathcal{C}$ together with a morphism $\pi : X \to X/G$ in $\mathcal{C}$ satisfying the universal property that:

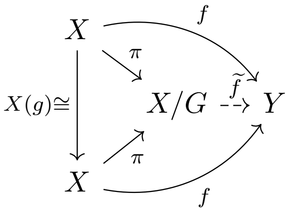
1. for all $g \in G$, the diagram below commutes in $\mathcal{C}$:



2. and, for every morphism $f : X \to Y$ in $\mathcal{C}$ such that, for all $g \in G$, the diagram below commutes in $\mathcal{C}$:



   there exists a unique morphism $\widetilde{f} : X/G \to Y$ such that, for all $g \in G$, the diagram below commutes in $\mathcal{C}$:



Equivalently, a group quotient object of a G-object X in a category $\mathcal{C}$ is a colimit of the functor $X : G \to \mathcal{C}$.

Remark 7.8.148. *As is the case for general colimits, a group quotient object of a G-object of a category $\mathcal{C}$ need not exist, and if it does, then it is unique up to canonical isomorphism in $\mathcal{C}$ provided by its universal property, thus allowing us to speak of 'the' group quotient object of said G-object.*

[22] A **monoid homomorphism** $f : M \to M'$ is a set map between two monoids M and M' which preserves the monoid operation and the unit element: we have $f(xy) = f(x)f(y) \in M'$ for all $x, y \in M$, as well as $f(1_M) = 1_{M'} \in M'$. Note that the permissible lack of two-sided inverses now forces us to require the preservation of the unit element on top of the preservation of the monoid operation, which is redundant when defining group homomorphisms. Thus, every group homomorphism is a monoid homomorphism, but the inclusion of the abelian monoid $(\mathbb{N}, \cdot)$ in the abelian monoid $(\mathbb{Z}, \cdot)$ is an abelian monoid homomorphism which is not an abelian group homomorphism, for neither its source nor its target is a group.

Example 7.8.149. The category Set of sets has all group quotient objects, which are group quotient sets.

Example 7.8.150. The category Top of topological spaces has all group quotient objects, which are orbit spaces.

With a G-object of a category $\mathcal{C}$, we may dually associate a fixed-point object of $\mathcal{C}$:

Definition 7.8.151 (Fixed-point object). A **fixed-point object** of a G-object X in a category $\mathcal{C}$ is an object X^G of $\mathcal{C}$ together with a morphism $\iota : X^G \to X$ in $\mathcal{C}$ satisfying the universal property that:

1. for all $g \in G$, the diagram below commutes in $\mathcal{C}$:

$$\begin{array}{ccc} & & X \\ & \overset{\iota}{\nearrow} & \\ X^G & & \big\downarrow{\scriptstyle \cong X(g)} \\ & \underset{\iota}{\searrow} & \\ & & X \end{array}$$

2. and, for every morphism $f : Y \to X$ in $\mathcal{C}$ such that, for all $g \in G$, the diagram below commutes in $\mathcal{C}$:

$$\begin{array}{ccc} & & X \\ & \overset{f}{\nearrow} & \\ Y & & \big\downarrow{\scriptstyle \cong X(g)} \\ & \underset{f}{\searrow} & \\ & & X \end{array}$$

   there exists a unique morphism $\widehat{f} : Y \to X^G$ such that, for all $g \in G$, the diagram below commutes in $\mathcal{C}$:

$$\begin{array}{ccccc} & & \overset{f}{\longrightarrow} & & X \\ & & & \overset{\iota}{\nearrow} & \\ Y & \overset{\widehat{f}}{\dashrightarrow} & X^G & & \big\downarrow{\scriptstyle \cong X(g)} \\ & & & \underset{\iota}{\searrow} & \\ & & \underset{f}{\longrightarrow} & & X \end{array}$$

Equivalently, a fixed-point object of a G-object X in a category $\mathcal{C}$ is a limit of the functor $X : G \to \mathcal{C}$.

Remark 7.8.152. *As is the case for general limits, a fixed-point object of a G-object of a category $\mathcal{C}$ need not exist, and if it does, then it is unique up to canonical isomorphism in $\mathcal{C}$ provided by its universal property, thus allowing us to speak of 'the' fixed-point object of said G-object.*

Remark 7.8.153. *'Duality' precisely translates to the fact that:*

1. *a group quotient object in a category $\mathcal{C}$ is equivalent to a fixed-point object in the opposite category $\mathcal{C}^{\mathrm{op}}$;*

2. *a fixed-point object in a category $\mathcal{C}$ is equivalent to a group quotient object in the opposite category $\mathcal{C}^{\mathrm{op}}$.*

Example 7.8.154. The category Set of sets has all fixed-point objects, which are fixed-point sets.

Example 7.8.155. The category Top of spaces has all fixed-point objects, which are fixed-point spaces.

7.8.7 Monomorphisms and epimorphisms and their preservation

We segue into studying monomorphisms and epimorphisms and various examples of such:

Definition 7.8.156 (**Monomorphism**). A **monomorphism** is a left-cancellable morphism $f : X \to Y$ in a category $\mathcal{C}$: for every pair of morphisms $g : Z \to X$ and $g' : Z \to X$ in $\mathcal{C}$ such that the diagram below commutes:

$$Z \underset{g'}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

that is, such that $fg = fg'$, we have $g = g'$.

Remark 7.8.157. *Equivalently, a monomorphism is a morphism $f : X \to Y$ in a category $\mathcal{C}$ such that the commutative square below is Cartesian in $\mathcal{C}$:*

$$\begin{array}{ccc} X & = & X \\ \| & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

Equivalently, a monomorphism is a morphism $f : X \to Y$ in a category $\mathcal{C}$ such that, for every object Z of $\mathcal{C}$, the set map of Hom sets $f_* : \mathrm{Hom}(Z, X) \to \mathrm{Hom}(Z, Y)$ given by post-composition with f in $\mathcal{C}$ is injective.

Example 7.8.158. Isomorphisms are monomorphisms.

Example 7.8.159. Composites, products, base changes, and retracts of monomorphisms are monomorphisms.

Example 7.8.160. If a composite morphism gf is a monomorphism, then so is f.

Example 7.8.161. Every morphism whose source object is terminal is a monomorphism.

Example 7.8.162. If f is a monomorphism in a category $\mathcal{C}$ and $\mathcal{D}$ is a subcategory of $\mathcal{C}$ containing f, then f is a monomorphism in $\mathcal{D}$.

Example 7.8.163. In Set, $\mathrm{Set_f}$, Top, Top_*, and $\mathcal{T}_2$, a map is a monomorphism if and only if it is injective.

Example 7.8.164. In R-Mod, an R-linear map is a monomorphism if and only if it is injective. Every injective R-linear map is left-cancellable as a set map, thus a monomorphism in R-Mod. Conversely, if $f : X \to Y$ is a monomorphism in R-Mod, then the commutative diagram below in R-Mod:

$$\ker(f) \underset{0}{\overset{i}{\rightrightarrows}} X \xrightarrow{f} Y$$

where i is the R-submodule inclusion of the kernel $\ker(f)$ of f, forces $\ker(f) = 0$, which is equivalent to f being injective. This argument applies verbatim to the categories Ab and Grp and the category $\mathrm{Grp_f}$ of finite groups.

Example 7.8.165. In the category CRing of commutative, unital rings and ring homomorphisms respecting the multiplicative unit, a ring homomorphism is a monomorphism if and only if it is injective, but the above argument fails in CRing: unless a ring homomorphism maps to the zero ring, its kernel is not a unital ring, for the preservation of the multiplicative unit by said ring homomorphism forces the unit out of said kernel.

Instead, we proceed as follows. Every injective ring homomorphism is left-cancellable as a set map, thus a monomorphism in CRing. Conversely, we suppose $f : R \to S$ is a monomorphism in CRing, and we show that f is injective. By way of contradiction, we assume there exist distinct $a, b \in R$ such that $f(a) = f(b) \in S$. We define a ring homomorphism $g : \mathbb{Z}[X] \to R$ by $g(X) := a$ and another ring homomorphism $g' : \mathbb{Z}[X] \to R$ by $g'(X) := b$, and $g \neq g'$ because $g(X) := a \neq b =: g'(X)$. However, we compute $fg(X) := f(a) = f(b) =: fg'(X)$, so we have $fg = fg'$ and $g \neq g'$, contradicting the hypothesis that f is a monomorphism in CRing. We conclude that f is injective, and that, in CRing, a ring homomorphism is a monomorphism if and only if it is injective.

Example 7.8.166. All coproduct morphisms are monomorphisms in Set, $\mathrm{Set_f}$, Top, and R-Mod, but that is not the case in CRing: the binary coproduct of the commutative, unital rings $\mathbb{Q}$ and $\mathbb{Z}/2\mathbb{Z}$ vanishes, so neither the coproduct morphism $\iota_{\mathbb{Q}}$ nor the coproduct morphism $\iota_{\mathbb{Z}/2\mathbb{Z}}$ is a monomorphism in CRing, for neither $\iota_{\mathbb{Q}}$ nor $\iota_{\mathbb{Z}/2\mathbb{Z}}$ is injective. Alternatively, one can verify that neither $\iota_{\mathbb{Q}}$ nor $\iota_{\mathbb{Z}/2\mathbb{Z}}$ is a monomorphism in CRing ad hoc, in similar fashion to our proof of the previous characterization of monomorphisms in CRing by their injectivity, but without resorting to it: if $g_0 : \mathbb{Z}[X] \to \mathbb{Q}$ is the ring homomorphism defined by $g_0(X) := 0$ and $g_1 : \mathbb{Z}[X] \to \mathbb{Q}$ is the ring homomorphism defined by $g_1(X) := 1$, then we have $\iota_{\mathbb{Q}} g_0 = \iota_{\mathbb{Q}} g_1 = 0$, but $g_0 \neq g_1$ because $g_0(X) := 0 \neq 1 =: g_1(X)$, so $\iota_{\mathbb{Q}}$ is not a monomorphism in CRing. Similarly, if $g'_{\overline{0}} : \mathbb{Z}[X] \to \mathbb{Z}/2\mathbb{Z}$ is the ring homomorphism defined by $g'_{\overline{0}}(X) := \overline{0}$ and $g'_{\overline{1}} : \mathbb{Z}[X] \to \mathbb{Z}/2\mathbb{Z}$ is the ring homomorphism defined by $g'_{\overline{1}}(X) := \overline{1}$, then we have $\iota_{\mathbb{Z}/2\mathbb{Z}} g'_{\overline{0}} = \iota_{\mathbb{Z}/2\mathbb{Z}} g'_{\overline{1}} = 0$, but $g'_{\overline{0}} \neq g'_{\overline{1}}$ because $g'_{\overline{0}}(X) := \overline{0} \neq \overline{1} =: g'_{\overline{1}}$, so $\iota_{\mathbb{Z}/2\mathbb{Z}}$ is not a monomorphism in CRing, either.

Example 7.8.167. In the full subcategory Div of Ab consisting of divisible abelian groups, the quotient map $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism, albeit not injective. Note that the above argument for Ab does not apply to q, for its kernel $\ker(q) = \mathbb{Z}$ is not divisible, so the inclusion of the kernel $i : \mathbb{Z} \to \mathbb{Q}$ is not a morphism in Div. Given a commutative diagram as below in Div:

$$W \underset{g'}{\overset{g}{\rightrightarrows}} \mathbb{Q} \xrightarrow{q} \mathbb{Q}/\mathbb{Z}$$

where $qg = qg'$, we show that $g - g' = 0$. By way of contradiction, we assume there exists $x \in W$ such that $g(x) - g'(x) \neq 0$, where such an element x must be non-zero. We know that $q\left(g(x) - g'(x)\right) = 0$ in $\mathbb{Q}/\mathbb{Z}$, so $n := g(x) - g'(x)$ must be a non-zero integer. Since W is divisible, we can divide x by the non-zero integer $2n$ to obtain an element $y \in W$ such that $(2n)y = x$, and we also know that $q\left(g(y) - g'(y)\right) = 0$ in $\mathbb{Q}/\mathbb{Z}$, so $g(y) - g'(y)$ must be an integer. At last, we arrive at the contradiction:

$$\frac{1}{2} = \frac{n}{2n} = \underbrace{\frac{1}{2n}\big(g(x) - g'(x)\big) = g(y) - g'(y)}_{g(y) = \frac{1}{2n}g(x) \text{ and } g'(y) = \frac{1}{2n}g'(x)} \in \mathbb{Z}$$

We conclude that $g - g' = 0$, so q is a monomorphism in Div. Note that q is not a monomorphism in Ab by the previous example and that our argument made heavy use of the additional divisibility property in Div. Lastly, note that injective group homomorphisms between divisible abelian groups are left-cancellable as set maps, thus being monomorphisms in Div.

Example 7.8.168 (Split monomorphisms). A **split monomorphism** is a left-invertible morphism $f : X \to Y$ in a category $\mathcal{C}$: there exists a left inverse morphism $g : Y \to X$ of f in $\mathcal{C}$ such that $gf = 1_X$.

Every split monomorphism is a monomorphism, and the two notions are equivalent in Set and in $\mathrm{Set_f}$, for a set map is injective if and only if it has a left inverse set map. However, in Top, the subspace inclusion $i : S^1 \to D^2$ of the unit circle S^1 as the boundary of the unit disk D^2 is a monomorphism, for it is injective, which is not split: S^1 is not a retract of D^2 because, at the level of fundamental groups, $\pi_1\left(S^1\right) \cong \mathbb{Z}$ is not a retract of the zero fundamental group $\pi_1\left(D^2\right)$ of the contractible unit disk D^2.

Hence, we have the following strict implications:

$$\text{isomorphism} \underset{\not\Longleftarrow}{\Longrightarrow} \text{split monomorphism} \underset{\not\Longleftarrow}{\Longrightarrow} \text{monomorphism}$$

The following result, important in its own right, will be recalled in our study of strict monomorphisms:

Proposition 7.8.169. *Every equalizer is a monomorphism.*

Proof. Let $f : \mathrm{eq}(h, h') \to X$ be an equalizer of a pair of morphisms $h : X \to Y$ and $h' : X \to Y$ in a category $\mathcal{C}$, so we have $hf = h'f$. Given a pair of morphisms $g : W \to \mathrm{eq}(h, h')$ and $g' : W \to \mathrm{eq}(h, h')$

in $\mathcal{C}$ such that the diagram below commutes in $\mathcal{C}$:

$$W \underset{g'}{\overset{g}{\rightrightarrows}} \mathrm{eq}(h,h') \xrightarrow{f} X$$

that is, such that $fg = fg'$, we write down the commutative diagram below in $\mathcal{C}$:

$$\begin{array}{ccccc} W & & & & \\ & \searrow^{fg=fg'} & & & \\ \mathrm{eq}(h,h') & \xrightarrow{f} & X & \underset{h'}{\overset{h}{\rightrightarrows}} & Y \end{array}$$

where $hfg = h'fg$ because $hf = h'f$. At last, we have the two commutative diagrams below in $\mathcal{C}$:

$$\begin{array}{ccccc} W & & & & \\ {\scriptstyle g}\downarrow & \searrow^{fg=fg'} & & & \\ \mathrm{eq}(h,h') & \xrightarrow{f} & X & \underset{h'}{\overset{h}{\rightrightarrows}} & Y \end{array} \qquad \begin{array}{ccccc} W & & & & \\ {\scriptstyle g'}\downarrow & \searrow^{fg=fg'} & & & \\ \mathrm{eq}(h,h') & \xrightarrow{f} & X & \underset{h'}{\overset{h}{\rightrightarrows}} & Y \end{array}$$

and the uniqueness in the universal property of equalizers implies that $g = g'$, so f is a monomorphism in $\mathcal{C}$. □

The dual notion to that of a monomorphism is that of an epimorphism:

Definition 7.8.170 (**Epimorphism**)**.** An **epimorphism** is a right-cancellable morphism $f : X \to Y$ in a category $\mathcal{C}$: for every pair of morphisms $g : Y \to Z$ and $g' : Y \to Z$ in $\mathcal{C}$ such that the diagram below commutes in $\mathcal{C}$:

$$X \xrightarrow{f} Y \underset{g'}{\overset{g}{\rightrightarrows}} Z$$

that is, such that $gf = g'f$, we have $g = g'$.

Remark 7.8.171. *Equivalently, an epimorphism is a morphism $f : X \to Y$ in a category $\mathcal{C}$ such that the commutative square below is co-Cartesian in $\mathcal{C}$:*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \| \\ Y & = & Y \end{array}$$

Equivalently, an epimorphism is a morphism $f : X \to Y$ in a category $\mathcal{C}$ such that, for every object Z of $\mathcal{C}$, the set map of Hom sets $f^* : \mathrm{Hom}\,(Y, Z) \to \mathrm{Hom}\,(X, Z)$ given by pre-composition with f in $\mathcal{C}$ is injective.

Remark 7.8.172. *'Duality' precisely translates to the fact that:*

1. *a monomorphism in a category $\mathcal{C}$ is equivalent to an epimorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$ and*

2. *an epimorphism in a category $\mathcal{C}$ is equivalent to a monomorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.*

Example 7.8.173. Isomorphisms are epimorphisms.

Example 7.8.174. Composites, coproducts, co-base changes, and retracts of epimorphisms are epimorphisms.

Example 7.8.175. If a composite morphism gf is an epimorphism, then so is g.

Example 7.8.176. Every morphism whose target object is initial is an epimorphism.

Example 7.8.177. If f is an epimorphism in a category $\mathcal{C}$ and $\mathcal{D}$ is a subcategory of $\mathcal{C}$ containing f, then f is an epimorphism in $\mathcal{D}$.

Example 7.8.178. In Set and in Set_f, a set map is an epimorphism if and only if it is surjective.

Example 7.8.179. In $\mathcal{T}_2$, the dense subspace inclusion $\iota : \mathbb{Q} \to \mathbb{R}$ is an epimorphism by virtue of its dense image, albeit not surjective. However, surjective continuous maps between Hausdorff spaces are right-cancellable as set maps, thus being epimorphisms in $\mathcal{T}_2$.

Example 7.8.180. In R-Mod, an R-linear map is an epimorphism if and only if it is surjective. Every surjective R-linear map is right-cancellable as a set map, thus an epimorphism in R-Mod. Conversely, if $f : X \to Y$ is an epimorphism in R-Mod, then the commutative diagram below in R-Mod:

$$X \xrightarrow{f} Y \underset{0}{\overset{q}{\rightrightarrows}} \operatorname{coker}(f)$$

where q is the quotient map to the cokernel $\operatorname{coker}(f)$ of f, forces $\operatorname{coker}(f) = 0$, so f is surjective. This argument applies verbatim to Ab, as well as the category Div of divisible abelian groups, for quotients of divisible abelian groups are divisible, so the quotient map to the cokernel in our argument is a morphism in Div.

Example 7.8.181. In Grp, a group homomorphism is an epimorphism if and only if it is surjective, but the above argument is invalid in Grp, for we cannot always form the cokernel of a group homomorphism because its image need not be normal in the target group: the inclusion homomorphism $i : \mathbb{Z}/2\mathbb{Z} \to A_5$ mapping $\overline{1}$ to the even involution $(1\,2)\,(3\,4)$ of A_5 does not have normal image in A_5 - in fact, A_5 is a simple group.

Instead, we proceed as follows. Every surjective group homomorphism is right-cancellable as a set map, thus an epimorphism in Grp. Conversely, if $f : X \to Y$ is an epimorphism in Grp, then we can always define $S := Y/\operatorname{im}(f)$, the set of left cosets of the image subgroup $\operatorname{im}(f)$ in Y, to which we can attach a disjoint element[23] $*$ to define the set $S_* := S \coprod \{*\}$. The left action of Y on S induces a group homomorphism $g : Y \to \operatorname{Sym}(S_*)$ to the symmetric group on the set S_*, and its image $\operatorname{im}(g)$ is a subgroup of the stabilizer subgroup $\operatorname{stab}(*)$ of $\operatorname{Sym}(S_*)$, so we have the ascending chain of subgroups $\operatorname{im}(g) \leq \operatorname{stab}(*) \leq \operatorname{Sym}(S_*)$. Let $\tau \in \operatorname{Sym}(S_*)$ be the transposition in $\operatorname{Sym}(S_*)$ transposing $*$ with the left coset $\operatorname{im}(f)$, and let c_τ be the inner automorphism of $\operatorname{Sym}(S_*)$ given by conjugation by $\tau \in \operatorname{Sym}(S_*)$. Let $g' := c_\tau g : Y \to \operatorname{Sym}(S_*)$ be the composite group homomorphism. Then, for every $x \in X$, we know that $gf(x) \in \operatorname{Sym}(S_*)$ stabilizes both $*$ and the left coset $\operatorname{im}(f)$, so $gf(x)$ commutes with the transposition τ in $\operatorname{Sym}(S_*)$, and we compute that:

$$g'f(x) := c_\tau gf(x) := \underbrace{\tau\,(gf(x))\,\tau^{-1} = (gf(x))\,\tau\tau^{-1}}_{\tau(gf(x))=(gf(x))\tau} = gf(x)$$

Thus, we have $g'f = gf$, which implies $g' = g$ because f is assumed to be an epimorphism in Grp. This means that, for every $y \in Y$, we have $g'(y) := c_\tau g(y) := \tau\,(g(y))\,\tau^{-1} = g(y)$ in $\operatorname{Sym}(S_*)$, which implies that $g(y)$ commutes with the transposition τ in $\operatorname{Sym}(S_*)$, and this further implies that $g(y)$ stabilizes the left coset $\operatorname{im}(f)$. From this, we infer that, for every $y \in Y$, we have $y \in \operatorname{im}(f)$. In other words, f is surjective, as required.

Note that the above argument circumvents the use of amalgamated products and works verbatim in the category Grp_f of finite groups: if X and Y are finite groups, then S_* is a finite set, and the symmetric group $\operatorname{Sym}(S_*)$ on S_* is also a finite group. Thus, in Grp_f, a group homomorphism is an epimorphism if and only if it is surjective.

[23] The practice of attaching a disjoint element to a set or a disjoint basepoint to a space (to make it pointed) is always valid by the non-existence of a universal set from set theory: for every set, we can find such a disjoint element not contained in said set.

Example 7.8.182. In Set and $\mathrm{Set_f}$, there is a pathological case when Cartesian product projection maps fail to be epimorphisms, that is, they fail to be surjective: if X is any non-empty finite set, then the binary product $X \times \emptyset$ is the empty set, so the Cartesian product projection map $p_X : X \times \emptyset \to X$ is the unique set map from the empty set to the non-empty finite set X, which is not surjective, thus not an epimorphism in Set or in $\mathrm{Set_f}$.

Example 7.8.183. In the category CRing of commutative, unital rings and ring homomorphisms respecting the multiplicative unit, the inclusion $j : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism, albeit not surjective. Consider a commutative diagram in CRing as below:

$$\mathbb{Z} \xrightarrow{j} \mathbb{Q} \underset{g'}{\overset{g}{\rightrightarrows}} W$$

that is, such that $gj = g'j$. Then, for all integers a and all non-zero integers b, we compute that:

$$g\left(\frac{a}{b}\right) = \underbrace{g(a) \cdot g\left(\frac{1}{b}\right) = g'(a) \cdot g\left(\frac{1}{b}\right)}_{g(a)=g'(a) \text{ because } a\in\mathbb{Z}} = g'(b) \cdot g'\left(\frac{a}{b}\right) \cdot g\left(\frac{1}{b}\right) =$$

$$= \underbrace{g'\left(\frac{a}{b}\right) \cdot g'(b) \cdot g\left(\frac{1}{b}\right) = g'\left(\frac{a}{b}\right) \cdot g(b) \cdot g\left(\frac{1}{b}\right)}_{g'(b)=g(b) \text{ because } b\in\mathbb{Z}} = \underbrace{g'\left(\frac{a}{b}\right) \cdot g(1) = g'\left(\frac{a}{b}\right)}_{g(1)=1}$$

Thus, we have $g = g'$, so j is an epimorphism in CRing. Note that surjective ring homomorphisms respecting the multiplicative unit are right-cancellable as set maps, thus being epimorphisms in CRing.

Example 7.8.184 (Split epimorphisms). A **split epimorphism** is a right-invertible morphism $f : X \to Y$ in a category $\mathcal{C}$: there exists a right inverse morphism $g : Y \to X$ of f in $\mathcal{C}$ such that $fg = 1_Y$.

Every split epimorphism is an epimorphism, and the two notions are equivalent in Set and in $\mathrm{Set_f}$ because a set map is surjective if and only if it has a right inverse set map. However, in Ab, the quotient abelian group homomorphism $\pi : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is an epimorphism, for it is surjective, which is not split: the only abelian group homomorphism from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$ is 0, and $\pi 0 = 0 \neq 1_{\mathbb{Z}/2\mathbb{Z}}$.

Hence, we have the following strict implications:

$$\text{isomorphism} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \text{split epimorphism} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \text{epimorphism}$$

The following dual result to that of proposition 7.8.169 will be recalled in our study of strict epimorphisms:

Proposition 7.8.185. *Every co-equalizer is an epimorphism.*

Our proof is the dual argument to that in our proof of proposition 7.8.169.

Proof. Let $f : Y \to \mathrm{coeq}(h, h')$ be a coequalizer of a pair of morphisms $h : X \to Y$ and $h' : X \to Y$ in a category $\mathcal{C}$, so we have $fh = fh'$. Given a pair of morphisms $g : \mathrm{coeq}(h, h') \to Q$ and $g' : \mathrm{coeq}(h, h') \to Q$ in $\mathcal{C}$ such that the diagram below commutes in $\mathcal{C}$:

$$Y \xrightarrow{f} \mathrm{coeq}(h, h') \underset{g'}{\overset{g}{\rightrightarrows}} Q$$

that is, such that $gf = g'f$, we write down the commutative diagram below in $\mathcal{C}$:

$$\begin{array}{ccccc} X & \underset{h'}{\overset{h}{\rightrightarrows}} & Y & \xrightarrow{f} & \mathrm{coeq}(h, h') \\ & & & \searrow^{gf=g'f} & \\ & & & & Q \end{array}$$

where $gfh = gfh'$ because $fh = fh'$. We have the two commutative diagrams below in $\mathcal{C}$:

$$\begin{array}{ccccc} X & \overset{h}{\underset{h'}{\rightrightarrows}} & Y & \xrightarrow{f} & \mathrm{coeq}(h, h') \\ & & & \searrow^{gf=g'f} & \downarrow g \\ & & & & Q \end{array} \qquad \begin{array}{ccccc} X & \overset{h}{\underset{h'}{\rightrightarrows}} & Y & \xrightarrow{f} & \mathrm{coeq}(h, h') \\ & & & \searrow^{gf=g'f} & \downarrow g' \\ & & & & Q \end{array}$$

and the uniqueness in the universal property of co-equalizers implies $g = g'$, so f is an epimorphism in $\mathcal{C}$. □

We proceed with studying the preservation and reflection of monomorphisms and epimorphisms:

Definition 7.8.186 (**Preservation of monomorphisms**)**.** A functor $G : \mathcal{C} \to \mathcal{C}'$ **preserves monomorphisms** if, for every monomorphism f of $\mathcal{C}$, its image morphism $G(f)$ under G is a monomorphism in $\mathcal{C}'$. It **reflects monomorphisms** if the converse implication is true: a morphism f of $\mathcal{C}$ is a monomorphism if its image morphism $G(f)$ under G is a monomorphism in $\mathcal{C}'$.

Example 7.8.187. The inclusion functor from $\mathrm{Set_f}$ to Set and the forgetful functors from Top, $R-$Mod, Ab, Grp, $\mathrm{Grp_f}$, or CRing to Set all preserve and reflect monomorphisms, as do the inclusion functors from Ab or $\mathrm{Grp_f}$ to Grp and the forgetful functor from $R-$Mod to Ab. However, neither the inclusion functor from Div to Ab nor that from Div to Grp, both of which reflect monomorphisms, preserves monomorphisms: the quotient map $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism in Div, but not in Ab or in Grp, for it is not injective. Lastly, the fundamental group functor $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ neither preserves nor reflects monomorphisms: the subspace inclusion $i : S^1 \to D^2$ of the unit circle S^1 as the boundary of the unit disk D^2 is a monomorphism in Top_* for it is injective, but it induces the terminal map of $\mathbb{Z}$ at the level of fundamental groups, which is not a monomorphism in Grp for it is not injective; conversely, the terminal map $g_{\mathbb{R}} : \mathbb{R} \to *$ is not a monomorphism in Top_* for it is not injective, but it induces the identity of the zero group at the level of fundamental groups.

Example 7.8.188. Every right adjoint functor preserves monomorphisms because it preserves Cartesian squares by virtue of theorem 7.8.136. This implies that the fundamental group functor $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ cannot possibly have a left adjoint, for it fails to preserve monomorphisms.

Example 7.8.189. Every faithful functor $G : \mathcal{C} \to \mathcal{C}'$ reflects monomorphisms: if $G(f)$ is a monomorphism in $\mathcal{C}'$ and we have $fg = fg'$ in $\mathcal{C}$, then we apply the functor G to $fg = fg'$ and deduce by the functoriality of G that $G(f)G(g) = G(f)G(g')$ in $\mathcal{C}'$, which implies $G(g) = G(g')$ in $\mathcal{C}'$ because $G(f)$ is a monomorphism in $\mathcal{C}'$, and this, in turn, implies $g = g'$ in $\mathcal{C}$, as required, because G is assumed to be a faithful functor.

For example, the inclusion functor from $\mathrm{Set_f}$ to Set and the forgetful functors from Top, $R-$Mod, Ab, Grp, $\mathrm{Grp_f}$, or CRing to Set all preserve and reflect monomorphisms, as do the inclusion functors from Ab or $\mathrm{Grp_f}$ to Grp and the forgetful functor from $R-$Mod to Ab. However, neither the inclusion functor from Div to Ab nor that from Div to Grp, both of which reflect monomorphisms, preserves monomorphisms, so faithful functors may fail to preserve monomorphisms.

Definition 7.8.190 (**Preservation of epimorphisms**)**.** A functor $G : \mathcal{C} \to \mathcal{C}'$ **preserves epimorphisms** if, for every epimorphism f of $\mathcal{C}$, its image morphism $G(f)$ under G is an epimorphism in $\mathcal{C}'$. It **reflects epimorphisms** if the converse implication is true: a morphism f of $\mathcal{C}$ is an epimorphism if its image morphism $G(f)$ under G is an epimorphism in $\mathcal{C}'$.

Example 7.8.191. The inclusion functor from $\mathrm{Set_f}$ to Set and the forgetful functors from $R-$Mod, Ab, Div, Grp, or $\mathrm{Grp_f}$ to Set all preserve and reflect epimorphisms, as do the inclusion functors from Ab or $\mathrm{Grp_f}$ to Grp and the forgetful functor from $R-$Mod to Ab. However, the forgetful functor from $\mathcal{T}_2$ to Set reflects but does not preserve epimorphisms: the dense subspace inclusion $\iota : \mathbb{Q} \to \mathbb{R}$ is an epimorphism in $\mathcal{T}_2$, but not in Set, for it is not surjective. Similarly, by the previous example, the forgetful functor from CRing to Set also reflects but does not preserve epimorphisms: the inclusion

$j : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism in CRing, but not in Set, for it is not surjective. Lastly, the fundamental group functor $\pi_1 : \text{Top}_* \to \text{Grp}$ neither preserves nor reflects epimorphisms: the universal cover $\widetilde{p} : \mathbb{R} \to S^1$ of the unit circle S^1 is an epimorphism in Top_* for it is surjective, but it induces the initial map of $\mathbb{Z}$ at the level of fundamental groups, which is not an epimorphism in Grp for it is not surjective; conversely, the continuous map $f_0 : * \to \mathbb{R}^2$ picking out the origin 0 of $\mathbb{R}^2$ is not an epimorphism in Top_*, for the identity map of $\mathbb{R}^2$ and the constant self-map of $\mathbb{R}^2$ at its origin both preserve the origin but are not equal, but it induces the identity map of the zero group at the level of fundamental groups.

Example 7.8.192. Every left adjoint functor preserves epimorphisms because it preserves co-Cartesian squares by virtue of corollary 7.8.137. This implies that the fundamental group functor $\pi_1 : \text{Top}_* \to \text{Grp}$ cannot possibly have a right adjoint, either, for it fails to preserve epimorphisms. The same applies to the forgetful functor from $\mathcal{T}_2$ to Set - in contrast to the forgetful functor from Top to Set, which has both a right adjoint and a left adjoint - and the forgetful functor from CRing to Set.

Example 7.8.193. Every faithful functor $G : \mathcal{C} \to \mathcal{C}'$ reflects epimorphisms: if $G(f)$ is an epimorphism in $\mathcal{C}'$ and we have $gf = g'f$ in $\mathcal{C}$, then we apply the functor G to $gf = g'f$ and deduce by the functoriality of G that $G(g)G(f) = G(g')G(f)$ in $\mathcal{C}'$, which implies $G(g) = G(g')$ in $\mathcal{C}'$ because $G(f)$ is an epimorphism in $\mathcal{C}'$, and this, in turn, implies $g = g'$ in $\mathcal{C}$, as required, because G is assumed to be a faithful functor.

For example, the inclusion functor from Set_f to Set and the forgetful functors from $R - \text{Mod}$, Ab, Div, Grp, or Grp_f to Set all preserve and reflect epimorphisms, as do the inclusion functors from Ab or Grp_f to Grp and the forgetful functor from $R - \text{Mod}$ to Ab. However, by the previous example, neither the forgetful functor from $\mathcal{T}_2$ to Set nor that from from CRing to Set, both of which reflect epimorphisms, preserves epimorphisms, so faithful functors may fail to preserve epimorphisms.

7.8.8 Balanced, Artinian, and Noetherian categories

We begin with a discussion on balanced categories and various examples of such:

Definition 7.8.194 (**Balanced category**)**.** A **balanced category** is a category in which a morphism is an isomorphism if and only if it is both a monomorphism and an epimorphism.

Remark 7.8.195. *Every isomorphism is both a monomorphism and an epimorphism, but the converse implication fails in some ubiquitous categories, as we explain in the examples below.*

Example 7.8.196. The categories Set, Set_f, R-Mod, Grp, and Grp_f all are balanced.

Example 7.8.197. The category $\mathcal{T}_2$ of Hausdorff spaces is not balanced: the subspace inclusion $\iota : \mathbb{Q} \to \mathbb{R}$ is both a monomorphism in $\mathcal{T}_2$, for it is injective, and an epimorphism in $\mathcal{T}_2$ by virtue of its dense image, but it is not an isomorphism in $\mathcal{T}_2$ - that is, it is not a homeomorphism - for it is not surjective, thus not bijective.

Example 7.8.198. The category CRing of commutative, unital rings and ring homomorphisms respecting the multiplicative unit is not balanced: the inclusion $j : \mathbb{Z} \to \mathbb{Q}$ is both a monomorphism in CRing, for it is injective, and an epimorphism in CRing, albeit not surjective, but it is not an isomorphism in CRing - that is, it is not a ring isomorphism - for it is not surjective, thus not bijective.

Example 7.8.199. The category Div of divisible abelian groups is not balanced: the quotient map $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is both a monomorphism in Div, albeit not injective, and an epimorphism in Div, for it is surjective, but it is not an isomorphism in Div - that is, it is not a group isomorphism - for it is not injective, thus not bijective.

We proceed with defining and studying Artinian categories and various examples of such:

Definition 7.8.200 (Artinian category). An **Artinian category** is a category $\mathcal{C}$ whose morphisms satisfy the **descending chain condition**, which states that, for every descending chain of monomorphisms in $\mathcal{C}$:

$$\cdots \longrightarrow X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

there exists a natural number $n \in \mathbb{N}$, which depends on the given chain, such that the given chain stabilizes at n: for every natural number $m \geq n$, the monomorphism f_m is an isomorphism in $\mathcal{C}$.

Remark 7.8.201. *The above definition and terminology are derived from the notion from commutative algebra of an Artinian commutative ring, which is defined to be a commutative ring whose ideals satisfy the descending chain condition, which states that, for every descending chain of ideals of said commutative ring:*

$$\cdots \subset I_2 \subset I_1 \subset I_0$$

there exists a natural number $n \in \mathbb{N}$, which depends on the given chain, such that the given chain stabilizes at n: for every natural number $m \geq n$, we have $I_m = I_n$.

Example 7.8.202. The category $\mathrm{Set}_{\mathrm{f}}$ of finite sets and the category $\mathrm{Grp}_{\mathrm{f}}$ of finite groups both are Artinian by cardinality considerations.

Example 7.8.203. However, the category Set of sets, the category Grp of groups, and the category Ab of abelian groups all are not Artinian, for the descending chain of inclusions of abelian groups:

$$\cdots \longrightarrow 4\mathbb{Z} \xrightarrow{i_2} 2\mathbb{Z} \xrightarrow{i_1} \mathbb{Z}$$

which are monomorphisms in Set, in Grp, and in Ab fails to stabilize. Endowing the above descending chain of inclusions with the discrete topology or the indiscrete topology provides a descending chain of monomorphisms in the category Top of topological spaces which fails to stabilize, so Top is also not Artinian.

We define and study the dual notion of a Noetherian category and various examples of such:

Definition 7.8.204 (Noetherian category). A **Noetherian category** is a category $\mathcal{C}$ whose morphisms satisfy the **ascending chain condition**, which states that, for every ascending chain of monomorphisms in $\mathcal{C}$:

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots$$

there exists a natural number $n \in \mathbb{N}$, which depends on the given chain, such that the given chain stabilizes at n: for every natural number $m \geq n$, the monomorphism f_m is an isomorphism in $\mathcal{C}$.

Remark 7.8.205. *The above definition and terminology are derived from the notion from commutative algebra of a Noetherian commutative ring, which is defined to be a commutative ring whose ideals satisfy the ascending chain condition, which states that, for every ascending chain of ideals of said commutative ring:*

$$I_0 \subset I_1 \subset I_2 \subset \cdots$$

there exists a natural number $n \in \mathbb{N}$, which depends on the given chain, such that the given chain stabilizes at n: for every natural number $m \geq n$, we have $I_m = I_n$.

Remark 7.8.206. *'Duality' precisely translates to the fact that:*

1. *if a category $\mathcal{C}$ is Artinian, then its opposite category $\mathcal{C}^{\mathrm{op}}$ is Noetherian, and*
2. *if a category $\mathcal{C}$ is Noetherian, then its opposite category $\mathcal{C}^{\mathrm{op}}$ is Artinian.*

Example 7.8.207. The category $\mathrm{Set}_{\mathrm{f}}$ of finite sets, the category $\mathrm{Grp}_{\mathrm{f}}$ of finite groups, the category Set of sets, the category Grp of groups, and the category Ab of abelian groups all are not Noetherian, for the ascending chain of inclusions of finite abelian groups:

$$0 \xrightarrow{i_1} \mathbb{Z}/2\mathbb{Z} \xrightarrow{i_2} \mathbb{Z}/4\mathbb{Z} \longrightarrow \cdots$$

which are monomorphisms in $\mathrm{Set_f}$, in $\mathrm{Grp_f}$, in Set, in Grp, and in Ab fails to stabilize. Endowing the above ascending chain of inclusions with the discrete topology or the indiscrete topology provides an ascending chain of monomorphisms in the category Top of topological spaces which fails to stabilize, so Top is not Noetherian.

7.8.9 Strict monomorphisms and strict epimorphisms and their preservation

We now study a strengthening of the notion of a monomorphism and a strengthening of the notion of an epimorphism, the latter featuring in the definition of a Galois category:

Definition 7.8.208 (Strict monomorphism). A **strict monomorphism** is a morphism $f : X \to Y$ in a category $\mathcal{C}$ such that:

1. the pushout square below exists in $\mathcal{C}$:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} \\ Y & \xrightarrow{j} & Y \cup_X Y \end{array}$$

2. and $f : X \to Y$ is an equalizer in $\mathcal{C}$ of the pair of pushout maps $i : Y \to Y \cup_X Y$ and $j : Y \to Y \cup_X Y$.

Remark 7.8.209. *Beware that the terminology for strict monomorphisms is ambiguous: they are sometimes called 'effective monomorphisms', and the term 'strict monomorphism' is sometimes used to name a different kind of strengthening of the notion of a monomorphism. We are only using the term 'strict monomorphism'.*

Example 7.8.210. Isomorphisms, as well as composites of strict monomorphisms, are strict monomorphisms.

Example 7.8.211. Every strict monomorphism is an equalizer, thus, by proposition 7.8.169, a monomorphism.

Example 7.8.212. In the category Div of divisible abelian groups, the quotient map $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism but not a strict monomorphism. We know the quotient map $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism in Div. We also know that q is an epimorphism in Div, for q is surjective, so the pushout at hand is:

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{q} & \mathbb{Q}/\mathbb{Z} \\ {\scriptstyle q}\downarrow & & \| \\ \mathbb{Q}/\mathbb{Z} & = & \mathbb{Q}/\mathbb{Z} \end{array}$$

We show $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is not an equalizer of the pair of pushout maps $1_{\mathbb{Q}/\mathbb{Z}} = 1_{\mathbb{Q}/\mathbb{Z}}$. Consider the commutative diagram in Div:

$$\begin{array}{ccccc} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{g} & \mathbb{Q}/\mathbb{Z} & = & \mathbb{Q}/\mathbb{Z} \\ & \nearrow_{q} & & & \\ \mathbb{Q} & & & & \end{array}$$

where $g\left(\overline{1}\right) := \overline{\frac{1}{2}}$. However, the only group homomorphism from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Q}$ is the zero homomorphism because $\mathbb{Q}$ is torsion-free, but the diagram below in Div does not commute:

$$\begin{array}{ccccc} \mathbb{Z}/2\mathbb{Z} & \xrightarrow{g} & \mathbb{Q}/\mathbb{Z} & = & \mathbb{Q}/\mathbb{Z} \\ {\scriptstyle 0}\downarrow & \nearrow_{q} & & & \\ \mathbb{Q} & & & & \end{array}$$

since $q0 = 0 \neq g$. Hence, $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is not an equalizer of the pair of pushout maps $1_{\mathbb{Q}/\mathbb{Z}} = 1_{\mathbb{Q}/\mathbb{Z}}$ in Div, thus failing to be a strict monomorphism in Div. Overall, the quotient map $q : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism but not a strict monomorphism in Div.

Example 7.8.213. In $\mathcal{T}_2$, the inclusion $\iota : \mathbb{Q} \to \mathbb{R}$ is a monomorphism but not a strict monomorphism. We know the inclusion $\iota : \mathbb{Q} \to \mathbb{R}$ is a monomorphism in $\mathcal{T}_2$, for it is injective. We also know that ι is an epimorphism in $\mathcal{T}_2$, for ι is a dense subspace inclusion, so the pushout at hand is:

$$\begin{array}{ccc} \mathbb{Q} & \xrightarrow{\iota} & \mathbb{R} \\ {\scriptstyle\iota}\downarrow & & \| \\ \mathbb{R} & = & \mathbb{R} \end{array}$$

We show $\iota : \mathbb{Q} \to \mathbb{R}$ is not an equalizer of the pair of pushout maps $1_{\mathbb{R}} = 1_{\mathbb{R}}$. Consider the commutative diagram in $\mathcal{T}_2$:

$$\begin{array}{ccccc} \mathbb{R} & = & \mathbb{R} & = & \mathbb{R} \\ & \nearrow_{\iota} & & & \\ \mathbb{Q} & & & & \end{array}$$

Since $\mathbb{R}$ is path-connected and $\mathbb{Q}$ is totally disconnected, every continuous map from $\mathbb{R}$ to $\mathbb{Q}$ is forced to be constant, and no constant map from $\mathbb{R}$ to $\mathbb{Q}$ fills out the above diagram of continuous maps so that it commutes. Hence, $\iota : \mathbb{Q} \to \mathbb{R}$ fails to be an equalizer of the pair of pushout maps $1_{\mathbb{R}} = 1_{\mathbb{R}}$ in $\mathcal{T}_2$, thus failing to be a strict monomorphism in $\mathcal{T}_2$. Overall, $\iota : \mathbb{Q} \to \mathbb{R}$ is a monomorphism but not a strict monomorphism in $\mathcal{T}_2$.

Proposition 7.8.214. *Let $f : X \to Y$ be a set map. The following are equivalent:*

1. *f is injective.*
2. *f is a monomorphism in* Set.
3. *f is a strict monomorphism in* Set.

Proof. We know that 1 and 2 are equivalent and that 3 implies 2, so it suffices to show 1 implies 3. Suppose the set map $f : X \to Y$ is injective. We know Set has all pushouts, including the pushout at hand:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle i}\downarrow \\ Y & \xrightarrow{j} & Y \cup_X Y \end{array}$$

where:

$$Y \cup_X Y := \frac{Y \coprod Y}{(f(x), 0) \sim (f(x), 1)\,,\ x \in X}$$

and i and j include in the first and second coordinate, respectively. We also know that an equalizer of the pair of pushout maps i and j in Set is:

$$\mathrm{eq}(i,j) \xrightarrow{\iota} Y \underset{j}{\overset{i}{\rightrightarrows}} Y \cup_X Y$$

where $\mathrm{eq}(i,j) := \{y \in Y : [y,0] = [y,1]\} = f(X)$ and $\iota : \mathrm{eq}(i,j) \to Y$ is the subset inclusion. The injective set map $f : X \to Y$ restricts to a bijection $f : X \xrightarrow{\approx} f(X) = \mathrm{eq}(i,j)$, and the diagram below commutes in Set:

$$\begin{array}{ccccc} \mathrm{eq}(i,j) & \xrightarrow{\iota} & Y & \underset{j}{\overset{i}{\rightrightarrows}} & Y \cup_X Y \\ {\scriptstyle f\approx}\uparrow & \nearrow^{f} & & & \\ X & & & & \end{array}$$

We conclude that $f : X \to Y$ is also an equalizer in Set of the pair of pushout maps i and j. Overall, $f : X \to Y$ is a strict monomorphism, and 1 implies 3, completing the proof. □

The result of proposition 7.8.214 holds by the exact same proof in the category $\mathrm{Set_f}$ of finite sets:

Proposition 7.8.215. *Let $f : X \to Y$ be a set map of finite sets. The following are equivalent:*

1. *f is injective.*
2. *f is a monomorphism in $\mathrm{Set_f}$.*
3. *f is a strict monomorphism in $\mathrm{Set_f}$.*

Overall, we have the following strict implications:

$$\text{isomorphism} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \text{strict monomorphism} \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \text{monomorphism}$$

The dual notion to that of a strict monomorphism is that of a strict epimorphism:

Definition 7.8.216 (Strict epimorphism). A **strict epimorphism** is a morphism $f : X \to Y$ in a category $\mathcal{C}$ such that:

1. the pullback square below exists in $\mathcal{C}$:

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{q} & X \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

2. and $f : X \to Y$ is a co-equalizer in $\mathcal{C}$ of the pair of pullback maps $p : X \times_Y X \to X$ and $q : X \times_Y X \to X$.

Remark 7.8.217. *Beware that the terminology for strict epimorphisms is ambiguous: they are sometimes called 'effective epimorphisms', and the term 'strict epimorphism' is sometimes used to name a different kind of strengthening of the notion of an epimorphism. We are only using the term 'strict epimorphism'.*

Remark 7.8.218. *'Duality' precisely translates to the fact that:*

1. *a strict monomorphism in a category $\mathcal{C}$ is equivalent to a strict epimorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$;*
2. *a strict epimorphism in a category $\mathcal{C}$ is equivalent to a strict monomorphism in the opposite category $\mathcal{C}^{\mathrm{op}}$.*

Example 7.8.219. Isomorphisms, as well as composites of strict epimorphisms, are strict epimorphisms.

Example 7.8.220. Every strict epimorphism is a co-equalizer, thus, by proposition 7.8.185, an epimorphism.

Example 7.8.221. In the category CRing of commutative, unital rings and ring homomorphisms respecting the multiplicative unit, the inclusion $j : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism but not a strict epimorphism. We know that $j : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism in CRing. We also know that j is a monomorphism in CRing, for it is injective, so the pullback at hand is:

$$\begin{array}{ccc} \mathbb{Z} & = & \mathbb{Z} \\ \big\| & & \big\downarrow{\scriptstyle j} \\ \mathbb{Z} & \xrightarrow{j} & \mathbb{Q} \end{array}$$

We show $j : \mathbb{Z} \to \mathbb{Q}$ is not a co-equalizer of the pair of pullback maps $1_{\mathbb{Z}} = 1_{\mathbb{Z}}$. Consider the commutative diagram in CRing:

$$\begin{array}{ccccc} \mathbb{Z} & = & \mathbb{Z} & \xrightarrow{j} & \mathbb{Q} \\ & & & \searrow_{\pi} & \\ & & & & \mathbb{Z}/2\mathbb{Z} \end{array}$$

where π is the quotient map. However, the only ring homomorphism from $\mathbb{Q}$ to $\mathbb{Z}/2\mathbb{Z}$ is the zero homomorphism because $\mathbb{Q}$ is a field, but the diagram below in CRing does not commute:

$$\begin{array}{ccccc} \mathbb{Z} & = & \mathbb{Z} & \xrightarrow{j} & \mathbb{Q} \\ & & & \searrow_{\pi} & \downarrow{\scriptstyle 0} \\ & & & & \mathbb{Z}/2\mathbb{Z} \end{array}$$

since $0j = 0 \neq \pi$. Hence, $j : \mathbb{Z} \to \mathbb{Q}$ fails to be a co-equalizer of the pair of pullback maps $1_{\mathbb{Z}} = 1_{\mathbb{Z}}$ in CRing, thus failing to be a strict epimorphism in CRing. Overall, the inclusion $j : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism but not a strict epimorphism in CRing.

Example 7.8.222. In $\mathcal{T}_2$, the inclusion $\iota : \mathbb{Q} \to \mathbb{R}$ is an epimorphism but not a strict epimorphism. We know the dense subspace inclusion $\iota : \mathbb{Q} \to \mathbb{R}$ is an epimorphism in $\mathcal{T}_2$. We also know that ι is a monomorphism in $\mathcal{T}_2$, for it is injective, so the pullback at hand is:

$$\begin{array}{ccc} \mathbb{Q} & = & \mathbb{Q} \\ \| & & \downarrow{\scriptstyle \iota} \\ \mathbb{Q} & \xrightarrow{\iota} & \mathbb{R} \end{array}$$

We show $\iota : \mathbb{Q} \to \mathbb{R}$ is not a co-equalizer of the pair of pullback maps $1_{\mathbb{Q}} = 1_{\mathbb{Q}}$. Consider the commutative diagram in $\mathcal{T}_2$:

$$\begin{array}{ccccc} \mathbb{Q} & = & \mathbb{Q} & \xrightarrow{\iota} & \mathbb{R} \\ & & & \diagdown\!\!\diagdown & \\ & & & & \mathbb{Q} \end{array}$$

Since $\mathbb{R}$ is path-connected and $\mathbb{Q}$ is totally disconnected, every continuous map from $\mathbb{R}$ to $\mathbb{Q}$ is forced to be constant, and no constant map from $\mathbb{R}$ to $\mathbb{Q}$ fills out the above diagram of continuous maps so that it commutes. Hence, $\iota : \mathbb{Q} \to \mathbb{R}$ fails to be a co-equalizer of the pair of pullback maps $1_{\mathbb{Q}} = 1_{\mathbb{Q}}$ in $\mathcal{T}_2$, thus failing to be a strict epimorphism in $\mathcal{T}_2$. Overall, $\iota : \mathbb{Q} \to \mathbb{R}$ is an epimorphism but not a strict epimorphism in $\mathcal{T}_2$. Note that, overall, $\iota : \mathbb{Q} \to \mathbb{R}$ is both a monomorphism and an epimorphism in $\mathcal{T}_2$, but not a strict monomorphism or a strict epimorphism or an isomorphism in $\mathcal{T}_2$.

Proposition 7.8.223. *Let $f : X \to Y$ be a set map. The following are equivalent:*

1. *f is surjective.*

2. *f is an epimorphism in* Set*.*

3. *f is a strict epimorphism in* Set*.*

Proof. We know that 1 and 2 are equivalent and that 3 implies 2, so it suffices to show 1 implies 3. Suppose the set map $f : X \to Y$ is surjective. We know Set has all pullbacks, including the pullback at hand:

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{f} & Y \end{array}$$

where:

$$X \times_Y X := \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\} \subset X \times X$$

and p and q project to the first and second coordinate, respectively. We also know that a co-equalizer of the pair of pullback maps p and q in Set is:

$$X \times_Y X \underset{q}{\overset{p}{\rightrightarrows}} X \xrightarrow{\pi} \mathrm{coeq}(p,q)$$

where $\mathrm{coeq}(p,q)$ is the quotient of X under the co-equalizer equivalence relation generated by decreeing that $x \sim x'$ when $f(x) = f(x')$, and $\pi : X \to \mathrm{coeq}(p,q)$ is the quotient map. By the universal property of quotients, there exists a unique set map $\phi : \mathrm{coeq}(p,q) \to Y$ such that the diagram below commutes in Set, and ϕ is surjective because f is surjective and ϕ is injective by the definition of our co-equalizer equivalence relation, so ϕ is bijective:

$$\begin{array}{ccccc} X \times_Y X & \underset{q}{\overset{p}{\rightrightarrows}} & X & \xrightarrow{\pi} & \mathrm{coeq}(p,q) \\ & & & \searrow^{f} & \downarrow{\scriptstyle \phi \approx} \\ & & & & Y \end{array}$$

We conclude that $f : X \to Y$ is also a co-equalizer in Set of the pair of pullback maps p and q. Overall, $f : X \to Y$ is a strict epimorphism in Set, and 1 implies 3, completing the proof. □

The result of proposition 7.8.223 holds by the exact same proof in the category $\mathrm{Set_f}$ of finite sets:

Proposition 7.8.224. *Let $f : X \to Y$ be a set map of finite sets. The following are equivalent:*

1. *f is surjective.*
2. *f is an epimorphism in* $\mathrm{Set_f}$*.*
3. *f is a strict epimorphism in* $\mathrm{Set_f}$*.*

Overall, we have the following strict implications:

$$\text{isomorphism} \underset{\not\Longleftarrow}{\Longrightarrow} \text{strict epimorphism} \underset{\not\Longleftarrow}{\Longrightarrow} \text{epimorphism}$$

We proceed with studying the preservation and reflection of strict monomorphisms and strict epimorphisms:

Definition 7.8.225 (Preservation of strict monomorphisms). A functor $G : \mathcal{C} \to \mathcal{C}'$ **preserves strict monomorphisms** if, for every strict monomorphism f of $\mathcal{C}$, its image morphism $G(f)$ under G is a strict monomorphism in $\mathcal{C}'$. If **reflects strict monomorphisms** if the converse implication is true: a morphism f of $\mathcal{C}$ is a strict monomorphism if its image morphism $G(f)$ under G is a strict monomorphism in $\mathcal{C}'$.

Example 7.8.226. Propositions 7.8.214 and 7.8.215 together imply that the inclusion functor from the category $\mathrm{Set_f}$ of finite sets to the category Set of sets preserves and reflects strict monomorphisms. However, the forgetful functor from $\mathcal{T}_2$ to Set does not reflect strict monomorphisms: the inclusion $\iota : \mathbb{Q} \to \mathbb{R}$ is a strict monomorphism in Set because it is injective, but ι is not a strict monomorphism in $\mathcal{T}_2$.

Definition 7.8.227 (Preservation of strict epimorphisms). A functor $G : \mathcal{C} \to \mathcal{C}'$ **preserves strict epimorphisms** if, for every strict monomorphism f of $\mathcal{C}$, its image morphism $G(f)$ under G is a strict epimorphism in $\mathcal{C}'$. If **reflects strict epimorphisms** if the converse implication is true: a morphism f of $\mathcal{C}$ is a strict epimorphism if its image morphism $G(f)$ under G is a strict epimorphism in $\mathcal{C}'$.

Example 7.8.228. Propositions 7.8.223 and 7.8.224 together imply that the inclusion functor from the category $\mathrm{Set_f}$ of finite sets to the category Set of sets preserves and reflects strict epimorphisms.

7.8.10 Reflection of isomorphisms

We conclude this appendix by defining and studying the reflection of isomorphisms by functors:

Definition 7.8.229 (**Reflection of isomorphisms**)**.** A functor $G : \mathcal{C} \to \mathcal{C}'$ **reflects isomorphisms** if a morphism f of $\mathcal{C}$ is an isomorphism whenever its image morphism $G(f)$ under G is an isomorphism of $\mathcal{C}'$.

Remark 7.8.230. *Every functor $G : \mathcal{C} \to \mathcal{C}'$ preserves isomorphisms. If $f : X \to Y$ is an isomorphism in $\mathcal{C}$ with unique two-sided inverse isomorphism $g : Y \to X$ in $\mathcal{C}$, then $G(f) : G(X) \to G(Y)$ is an isomorphism in $\mathcal{C}'$ with unique two-sided inverse isomorphism $G(g) : G(Y) \to G(X)$ in $\mathcal{C}'$ as the two functoriality axioms for G yield:*

$$G(g)G(f) = G(gf) = G(1_X) = 1_{G(X)}$$

$$G(f)G(g) = G(fg) = G(1_Y) = 1_{G(Y)}$$

Example 7.8.231. Fully faithful functors, such as equivalences of categories, reflect isomorphisms.

Example 7.8.232. The inclusion functor from $\mathrm{Set_f}$ to Set, the forgetful functors from Ab, CRing, Div, Grp, $\mathrm{Grp_f}$, or R-Mod to Set, the inclusion functors from Ab, Div, or $\mathrm{Grp_f}$ to Grp, the inclusion functor from Div to Ab, and the forgetful functor from R-Mod to Ab all reflect isomorphisms.

Example 7.8.233. The forgetful functor from Top to Set does not reflect isomorphisms: neither the continuous bijection $f : [0,1) \xrightarrow{\approx} S^1$ defined by $f(t) := e^{2\pi i t}$ nor the continuous bijection $1_{S^1} : S^1_{\mathrm{d}} \xrightarrow{\approx} S^1_{\mathrm{i}}$, the identity from the circle with the discrete topology S^1_{d} to the circle with the indiscrete topology S^1_{i}, is a homeomorphism. However, both the left adjoint of the forgetful functor from Top to Set, which is the discrete topology functor from Set to Top, and the right adjoint of the forgetful functor from Top to Set, which is the indiscrete topology functor from Set to Top, reflect isomorphisms, so reflection of isomorphisms does not play well with adjunctions. Lastly, note that the failure of the forgetful functor from Top to Set to reflect isomorphisms is rectified if one restricts to the set-valued forgetful functor from the full subcategory $\mathcal{T}_2^{\mathrm{c}}$ of Top consisting of compact and Hausdorff spaces, for every continuous bijection with compact source and Hausdorff target is a homeomorphism.

Example 7.8.234. The fundamental group functor $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$ fails to reflect isomorphisms: the terminal map $g_{\mathbb{R}} : \mathbb{R} \to *$ induces the identity of the zero group at the level of fundamental groups, but is not a homeomorphism.

7.9 Table of standard categories

We follow the example of [7], p. 293, in assembling a table of standard categories appearing in these notes:

Ab	Abelian groups and abelian group homomorphisms
Cat	Small categories and functors between them
$\mathrm{Cov}_{\mathrm{f}}^{S}$	Finite-sheeted covering spaces of a connected space[24] S, and finite-sheeted covering space maps
CRing	Commutative, unital rings and ring homomorphisms preserving the multiplicative unit
$\mathbb{C} - \mathrm{Vect}_{\mathrm{fd}}$	Finite-dimensional complex vector spaces and $\mathbb{C}$-linear maps between them
Div	Divisible abelian groups and abelian group homomorphisms between them
$\mathrm{Fét}_S$	Finite étale covers of a connected scheme S, and scheme morphisms of finite étale covers[25]
Grp	Groups and group homomorphisms.
$\mathrm{Grp}_{\mathrm{f}}$	Finite groups and group homomorphisms between them
$\mathrm{Grp}_{\neq 0}$	Non-zero groups and group homomorphisms between them
$G - \mathrm{Set}_{\mathrm{f}}$	Finite and discrete G-spaces, where G is a group, and G-equivariant continuous maps
MS	Metric spaces and continuous maps between them
R-Mod	Left modules over a unital ring R, and R-linear maps
Set	Sets and set maps
$\mathrm{Set}_{\mathrm{f}}$	Finite sets and set maps between them
$\mathrm{Set}_{\neq 1}$	Non-singleton sets and set maps between them
$\mathrm{Set}_{\geq 1}$	Non-empty sets and set maps between them
Top	Topological spaces and continuous maps (also called the 'continuous category'[26])
Top_*	Pointed topological spaces and pointed continuous maps
$\mathrm{Top}_{\geq 2}$	Simply connected spaces and continuous maps between them
$\mathcal{T}_2$	Hausdorff spaces and continuous maps between them
$\mathcal{T}_2^{\mathrm{c}}$	Compact and Hausdorff spaces and continuous maps between them

[24] This space is also called the 'base space' of the covering space, a term that has the same meaning in the study of vector bundles, fiber bundles, and fibrations. Then, the source space of said covering space is called the 'total space' of said covering space.

[25] The latter are precisely morphisms of schemes over S, so $\mathrm{Fét}_S$ is a full subcategory of the overcategory - also called the 'slice category' - of schemes and morphisms of schemes over the connected scheme S.

[26] This terminology is used in differential topology to distinguish Top from the 'smooth category' of smooth manifolds and maps.

Bibliography

[1] T. J. Ford, *Separable Algebras* (American Mathematical Society, Rhode Island, 2017)

[2] Q. Liu, *Algebraic Geometry and Arithmetic Curves* (Oxford University Press, 2002)

[3] M. A. Knus, *Quadratic and Hermitian Forms over Rings* (Springer-Verlag, 1990)

[4] M. Raynaud, *Anneaux Locaux Henséliens* (Springer-Verlag, 2006)

[5] W. G. Dwyer and J. Spalinski, *Homotopy theories and model categories* (Elsevier Science B.V., 1995)

[6] Alexander Grothendieck and Michèle Raynaud, *Revêtements étales et groupe fondamental* (Springer-Verlag Berlin Heidelberg, 1971)

[7] Saunders Mac Lane, *Categories for the working mathematician* (Springer-Verlag New York, 1978)

[8] Emily Riehl, *Category theory in context* (Dover Publications, 2016)

[9] Harold Simmons, *An introduction to category theory* (Cambridge University Press, 2011)

[10] Saunders Mac Lane, *Homology* (Springer-Verlag Berlin-Göttingen-Heidelberg, 1963)