Étale cohomology: starting points

P. Deligne (notes by J. F. Boutot)

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Abstract

At the AMS Summer Institute in Algebraic Geometry in 1974, Deligne gave a series of lectures "Inputs of étale cohomology" intended to explain the basic étale cohomology required for his recent proof of the Weil conjectures. The lectures were written up by Boutot (in French) and published in SGA $4\frac{1}{2}$ (Lecture Notes in Math., Springer, 1977). This is a translation. It is available at www.jmilne.org/math/ under Documents. Corrections should be sent to the email address on that page. The original numbering has been retained. All footnotes have been added by the translator.

Original preface

This work contains the notes for six lectures given by P. Deligne at Arcata in August 1974 (AMS Summer School[1] on algebraic geometry) under the title "Inputs of étale cohomology". The seventh lecture became "Rapport sur la formule des traces", published in SGA $4\frac{1}{2}$. The purpose of the lectures was to give the proofs of the fundamental theorems in étale cohomology, freed from the gangue[2] of nonsense that surrounds them in SGA 4. We have not sought to state the theorems in their most general form, nor followed the reductions, sometimes clever, that their proof requires. We have on the contrary emphasized the "irreducible" case, which, once the reductions have been made, remains to be treated.

We hope that this text, which makes no claim to originality, will help the reader consult with profit the three volumes of SGA 4.

Conventions. We consider only schemes that are quasi-compact (= finite union of open affines) and quasi-separated (= such that the intersection of two open affines is quasi-compact), and we simply call them schemes.

[1] Actually, Institute.

[2] The commercially valueless material in which ore is found (Oxford English Dictionary).

Contents

I Grothendieck topologies

Grothendieck's topologies first appeared as the foundation of his theory of descent (cf. SGA 1, VI, VIII); their use in the corresponding cohomology theories came later. The same path is followed here: by formalizing the classical notions of localization, local property, and patching (§1, 2, 3), we obtain the general concept of a Grothendieck topology (§6). To justify its introduction into algebraic geometry, we prove a faithfully flat descent theorem (§4), generalizing the classical theorem 90 of Hilbert (§5).

The reader will find a more complete, but concise, exposition of this formalism in Giraud 1964. The notes of M. Artin: "Grothendieck topologies" Artin 1962 (Chapters I to III) also remain useful. The 866 pages of Exposés I to VI of SGA 4 are valuable when considering exotic topologies, such as the one that gives rise to crystalline cohomology; in working with étale topology, so close to the classical intuition, it is not strictly necessary to read them.

1 Sieves (Cribles)

Let X be a topological space and $f : X \to \mathbb{R}$ a real-valued function on X. The continuity of f is a local property. In other words, if f is continuous on every sufficiently small open[3] of X, then f is continuous on the whole of X. To formalize the notion of "local property" we introduce some definitions.

We say that a set[4] $\mathcal{U}$ of opens of X is a *sieve* if, for all $U \in \mathcal{U}$ and $V \subset U$, we have $V \in \mathcal{U}$. We say that a sieve is *covering* if the union of the opens belonging to it equals X.

The sieve generated by a family $\{U_i\}$ of opens of X is defined to be the set of open subsets U of X such that U is contained in some U_i.

We say that a property $P(U)$, defined for all open U in X, is *local* if, for every open U of X and every covering sieve $\mathcal{U}$ of U, $P(U)$ is true if and only if $P(V)$ is true for all $V \in \mathcal{U}$. For example, for a map $f : X \to \mathbb{R}$, the property "f is continuous on U" is local.

2 Sheaves (Faisceaux)

We make precise the notion of a function given locally on X.

2.1 The sieve point of view

Let $\mathcal{U}$ be a sieve of opens of X. We call a function given $\mathcal{U}$-locally on X the data of, for every $U \in \mathcal{U}$, a function f_U on U such that, if $V \subset U$, then $f_V = f_U|V$.

2.2 The Čech point of view

If the sieve $\mathcal{U}$ is generated by a family of opens U_i of X, then to give a function $\mathcal{U}$-locally on X amounts to giving a function f_i on each U_i such that $f_i|U_i \cap U_j = f_j|U_i \cap U_j$ for all i, j.

In other words, if we let $Z = \coprod U_i$, then to give a function $\mathcal{U}$-locally amounts to giving a function on Z that is constant on the fibres of the natural projection $Z \to X$.

The continuous functions form a sheaf. This means that for every covering sieve $\mathcal{U}$ of an open V of X and every function $\{f_U\}$ given $\mathcal{U}$-locally on V such that each

[3] "ouvert" isn't a noun in French, but the French mathematicians treat it as if it were. I've copied the solecism into English (should be "open subset").

[4] Underlined mathcal $\underline{\mathcal{U}}$ in the original has been replaced by mathcal $\mathcal{U}$.

f_U is continuous on U, there exists a unique continuous function f on V such that $f|U = f_U$ for all $U \in \mathcal{U}$.

3 Stacks (Champs)

We make precise the notion of vector bundle (fibré vectoriel) given locally on X.

3.1 The sieve point of view

Let $\mathcal{U}$ be a sieve of opens of X. We call a vector bundle given $\mathcal{U}$-locally on X the data of

(a) a vector bundle E_U on each $U \in \mathcal{U}$,

(b) if $V \subset U$, an isomorphism $\rho_{U,V} : E_V \xrightarrow{\sim} E_U|V$, these being such that

(c) if $W \subset V \subset U$, then the diagram

$$\begin{tikzcd} E_W \arrow[rr, "\rho_{U,W}"] \arrow[dr, "\rho_{V,W}"'] & & E_U|W \\ & E_V|W \arrow[ur, "\rho_{U,V}|W"'] & \end{tikzcd}$$

commutes, that is, $\rho_{U,W} = (\rho_{U,V}$ restricted to $W) \circ \rho_{V,W}$.

3.2 The Čech point of view

If the sieve $\mathcal{U}$ is generated by a family of opens U_i of X, then to give a vector bundle $\mathcal{U}$-locally on X amounts to giving:

(a) a vector bundle E_i on each U_i,

(b) if $U_{ij} = U_i \cap U_j = U_i \times_X U_j$, an isomorphism $\rho_{ji}: E_i|U_{ij} \xrightarrow{\sim} E_j|U_{ij}$, these being such that

(c) if $U_{ijk} = U_i \times_X U_j \times_X U_k$, then the diagram

$$\begin{tikzcd} E_i|U_{ijk} \arrow[rr, "\rho_{ki}|U_{ijk}"] \arrow[dr, "\rho_{ji}|U_{ijk}"'] & & E_k|U_{ijk} \\ & E_j|U_{ijk} \arrow[ur, "\rho_{kj}|U_{ijk}"'] & \end{tikzcd}$$

commutes, that is, $\rho_{ki} = \rho_{kj} \circ \rho_{ji}$ on U_{ijk}.

In other words, if $Z = \coprod U_i$ and we let $\pi: Z \to X$ be the natural projection, then to give a vector bundle $\mathcal{U}$-locally on X amounts to giving

(a) a vector bundle E on Z,

(b) if x and y are two points of Z such that $\pi(x) = \pi(y)$, an isomorphism $\rho_{yx}: E_x \xrightarrow{\sim} E_y$ between the fibres of E at x and y depending continuously on (x, y), these being such that

(c) if x, y, and z are three points of Z such that $\pi(x) = \pi(y) = \pi(z)$, then $\rho_{zx} = \rho_{zy} \circ \rho_{yx}$.

A vector bundle E on X defines a vector bundle given $\mathcal{U}$-locally $E_{\mathcal{U}}$, namely, the system of restrictions E_U of E to the objects of $\mathcal{U}$. The fact that the notion of vector bundle is local can be expressed as follows: for every covering sieve $\mathcal{U}$ of X, the functor $E \mapsto E_{\mathcal{U}}$ from vector bundles on X to vector bundles given $\mathcal{U}$-locally is an equivalence of categories.

If in §1, we replace "open of X" by "subset of X", we get the notion of a sieve of subspaces of X. In this context also there are patching theorems. For example, let X be a normal space and $\mathcal{C}$ a sieve of subspaces of X generated by a finite locally closed covering of X; then the functor $E \mapsto E_{\mathcal{C}}$ from vector bundles on X to vector bundles given $\mathcal{C}$-locally is an equivalence of categories.

In algebraic geometry, it is useful to also consider "sieves of spaces above X"; this is what we will see in the next paragraph.

4 *Faithfully flat descent*

4.1. In the setting of schemes, the Zariski topology is not fine enough for the study of nonlinear problems, and one is led to replace the open immersions in the preceding definitions with more general morphisms. From this point of view, descent techniques appear as localization techniques. Thus the following statement of descent can be paraphrased by saying that the properties considered are local for the faithfully flat topology. [A morphism of schemes is said to be faithfully flat if it is flat and surjective.]

PROPOSITION 4.2. *Let A be a ring and B a faithfully flat A-algebra. Then*

(i) *A sequence $\Sigma = (M' \to M \to M'')$ of A-modules is exact if the sequence $\Sigma_{(B)}$, deduced from Σ by extension of scalars to B, is exact.*

(ii) *An A-module M is of finite type (resp. finite presentation, flat, locally free of finite rank, invertible (i.e. locally free of rank one) if the B-module $M_{(B)}$ is.*

PROOF. (i) As the functor $M \mapsto M_{(B)}$ is exact (flatness of B), it suffices to show that, if an A-module N is nonzero, then $N_{(B)}$ is nonzero. If N is nonzero, then it contains a nonzero monogenic submodule $A/\mathfrak{a}$, and so $N_{(B)}$ contains a monogenic submodule $(A/\mathfrak{a})_{(B)} = B/\mathfrak{a}B$, which is nonzero by the surjectivity of the structure morphism $\varphi\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ (if $V(\mathfrak{a})$ is nonempty, then $\varphi^{-1}(V(\mathfrak{a})) = V(\mathfrak{a}B)$ is nonempty).

(ii) For any finite family (x_i) of elements of $M_{(B)}$, there exists a finitely generated submodule M' of M such that $M'_{(B)}$ contains the x_i. If $M_{(B)}$ is of finite type and the x_i generate $M_{(B)}$, then $M'_{(B)} = M_{(B)}$, so $M' = M$ and M is of finite type.

If $M_{(B)}$ is of finite presentation, then, from the above, we can find a surjection $A^n \to M$. If N is the kernel of this surjection, then the B-module $N_{(B)}$ is of finite type, so N is, and M is of finite presentation. The assertion for "flat" follows immediately from (i). The condition "locally free of finite rank" means "flat and of finite presentation", and the rank can be tested after extension of scalars to a field. □

4.3. Let X be a scheme and $\mathcal{S}$ a class of X-schemes stable under fibre products over X. A class $\mathcal{U} \subset \mathcal{S}$ is a *sieve* on X (relative to $\mathcal{S}$) if, for every morphism $\varphi: V \to U$ of X-schemes with $U, V \in \mathcal{S}$ and $U \in \mathcal{U}$, we have $V \in \mathcal{U}$. The sieve *generated* by a family $\{U_i\}$ of X-schemes in $\mathcal{S}$ is the class of $V \in \mathcal{S}$ such that there exists a morphism of X-schemes from V into one of the U_i.

4.4. Let $\mathcal{U}$ be a sieve on X. We define a quasi-coherent module given $\mathcal{U}$-locally on X to be the data of

(a) a quasi-coherent module[5] $\mathcal{E}_U$ on each $U \in \mathcal{U}$,

(b) for every $U \in \mathcal{U}$ and every morphism $\varphi: V \to U$ of X-schemes in $\mathcal{S}$, an isomorphism $\rho_\varphi: \mathcal{E}_V \xrightarrow{\sim} \varphi^*\mathcal{E}_U$, these being such that

(c) if $\psi: W \to V$ is a morphism of X-schemes in $\mathcal{S}$, then the diagram

$$\begin{tikzcd}
\mathcal{E}_W \arrow[rr, "\rho_{\varphi\circ\psi}"] \arrow[dr, "\rho_\psi"'] & & \psi^*\varphi^*\mathcal{E}_U \\
& \psi^*\mathcal{E}_V \arrow[ur, "\psi^*\rho_\psi"'] &
\end{tikzcd}$$

commutes, in other words, $\rho_{\varphi\circ\psi} = (\psi^*\rho_\varphi) \circ \rho_\psi$.

A quasi-coherent module $\mathcal{E}$ on X defines a quasi-coherent module $\mathcal{E}_{\mathcal{U}}$ given $\mathcal{U}$-locally, namely, for $\varphi_U: U \to X$ take the quasi-coherent module $\varphi_U^* E$, and for a morphism $\psi: V \to U$ take the restriction isomorphism ρ_ψ to be the canonical isomorphism $\mathcal{E}_V = (\varphi_U \circ \psi)^*\mathcal{E} \xrightarrow{\sim} \psi^*\varphi_U^*\mathcal{E} = \psi^*\mathcal{E}_U$.

THEOREM 4.5. *Let $\{U_i\} \in \mathcal{S}$ be a finite family of X-schemes flat over X such that X is the union of the images of the U_i, and let $\mathcal{U}$ be the sieve generated by $\{U_i\}$. Then the functor $\mathcal{E} \mapsto \mathcal{E}_{\mathcal{U}}$ is an equivalence from the category of quasi-coherent modules on X to the category of quasi-coherent modules given $\mathcal{U}$-locally.*

PROOF. We treat only the case where X is affine and $\mathcal{U}$ is generated by an affine X-scheme U faithfully flat over X. The reduction to this case is formal. Let $X = \mathrm{Spec}(A)$ and $U = \mathrm{Spec}(B)$.

If the morphism $U \to X$ has a section, then X belongs to the sieve $\mathcal{U}$, and the assertion is obvious. We will reduce the general case to this case.

A quasi-coherent module given $\mathcal{U}$-locally defines modules M', M'', and M''' on U, $U \times_X U$, and $U \times_X U \times_X U$, and isomorphisms $\rho : p^*M^\bullet \simeq M^\bullet$ for every projection morphism p between these spaces. There is a *cartesian diagram*

$$M^* : M' \rightrightarrows M'' \ \overrightarrow{\underrightarrow{\to}}\ M'''$$

over

$$U_* : U \leftleftarrows U \times_X U \ \overleftarrow{\underleftarrow{\leftarrow}}\ U \times_X U \times_X U.$$

[5]We use script (eucal) instead of roman for sheaves.

Conversely, M^* determines the module given $\mathcal{U}$-locally: for $V \in \mathcal{U}$, there exists $\varphi: V \to U$, and we put $M_U = \varphi^* M'$; for $\varphi_1, \varphi_2: V \to U$, we have natural identifications $\varphi_1^* M' \simeq (\varphi_1 \times \varphi_2)^* M'' \simeq \varphi_2^* M'$, and we see using M''' that these identifications are compatible, and so the definition is valid. In summary, to give a coherent module $\mathcal{U}$-locally is the same as giving a cartesian diagram M^* over U_*.

Let us translate this into algebraic terms: to give M^* amounts to giving a cartesian diagram of modules

$$\begin{tikzcd} M' \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & M'' \arrow[r, shift left=2, "\partial_0"] \arrow[r, "\partial_1" description] \arrow[r, shift right=2, "\partial_2"'] & M''' \end{tikzcd}$$

over the diagram of rings

$$\begin{tikzcd} B \arrow[r, shift left, "\partial_0"] \arrow[r, shift right, "\partial_1"'] & B \otimes_A B \arrow[r, shift left=2, "\partial_0"] \arrow[r, "\partial_1" description] \arrow[r, shift right=2, "\partial_2"'] & B \otimes_A B \otimes_A B. \end{tikzcd}$$

[More precisely: we have $\partial_i(bm) = \partial_i(b) \cdot \partial_i(m)$, the usual identities such as $\partial_0 \partial_1 = \partial_0 \partial_0$ are true, and "cartesian" means that the morphisms $\partial_i: M' \otimes_{B,\partial_i} (B \otimes_A B) \to M''$ and $M'' \otimes_{B \otimes_A B, \partial_i} (B \otimes_A B \otimes_A B) \to M'''$ are isomorphisms.]

Now $\mathcal{E} \mapsto \mathcal{E}_{\mathcal{U}}$ becomes the functor sending an A-module M to

$$M^* = \left(M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B \Rrightarrow M \otimes_A B \otimes_A B \otimes_A B \right)$$

It admits the functor

$$(M' \rightrightarrows M'' \Rrightarrow M''') \mapsto \operatorname{Ker}\,(M' \rightrightarrows M'')$$

as a right adjoint. We have to prove that the adjunction arrows

$$M \to \operatorname{Ker}(M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B)$$

and

$$\operatorname{Ker}(M' \rightrightarrows M'') \otimes_A B \to M'$$

are isomorphisms. According to 4.2(i), it suffices to do this after a faithfully flat base change $A \to A'$ (B becoming $B' = B \otimes_A A'$). Taking $A' = B$ brings us back to the case where $U \to X$ admits a section. □

5 A special case: Hilbert's theorem 90

5.1. Let k be a field, k' a Galois extension of k, and $G = \operatorname{Gal}(k'/k)$. Then the homomorphism

$$k' \otimes_k k' \to \prod_{\sigma \in G} k'$$
$$x \otimes y \mapsto \{x \cdot \sigma(y)\}_{\sigma \in G}$$

is bijective.

It follows that to give a module locally for the sieve generated by $\mathrm{Spec}(k')$ over $\mathrm{Spec}(k)$ is the same as giving a k'-vector space with a semi-linear action of G, i.e.,

(a) a k'-vector space V',

(b) for all $\sigma \in G$, an endomorphism φ_σ of the underlying group of V' such that

$$\varphi_\sigma(\lambda v) = \sigma(\lambda)\varphi_\sigma(v), \text{ for all } \lambda \in k' \text{ and } v \in V',$$

satisfying the condition

(c) for all $\sigma, \tau \in G$, we have $\varphi_{\tau\sigma} = \varphi_\tau \circ \varphi_\sigma$.

Let $V = V'^G$ be the group of invariants for this action of G. It is a k-vector space and, according to Theorem 4.5, we have

PROPOSITION 5.2. *The inclusion of V into V' defines an isomorphism $V \otimes_k k' \xrightarrow{\sim} V'$.*

In particular, if V' has dimension 1 and $v' \in V$ is nonzero, then φ_σ is determined by the constant [6] $c(\sigma) \in k'^\times$ such that $\varphi_\sigma(v') = c(\sigma)v'$, and the condition c) becomes

$$c(\tau\sigma) = c(\tau) \cdot \tau(c(\sigma)).$$

According to the proposition, there is a nonzero invariant vector $v = \mu v'$, $\mu \in k'^\times$. Therefore, for all $\sigma \in G$,

$$c(\sigma) = \mu \cdot \sigma(\mu^{-1}).$$

In other words, every 1-cocycle of G with values in $k'^\times$ is a coboundary:

COROLLARY 5.3. *We have $H^1(G, k'^\times) = 0$.*

6 Grothendieck topologies

We now rewrite the definitions of the preceding paragraphs in an abstract framework that encompasses both the case of topological spaces and that of schemes.

6.1. Let $\mathcal{S}$ be a category and U an object of $\mathcal{S}$. We call a *sieve* on U a subset $\mathcal{U}$ of $\mathrm{Ob}(\mathcal{S}/U)$ such that, if $\varphi\colon V \to U$ belongs to $\mathcal{U}$ and $\psi\colon W \to V$ is a morphism in $\mathcal{S}$, then $\varphi \circ \psi\colon W \to U$ belongs to $\mathcal{U}$.

If $\{\varphi_i : U_i \to U\}$ is a family of morphisms, then the sieve generated by the U_i is defined to be the set of morphisms $\varphi\colon V \to U$ which factor through one of the φ_i.

If $\mathcal{U}$ is a sieve on U and $\varphi\colon V \to U$ is a morphism, then the restriction $\mathcal{U}_V$ of $\mathcal{U}$ to V is defined to be the sieve on V consisting of the morphisms $\psi\colon W \to V$ such that $\varphi \circ \psi\colon W \to U$ belongs to $\mathcal{U}$.

6.2. A *Grothendieck topology* on $\mathcal{S}$ is the data of a set $C(U)$ of sieves for every object U of $\mathcal{S}$, called the covering sieves, such that the following axioms are satisfied:

(a) The sieve generated by the identity morphism of U is covering.

[6] A^* (group of units in the ring A) in the original has been replaced by $A^\times$.

(b) If $\mathcal{U}$ is a covering sieve U and $V \to U$ is a morphism, then the sieve $\mathcal{U}_V$ is covering.

(c) A locally covering sieve is covering. In other words, if $\mathcal{U}$ is a covering sieve on U and $\mathcal{U}'$ is a sieve on U such that, for all $V \to U$ belonging to $\mathcal{U}$, the sieve $\mathcal{U}'_V$ on V is covering, then $\mathcal{U}'$ is covering.

A *site* is defined to be a category equipped with a Grothendieck topology.

6.3. Let $\mathcal{S}$ be a site. A *presheaf* $\mathcal{F}$ on $\mathcal{S}$ is a contravariant functor from $\mathcal{S}$ to the category of sets. A section of $\mathcal{F}$ over an object U is an element of $\mathcal{F}(U)$. For a morphism $V \to U$ and an $s \in \mathcal{F}(U)$, we let $s|V$ (s restricted to V) denote the image of s in $\mathcal{F}(V)$.

Let $\mathcal{U}$ be a sieve on U. We call a section of $\mathcal{F}$ given $\mathcal{U}$-locally the data of, for every $V \to U$ belonging to $\mathcal{U}$, a section $s_V \in \mathcal{F}(V)$ such that, for every morphism $W \to V$, we have $s_V|W = s_W$. We say that $\mathcal{F}$ is a *sheaf* if, for every object U of $\mathcal{S}$, every covering sieve $\mathcal{U}$ on U, and every section $\{s_V\}$ given $\mathcal{U}$-locally, there is a unique section $s \in \mathcal{F}(U)$ such that $s|V = s_V$ for all $V \to U$ belonging to $\mathcal{U}$.

We define in a similar way *abelian sheaves* by replacing the category of sets by that of abelian groups. One shows that the category of abelian sheaves on $\mathcal{S}$ is a abelian category with enough injectives. A sequence $\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H}$ of sheaves is exact if, for every object U of $\mathcal{S}$ and every $s \in \mathcal{G}(U)$ such that $g(s) = 0$, there exists locally a t such that $f(t) = s$, i.e., there exists a covering sieve $\mathcal{U}$ of U and sections $t_V \in \mathcal{F}(V)$ for all $V \in \mathcal{U}$ such that $f(t_V) = s|V$.

6.4 Examples

We have seen two above.

(a) Let X be a topological space and $\mathcal{S}$ the category whose objects are the opens of X and whose morphisms are the natural inclusions. The Grothendieck topology on $\mathcal{S}$ corresponding to the usual topology X is that for which a sieve on an open U of X is covering if the union of its opens equals U. It is clear that the category of sheaves on $\mathcal{S}$ is equivalent to the category of sheaves on X in the usual sense.

(b) Let X be a scheme and $\mathcal{S}$ the category of schemes over X. We define the fpqc (faithfully flat quasi-compact) topology on $\mathcal{S}$ to be the Grothendieck topology for which a sieve on an X-scheme U is covering if it is generated by a finite family of flat morphisms whose images cover U.

6.5 Cohomology

We will always assume that the category $\mathcal{S}$ has a final object X. The group of global sections of an abelian sheaf $\mathcal{F}$, denoted $\Gamma\mathcal{F}$ or $H^0(X, \mathcal{F})$, is defined to be the group $\mathcal{F}(X)$. The functor $\mathcal{F} \mapsto \Gamma\mathcal{F}$ from the category of abelian sheaves on $\mathcal{S}$ to the category of abelian groups is left exact, and we denote its derived functors (or satellites) by $H^i(X, -)$. These cohomology groups represent the obstructions to passing from the local to the global. By definition, if

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is an exact sequence of abelian sheaves, then there is a long exact cohomology sequence:

$$0 \to H^0(X,\mathcal{F}) \to H^0(X,\mathcal{G}) \to H^0(X,\mathcal{H}) \to H^1(X,\mathcal{F}) \to \cdots$$
$$\cdots \to H^n(X,\mathcal{F}) \to H^n(X,\mathcal{G}) \to H^n(X,\mathcal{H}) \to H^{n+1}(X,\mathcal{F}) \to \cdots$$

6.6 [7]

Given an abelian sheaf $\mathcal{F}$ on $\mathcal{S}$, we define an $\mathcal{F}$-*torsor* to be a sheaf $\mathcal{G}$ endowed with an action $\mathcal{F} \times \mathcal{G} \to \mathcal{G}$ of $\mathcal{F}$ such that locally (after restriction to all the objects of a covering sieve of the final object X) $\mathcal{G}$ equipped with the $\mathcal{F}$-action is isomorphic to $\mathcal{F}$ equipped with the canonical action $\mathcal{F} \times \mathcal{F} \to \mathcal{F}$ by translations.

The set $H^1(X,\mathcal{F})$ can be interpreted as the set of isomorphism classes of $\mathcal{F}$-torsors.

II The étale topology

We specialize the definitions of the preceding chapter to the case of the étale topology of a scheme X (§1, 2, 3). The corresponding cohomology coincides in the case that X is the spectrum of a field K with the Galois cohomology of K (§4).

1 The étale topology

We begin by reviewing the notion of an étale morphism.

DEFINITION 1.1. Let A be a (commutative) ring. An A-algebra B is said to be *étale* if it is of finite presentation and if the following equivalent conditions are satisfied:

(a) For every A-algebra C and ideal J of square zero in C, the canonical map

$$\mathrm{Hom}_{A\text{-alg}}(B,C) \to \mathrm{Hom}_{A\text{-alg}}(B,C/J)$$

is a bijection.

(b) B is a flat A-module and $\Omega_{B/A} = 0$ (here $\Omega_{B/A}$ denotes the module of relative differentials).

(c) Let $B = A[X_1,\ldots,X_n]/I$ be a presentation of B. Then, for every prime ideal $\mathfrak{p}$ of $A[X_1,\ldots,X_n]$ containing I, there exist polynomials $P_1,\ldots,P_n \in I$ such that $I_{\mathfrak{p}}$ is generated by the images of $P_1,\ldots,P_n$ and $\det(\partial P_i/\partial X_j) \notin \mathfrak{p}$.

[Cf. SGA 1, Exposé I, or Raynaud 1970, Chapitre V.]

We say that a morphism of schemes $f : X \to S$ is *étale* if, for all $x \in X$, there exists an affine open neighbourhood $U = \mathrm{Spec}(A)$ of $f(x)$ and an affine open neighbourhood $V = \mathrm{Spec}(B)$ of x in $X \times_S U$ such that B is an étale A-algebra.

[7] The original repeats the number 6.5.

1.2 Examples

(a) When A is a field, an A-algebra B is étale if and only if it is a finite product of separable extensions of A.

(b) When X and S are schemes of finite type over $\mathbb{C}$, a morphism $f:X \to S$ is étale if and only if the associated analytic map $f^{\text{an}}: X^{\text{an}} \to S^{\text{an}}$ is a local isomorphism.

1.3 Sorite

(a) (base change) If $f:X \to S$ is an étale morphism, then so also is $f_{S'}: X \times_S S' \to S'$ for any morphism $S' \to S$.

(b) (composition) The composite of two étale morphisms is an étale morphism.

(c) If $f:X \to S$ and $g:Y \to S$ are two étale morphisms, then every S-morphism from X to Y is étale.

(d) (descent) Let $f:X \to S$ be a morphism. If there exists a faithfully flat morphism $S' \to S$ such that $f_{S'}: X \times_S S' \to S'$ is étale, then f is étale.

1.4. Let X be a scheme and $\mathcal{S}$ the category of étale X-schemes. According to (1.3.c) every morphism in $\mathcal{S}$ is étale. We define the *étale topology* on $\mathcal{S}$ to be the topology for which a sieve over U is covering if it is generated by a finite family of morphisms $\varphi_i: U_i \to U$ such the union of the images of the φ_i covers U. We define the *étale site* of X, denoted X_{et}, to be $\mathcal{S}$ endowed with the étale topology.

2 *Examples of sheaves*

2.1 Constant sheaf

Let C be an abelian group, and suppose for simplicity that X is noetherian. We let $\underline{C}_X$ (or just C if there is no ambiguity) denote the sheaf defined by $U \mapsto C^{\pi_0(U)}$, where $\pi_0(U)$ is the (finite) set of connected components of U. The most important case will be $C = \mathbb{Z}/n$. By definition

$$H^0(X, \mathbb{Z}/n) = (\mathbb{Z}/n)^{\pi_0(X)}.$$

In addition, $H^1(X, \mathbb{Z}/n)$ is the set of isomorphism classes of $\mathbb{Z}/n$-torsors (I.6.6), i.e., of Galois finite étale coverings of X with group $\mathbb{Z}/n$. In particular, if X is connected and $\pi_1(X)$ is its fundamental group for some chosen base point, then

$$H^1(X, \mathbb{Z}/n) = \operatorname{Hom}(\pi_1(X), \mathbb{Z}/n).$$

2.2 Multiplicative group

We denote by $\mathbb{G}_{m,X}$ (or $\mathbb{G}_m$ if there is no ambiguity) the sheaf $U \mapsto \Gamma(U, \mathcal{O}_U^\times)$. It is indeed a sheaf thanks to the faithfully flat descent theorem (I.4.5). We have by definition

$$H^0(X, \mathbb{G}_m) = H^0(X, \mathcal{O}_X)^\times.$$

In particular, if X is reduced, connected, and proper over an algebraically closed field k, then

$$H^0(X, \mathbb{G}_m) = k^\times.$$

PROPOSITION 2.3. *There is an isomorphism*

$$H^1(X, \mathbb{G}_m) = \mathrm{Pic}(X),$$

where Pic(X) *is the group of isomorphism classes of invertible sheaves on* X.

PROOF. Let $*$ be the functor which, to an invertible sheaf $\mathcal{L}$ over X, attaches the following presheaf $\mathcal{L}^*$ on X_{et}: for $\varphi\colon U \to X$ étale,

$$\mathcal{L}^*(U) = \mathrm{Isom}_U(\mathcal{O}_U, \varphi^*\mathcal{L}).$$

According to I.4.2(i) and I.4.5[8] (full faithfulness), this presheaf is a sheaf; it is even a $\mathbb{G}_m$-torsor. We see immediately that

(a) the functor $*$ is compatible with (étale) localization;

(b) it induces an equivalence of the category of trivial invertible sheaves (i.e., isomorphic to $\mathcal{O}_X$) with the category of trivial $\mathbb{G}_m$-torsors: $\mathcal{L}$ is trivial if and only if $\mathcal{L}^*$ is.

Moreover, according to I.4.2(ii) and I.4.5,

(c) the notion of invertible sheaf is local for the étale topology.

It follows formally from a), b), c) that $*$ is an equivalence from the category of invertible sheaves on X to the category of $\mathbb{G}_m$-torsors on X_{et}. It induces the isomorphism sought. The inverse equivalence can be constructed as follows: if $\mathcal{T}$ is a $\mathbb{G}_m$-torsor, then there exists a finite étale covering $\{U_i\}$ of X such that the torsors $\mathcal{T}|U_i$ are trivial; $\mathcal{T}$ is then trivial on every V étale over X belonging to the sieve $\mathcal{U} \subset X_{\mathrm{et}}$ generated by $\{U_i\}$. On each $V \in \mathcal{U}$, $\mathcal{T}|V$ corresponds to an invertible sheaf $\mathcal{L}_V$ (by b)) and the $\mathcal{L}_V$ constitute an invertible sheaf given $\mathcal{U}$-locally $\mathcal{L}_{\mathcal{U}}$ (by a)). By c), the latter comes from an invertible sheaf $\mathcal{L}(\mathcal{T})$ on X, and $\mathcal{T} \mapsto \mathcal{L}(\mathcal{T})$ is the inverse of $*$ sought. □

2.4 Roots of unity

For an integer $n > 0$, we define the sheaf of nth roots of unity, denoted μ_n, to be the kernel of "raising to the nth power" on $\mathbb{G}_m$. If X is a scheme over a separably closed field k and n is invertible in k, then the choice of a primitive nth root of unity $\zeta \in k$ defines an isomorphism $i \mapsto \zeta^i$ of $\mathbb{Z}/n$ with μ_n.

The relationship between cohomology with coefficients in μ_m and cohomology with coefficients in $\mathbb{G}_m$ is given by the exact cohomology sequence deduced from

KUMMER THEORY 2.5. *If n is invertible on X, then raising to the nth power in $\mathbb{G}_m$ is a sheaf epimorphism, so there is an exact sequence*

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0.$$

[8] The original had 4.2 and 4.5. Similar corrections are made elsewhere.

PROOF. Let $U \to X$ be an étale morphism and let $a \in \mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U^\times)$. As n is invertible on U, the equation $T^n - a = 0$ is separable, i.e., $U' = \operatorname{Spec}(\mathcal{O}_U[T]/(T^n - a))$ is étale over U. As $U' \to U$ is surjective and a admits an nth root on U', we obtain the result. □

3 Fibres, direct images

3.1. A *geometric point* of X is a morphism $\overline{x} \to X$, where $\overline{x}$ is the spectrum of a separably closed field $k(\overline{x})$. By an abuse of language, we denote it by $\overline{x}$, the morphism $\overline{x} \to X$ being understood. If x is the image of $\overline{x}$ in X, then we say that $\overline{x}$ is centred at x. When the field $k(\overline{x})$ is an algebraic extension of the residue field $k(x)$, we say that $\overline{x}$ is an *algebraic* geometric point of X.

We define an *étale neighbourhood* of $\overline{x}$ to be a commutative diagram

$$\begin{tikzcd} & U \arrow[d] \\ \overline{x} \arrow[ur] \arrow[r] & X, \end{tikzcd}$$

where $U \to X$ is an étale morphism.

The *strict localization* of X at $\overline{x}$ is the ring $\mathcal{O}_{X,\overline{x}} = \varinjlim \Gamma(U, \mathcal{O}_U)$, where the inductive limit is over the étale neighbourhoods of $\overline{x}$. It is a strictly henselian local ring whose residue field is the separable closure of the residue field $k(x)$ of X at x in $k(\overline{x})$. It plays the role of the local ring for the étale topology.

3.2. Given a sheaf $\mathcal{F}$ on X_{et}, we define the *fibre* of $\mathcal{F}$ at $\overline{x}$ to be the set (resp. the group, ...) $\mathcal{F}_{\overline{x}} = \varinjlim \mathcal{F}(U)$, where the inductive limit is again over the étale neighbourhoods of $\overline{x}$.

In order for a homomorphism $\mathcal{F} \to \mathcal{G}$ of sheaves to be a mono-/epi-/isomorphism, it is necessary and sufficient that this is so of the morphisms $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}}$ induced on the fibres at all the geometric points of X. When X is of finite type over an algebraically closed field, it suffices to check this for the rational points of X.

3.3. If $f : X \to Y$ is a morphism of schemes and $\mathcal{F}$ a sheaf on X_{et}, then the *direct image* $f * \mathcal{F}$ of $\mathcal{F}$ by f is the sheaf on Y_{et} defined by $f_*\mathcal{F}(V) = \mathcal{F}(X \times_Y V)$ for all V étale over Y.

The functor $f_* : \mathrm{Sh}(X_{\text{et}}) \to \mathrm{Sh}(Y_{\text{et}})$ is left exact. Its right derived functors $R^q f_*$ are called its higher direct images. If $\overline{y}$ is a geometric point of Y, we have

$$(R^q f_*\mathcal{F})_{\overline{y}} = \varinjlim H^q(V \times_Y X, \mathcal{F}),$$

the inductive limit being over the étale neighbourhoods V of $\overline{y}$.

Let $\mathcal{O}_{Y,\overline{y}}$ be the strict localization of Y at $\overline{y}$, let $\widetilde{Y} = \operatorname{Spec}(\mathcal{O}_{Y,\overline{y}})$, and let $\widetilde{X} = X \times_Y \widetilde{Y}$. We can extend $\mathcal{F}$ to $\widetilde{X}_{\text{et}}$ (this is a special case of the general notion of an inverse image) as follows: if $\widetilde{U}$ is a scheme étale over $\widetilde{X}$, then there exists an étale neighbourhood V of $\overline{y}$ and a scheme U étale over $X \times_Y V$ such that $\widetilde{U} = U \times_V \widetilde{Y}$; we put

$$\mathcal{F}(\widetilde{U}) = \varinjlim \mathcal{F}\left(U \times_V V'\right),$$

the inductive limit being over the étale neighbourhoods V' of $\bar{y}$ which dominate V. With this definition, we have

$$\left(R^q f_* \mathcal{F}\right)_{\bar{y}} = H^q(\widetilde{X}, \mathcal{F}).$$

The functor f_* has a left adjoint f^*, the "inverse image" functor. If $\bar{x}$ is a geometric point of X and $f(\bar{x})$ its image in Y, we have $(f^*\mathcal{F})_{\bar{x}} = \mathcal{F}_{f(\bar{x})}$. This formula shows that f^* is an exact functor. The functor $f*$ thus transforms injective sheaves into injective sheaves, and the spectral sequence of the composite functor $\Gamma \circ f_*$ (resp. $g_* \circ f_*$) gives the

LERAY SPECTRAL SEQUENCE 3.4. *Let $\mathcal{F}$ be an abelian sheaf on X_{et}, and let $f : X \to Y$ be a morphism of schemes (resp. let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes). We have a spectral sequence*

$$E_2^{pq} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

$$\text{(resp.} \qquad E_2^{pq} = R^p g_* R^q f_* \mathcal{F} \Rightarrow R^{p+q}(gf)_* \mathcal{F}).$$

COROLLARY 3.5. *If $R^q f_* \mathcal{F} = 0$ for all $q > 0$, then $H^p(Y, f_*\mathcal{F}) = H^p(X, \mathcal{F})$ (resp. $R^p g_*(f_*\mathcal{F}) = R^p(gf)_*\mathcal{F})$ for all $p \geqslant 0$.*

This applies in particular in the following case:

PROPOSITION 3.6. *Let $f : X \to Y$ be a finite morphism (or, by passage to the limit, an integral morphism) and $\mathcal{F}$ an abelian sheaf on X. Then $R^q f_* \mathcal{F} = 0$, for all $q > 0$.*

Indeed, let $\bar{y}$ be a geometric point of Y, $\widetilde{Y}$ the spectrum of the strict localization of Y at y, and $\widetilde{X} = X \times_Y \widetilde{Y}$. According to the above, it suffices show that $H^q(\widetilde{X}, \mathcal{F}) = 0$ for all $q > 0$. But $\widetilde{X}$ is the spectrum of a product of strictly Henselian local rings (cf. Raynaud 1970, Chapter I), and the functor $\Gamma(\widetilde{X}, -)$ is exact because every étale surjective map to $\widetilde{X}$ admits a section, whence the assertion.

4 *Galois cohomology*

For $X = \mathrm{Spec}(K)$, the spectrum of a field, we shall see that étale cohomology can be identified with Galois cohomology.

4.1. Let us begin with a topological analogy. If K is the field of functions of an integral affine algebraic variety $Y = \mathrm{Spec}(A)$ over $\mathbb{C}$, then

$$K = \varinjlim_{f \in A} A[1/f].$$

In other words, $X = \varprojlim U$, where U runs over all open sets of Y. We know that there are arbitrarily small Zariski opens which are $K(\pi, 1)$'s for the classical topology. Therefore, we should not be surprised if we can consider $\mathrm{Spec}(K)$ itself to be a $K(\pi, 1)$, π being the fundamental group (in the algebraic sense) of $X = \mathrm{Spec}(K)$, i.e., the Galois group of $\overline{K}/K$, where $\overline{K}$ is a separable closure of K.

4.2. More precisely, let K be a field, $\bar{K}$ a separable closure of K, and $G = \mathrm{Gal}(\bar{K}/K)$ the Galois group with its natural topology. To any finite étale K-algebra A (finite product of separable extensions of K), attach the finite set $\mathrm{Hom}_K(A, \bar{K})$. The Galois group G acts on this set through a discrete (hence finite) quotient. If $A = K[T]/(F)$, then this set can be identified with the set of roots of the polynomial F in $\bar{K}$. Galois theory, in the form given to it by Grothendieck says that,

PROPOSITION 4.3. *The functor*

$$\left\{\begin{array}{l}\text{finite étale}\\ K\text{-algebras}\end{array}\right\} \to \left\{\begin{array}{l}\text{finite sets on which}\\ G \text{ acts continuously}\end{array}\right\},$$

which to an étale algebra A attaches $\mathrm{Hom}_K(A, \bar{K})$, is an anti-equivalence of categories.

We can deduce from this an analogous description of the sheaves for the étale topology on $\mathrm{Spec}(K)$,

PROPOSITION 4.4. *The functor*

$$\left\{\begin{array}{l}\text{étale sheaves}\\ \text{on } \mathrm{Spec}(K)\end{array}\right\} \to \left\{\begin{array}{l}\text{sets on which}\\ G \text{ acts continuously}\end{array}\right\},$$

which to a sheaf $\mathcal{F}$ attaches its fibre $\mathcal{F}_{\bar{K}}$ at the geometric point $\mathrm{Spec}(\bar{K})$, is an equivalence of categories.

The group G is said to act continuously on a set E if the stabilizer of every element of E is an open subgroup of G. The functor in the reverse direction has an obvious description: let A a finite étale K-algebra, $U = \mathrm{Spec}(A)$, and $U(\bar{K}) = \mathrm{Hom}_K(A, \bar{K})$ the G-set corresponding to A; then $\mathcal{F}(U) = \mathrm{Hom}_{G\text{-sets}}(U(\bar{K}), \mathcal{F}_{\bar{K}})$.

In particular, if $X = \mathrm{Spec}(K)$, then $\mathcal{F}(X) = \mathcal{F}_{\bar{K}}^G$. When we consider only abelian sheaves, we get, by passage to the derived functors, canonical isomorphisms

$$H^q(X_{\mathrm{et}}, \mathcal{F}) = H^q(G, \mathcal{F}_{\bar{K}})$$

4.5 Examples

(a) The constant sheaf $\mathbb{Z}/n$ corresponds to $\mathbb{Z}/n$ with the trivial action of G.

(b) The sheaf of nth roots of unity μ_n corresponds to the group $\mu_n(\bar{K})$ of nth roots of unity in $\bar{K}$ with the natural action of G.

(c) The sheaf $\mathbb{G}_m$ corresponds to the group $\bar{K}^\times$ with the natural action of G.

III The cohomology of curves

In the case of topological spaces, unwindings[9] using the Künneth formula and simplicial decompositions allow one to reduce the calculation of the cohomology to that of the interval $I = [0, 1]$ for which we have $H^0(I, \mathbb{Z}) = \mathbb{Z}$ and $H^q(I, \mathbb{Z}) = 0$ for $q > 0$.

[9] "dévissages" in the original. The word "dévissage", literally "unscrewing", is used by French mathematicians to mean the reduction of a problem to a special case, often by fairly standard arguments.

In our case, the unwindings lead to more complicated objects, namely, to curves over an algebraically closed field. We will calculate their cohomology in this chapter. The situation is more complex than in the topological case because the cohomology groups are zero only for $q > 2$. The essential ingredient for the calculations is the vanishing of the Brauer group of the function field of such a curve (Tsen's theorem, §2).

1 The Brauer group

First recall the classical definition:

DEFINITION 1.1. Let K be a field and A a K-algebra of finite dimension. We say that A is a *central simple algebra* over K if it satisfies the following equivalent conditions:

(a) A has no nontrivial two-sided ideal and its centre is K.

(b) There exists a finite Galois extension K'/K such that $A_{K'} = A \otimes_K K'$ is isomorphic to a matrix algebra over K'.

(c) A is K-isomorphic to a matrix algebra over a skew field with centre K.

Two such algebras are said to be equivalent if the skew fields associated with them by c) are K-isomorphic. When the algebras have the same dimension, this is the same as requiring that they are K-isomorphic. Tensor product defines by passage to the quotient an abelian group structure on the set of equivalence classes. It is this group that we classically call *the Brauer group* of K and that we denote $\mathrm{Br}(K)$.

1.2. We let $\mathrm{Br}(n,K)$ denote the set of K-isomorphism classes of K-algebras A such that there exists a finite Galois extension K' of K for which $A_{K'}$ is isomorphic to the algebra $M_n(K')$ of $n \times n$ matrices over K'. By definition $\mathrm{Br}(K)$ is the union of the subsets $\mathrm{Br}(n,K)$ for $n \in \mathbb{N}$. Let $\bar{K}$ be a separable closure of K and $G = \mathrm{Gal}(\bar{K}/K)$. Then $\mathrm{Br}(n,K)$ is the set of "forms" of $M_n(\bar{K})$, and so it is canonically isomorphic to $H^1(G, \mathrm{Aut}(M_n(\bar{K})))$. Every automorphism of $M_n(\bar{K})$ is inner. Therefore the group $\mathrm{Aut}(M_n(\bar{K}))$ can be identified with the projective linear group $\mathrm{PGL}(n,\bar{K})$, and there is a canonical bijection

$$\theta_n\colon \mathrm{Br}(n,K) \xrightarrow{\sim} H^1(G, \mathrm{PGL}(n,\bar{K})).$$

On the other hand, the exact sequence

$$1 \to \bar{K}^\times \to \mathrm{GL}(n,\bar{K}) \to \mathrm{PGL}(n,\bar{K}) \to 1, \tag{1.2.1}$$

defines a coboundary map

$$\Delta_n\colon H^1(G, \mathrm{PGL}(n,\bar{K})) \to H^2(G,\bar{K}^\times).$$

On composing θ_n with Δ_n, we get a map

$$\delta_n\colon \mathrm{Br}(n,K) \to H^2(G,\bar{K}^\times).$$

It is easily checked that the maps δ_n are compatible among themselves and define a group homomorphism

$$\delta\colon \mathrm{Br}(K) \to H^2(G,\bar{K}^\times).$$

PROPOSITION 1.3. *The homomorphism* $\delta\colon \mathrm{Br}(K) \to H^2(G, \bar{K}^\times)$ *is bijective.*

This is a consequence of the next two lemmas.

LEMMA 1.4. *The map* $\Delta_n\colon H^1\big(G, \mathrm{PGL}(n, \bar{K})\big) \to H^2(G, \bar{K}^\times)$ *is injective.*

According to Serre 1965, cor. to prop. I-44, it suffices to check that, if the exact sequence (1.2.1) is twisted by an element of $H^1\big(G, \mathrm{PGL}(n, \bar{K})\big)$, then the H^1 of the middle group is trivial. This middle group is the group of $\bar{K}$-points of the multiplicative group of a central simple algebra A of degree n^2 over K. To prove that $H^1(G, A^\times_{\bar{K}}) = 0$, we interpret $A^\times$ as the group of automorphisms of a free A-module L of rank 1 and H^1 as the set of "forms" of L; these are A-modules of rank n^2 over K, which are automatically free.

LEMMA 1.5. *Let* $\alpha \in H^2(G, \bar{K}^\times)$, *let* K' *be a finite extension of* K *contained in* $\bar{K}$ *with* $[K' : K] = n$, *and let* $G' = \mathrm{Gal}(\bar{K}/K')$. *If the image of* α *in* $H^2(G', \bar{K}^\times)$ *is zero, then* α *belongs to the image of* Δ_n.

Note first that we have

$$H^2(G', \bar{K}^\times) \simeq H^2\big(G, (\bar{K} \otimes_K K'^\times\big).$$

(From a geometric point of view, if we let $x = \mathrm{Spec}(K)$ and $x' = \mathrm{Spec}(K')$, and we let $\pi\colon x' \to x$ denote the canonical morphism, then $R^q\pi_*(\mathbb{G}_{m,x'}) = 0$ for $q > 0$ and hence $H^q(x', \mathbb{G}_{m,X'}) \simeq H^q(x, \pi_*\mathbb{G}_{m,X'})$ for $q \geqslant 0$.)

Furthermore, the choice of a basis for K' as a vector space over K allows us to define a homomorphism

$$(\bar{K} \otimes_K K')^\times \to \mathrm{GL}(n, \bar{K}),$$

which sends an element x to the endomorphism "multiplication by x" of $\bar{K} \otimes_K K'$. We then have a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & \bar{K}^\times \arrow[r] \arrow[d, equal] & (\bar{K} \otimes_K K')^\times \arrow[r] \arrow[d] & (\bar{K} \otimes_K K')^\times/\bar{K}^\times \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \bar{K}^\times \arrow[r] & \mathrm{GL}(n, \bar{K}) \arrow[r] & \mathrm{PGL}(n, \bar{K}) \arrow[r] & 1
\end{tikzcd}$$

The lemma now follows from the commutative diagram deduced from the above diagram by passing to the cohomology

$$\begin{tikzcd}
H^1(G, (\bar{K} \otimes_K K')^\times/\bar{K}^\times) \arrow[r] \arrow[d] & H^2(G, \bar{K}^\times) \arrow[r] \arrow[d, equal] & H^2(G, (\bar{K} \otimes_K K')^\times) \\
H^1(G, \mathrm{PGL}(n, \bar{K})) \arrow[r, "\Delta_n"] & H^2(G, \bar{K}^\times).
\end{tikzcd}$$

The knowledge of the Brauer group, in particular its vanishing, is extremely important in Galois cohomology as the following proposition shows.

PROPOSITION 1.6. *Let K a field, $\bar{K}$ a separable closure of K, and $G = \mathrm{Gal}(\bar{K}/K)$. Suppose that $\mathrm{Br}(K') = 0$ for every finite extension K' of K. Then*

(i) $H^q(G, \bar{K}^\times) = 0$ for all $q > 0$.

(ii) $H^q(G, F) = 0$ for all torsion G-modules F and all $q \geqslant 2$.

For the proof, cf. Serre 1965 or Serre 1968.

2 Tsen's theorem

DEFINITION 2.1. A field K is said to be C_1 if every nonconstant homogeneous polynomial $f(x_1, \ldots, x_n)$ of degree $d < n$ has a non-trivial zero.

PROPOSITION 2.2. *If K is a C_1 field, then $\mathrm{Br}(K) = 0$.*

We have to show that every skew field D, finite over K with centre K, equals K. Let r^2 be the degree of D over K and $\mathrm{Nrd}\colon D \to K$ the reduced norm. [Locally for the étale topology on K, D is isomorphic — non canonically — to a matrix algebra M_r and the determinant map on M_r defines a reduced norm map on D. This is independent of the isomorphism chosen between D and M_r because all automorphisms of M_r are inner and similar matrices have the same determinant. This map, defined locally for the étale topology, descends because of its local uniqueness, to a map $\mathrm{Nrd}\colon D \to K$.]

The only zero of Nrd is the zero element of D because, if $x \neq 0$, then $\mathrm{Nrd}(x) \cdot \mathrm{Nrd}(x^{-1}) = 1$. On the other hand, if $\{e_1, \ldots, e_{r^2}\}$ is a basis for D over K and $x = \sum x_i e_i$, then the function $\mathrm{Nrd}(x)$ is a homogeneous polynomial $\mathrm{Nrd}(x_1, \ldots, x_{r^2})$ of degree r [this is clear locally for the étale topology]. As K is C_1, we have $r^2 \leqslant r$, which implies that $r = 1$ and $D = K$.

THEOREM 2.3 (TSEN). *Let K be an extension of transcendence degree 1 of an algebraically closed field k. Then K is C_1.*

Suppose first that $K = k(X)$. Let

$$f(T) = \sum a_{i_1,\ldots i_n} T_1^{i_1} \cdots T_n^{i_n}$$

be a homogeneous polynomial of degree $d < n$ with coefficients in $k(X)$. After multiplying the coefficients by a common denominator we may suppose that they are in $k[X]$. Let $\delta = \sup\deg(a_{i_1 \ldots i_n})$. We search by the method of undetermined coefficients for a non-trivial zero in $k[X]$ by writing each T_i $(i = 1, \ldots, n)$ as a polynomial of degree N in X. Then the equation $f(T) = 0$ becomes a system of homogeneous equations in the $n \times (N + 1)$ coefficients of the polynomials $T_i(X)$ expressing the vanishing of the polynomial coefficients in X obtained by replacing T_i with $T_i(X)$. This polynomial is of degree at most $\delta + ND$, so there are $\delta + Nd + 1$ equations in $n \times (N + 1)$ variables. As k is algebraically closed this system has a non-trivial solution if $n(N + 1) > Nd + \delta + 1$, which will be the case for N sufficiently large when $d < n$.

It is clear that, to prove the theorem in the general case, it suffices to prove it when K is a finite extension of a pure transcendental extension $k(X)$ of k. Let $f(T) = f(T_1,\dots,T_n)$ be a homogeneous polynomial of degree $d < n$ with coefficients in K. Let $s = [K{:}k(X)]$ and let $e_1,\dots,e_s$ be a basis for K over $k(X)$. Introduce sn new variables U_{ij} such that $T_i = \sum U_{ij}e_j$. In order for the polynomial $f(T)$ to have a non-trivial zero in K, it suffices that the polynomial $g(X_{ij}) = N_{K/k}(f(T))$ have a nontrivial zero in $k(X)$. But g is a homogeneous polynomial of degree sd in sn variables, whence the result.

COROLLARY 2.4. *Let K be an extension of transcendence degree 1 of an algebraically closed field k. Then the étale cohomology groups $H^q(\mathrm{Spec}(K),\mathbb{G}_m)$ are zero for $q > 0$.*

3 The cohomology of smooth curves

Henceforth, and unless expressly mentioned otherwise, the cohomology groups considered are the étale cohomology groups.

PROPOSITION 3.1. *Let k be an algebraically closed field and X a connected nonsingular projective curve over k. Then*

$$\begin{aligned} H^0(X,\mathbb{G}_m) &= k^\times,\\ H^1(X,\mathbb{G}_m) &= \mathrm{Pic}(X),\\ H^q(X,\mathbb{G}_m) &= 0 \text{ for } q \geqslant 2. \end{aligned}$$

Let η be the generic point of X, $j{:}\eta \to X$ the canonical morphism, and $\mathbb{G}_{m,\eta}$ the multiplicative group of the field of fractions $k(X)$ of X.[10] For a closed point x of X, let $i_x{:}x \to X$ be the canonical immersion and $\mathbb{Z}_x$ the constant sheaf with value $\mathbb{Z}$ on x. Then $j_*\mathbb{G}_{m,\eta}$ is the sheaf of nonzero meromorphic functions on X and $\bigoplus_x i_{x*}\mathbb{Z}_x$ is the sheaf of divisors. We therefore have an exact sequence of sheaves

$$0 \longrightarrow \mathbb{G}_m \longrightarrow j_*\mathbb{G}_{m,\eta} \xrightarrow{\text{div}} \bigoplus\nolimits_x i_{x*}\mathbb{Z}_x \longrightarrow 0. \tag{3.1.1}$$

LEMMA 3.2. *We have $R^q j_*\mathbb{G}_{m,\eta} = 0$ for all $q > 0$.*

It suffices to show that the fibre of this sheaf at every closed point x of X is zero. If $\widetilde{\mathcal{O}}_{X,x}$ is the henselization of X at x and K the field of the fractions of $\widetilde{\mathcal{O}}_{X,x}$, then we have

$$\mathrm{Spec}(K) = \eta \times_X \mathrm{Spec}(\widetilde{\mathcal{O}}_{X,x}),$$

and so $(R^q j_*\mathbb{G}_{m,\eta})_x = H^q(\mathrm{Spec}(K),\mathbb{G}_m)$. But K is an algebraic extension of $k(X)$, and hence an extension of transcendence degree 1 of k. The lemma now follows from Corollary 2.4.

LEMMA 3.3. *We have $H^q(X, j_*\mathbb{G}_{m,\eta}) = 0$ for all $q > 0$.*

[10]Better, $\mathbb{G}_{m,\eta}$ is the sheaf on η_{et} defined by $\mathbb{G}_m$.

Indeed from 3.2 and the Leray spectral sequence for j, we deduce that

$$H^q(X, j_*\mathbb{G}_{m,\eta}) = H^q(\eta, \mathbb{G}_{m,\eta})$$

for all $q \geqslant 0$, and the second term is zero for $q > 0$ by 2.4.

LEMMA 3.4. *We have* $H^q\left(X, \bigoplus_{x\in X} i_{x*}\mathbb{Z}_x\right) = 0$ *for all* $q > 0$.

Indeed, for a closed point x of X, we have $R^q i_{x*}\mathbb{Z}_x = 0$ for $q > 0$ because i_x is a finite morphism 3.6, and so

$$H^q(X, i_{x*}\mathbb{Z}_x) = H^q(x, \mathbb{Z}_x).$$

The second term is zero for $q > 0$ because x is the spectrum of an algebraically closed field. [The lemma is true more generally for all "skyscraper" sheaves on X.]

We deduce from the preceding lemmas and the exact sequence (3.1.1), equalities

$$H^q(X, \mathbb{G}_m) = 0 \text{ for } q \geqslant 2,$$

and an exact cohomology sequence in low degrees

$$1 \to H^0(X, \mathbb{G}_m) \to H^0(X, j_*\mathbb{G}_{m,\eta}) \to H^0(X, \bigoplus_x i_{x*}\mathbb{Z}_x) \to H^1(X, \mathbb{G}_m) \to 1,$$

which is nothing but the exact sequence

$$1 \to k^\times \to k(X)^\times \to \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 1.$$

From Proposition 3.1 we deduce that the cohomology groups of X with values in $\mathbb{Z}/n$, n prime to the characteristic of k, have the expected values.

COROLLARY 3.5. *If X has genus g and n is invertible in k, then the groups $H^q(X, \mathbb{Z}/n)$ are zero for $q > 2$, and free over $\mathbb{Z}/n$ of rank $1, 2g, 1$ for $q = 0, 1, 2$. Replacing $\mathbb{Z}/n$ with the isomorphic group μ_n, we get canonical isomorphisms*

$$\begin{aligned} H^0(X, \mu_n) &= \mu_n \\ H^1(X, \mu_n) &= \mathrm{Pic}^0(X)_n \\ H^2(X, \mu_n) &= \mathbb{Z}/n. \end{aligned}$$

As the field k is algebraically closed, $\mathbb{Z}/n$ is isomorphic (noncanonically) to μ_n. From the Kummer exact sequence

$$0 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 0,$$

and Proposition 3.1, we deduce the equalities,

$$H^q(X, \mathbb{Z}/n) = 0 \text{ for } q > 2,$$

and, in low degrees, exact sequences

$$0 \longrightarrow H^0(X, \mu_n) \longrightarrow k^\times \xrightarrow{\ n\ } k^\times \longrightarrow 0$$

$$0 \longrightarrow H^1(X, \mu_n) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\ n\ } \mathrm{Pic}(X) \longrightarrow H^2(X, \mu_n) \longrightarrow 0.$$

We also have an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

and $\mathrm{Pic}^0(X)$ can be identified with the group of k-rational points of an abelian variety of dimension g, the Jacobian X. In such a group, multiplication by n is surjective with kernel a $\mathbb{Z}/n\mathbb{Z}$-free module of rank $2g$ (because n is invertible in k); whence the corollary.

A clever unwinding, using the "trace method," allows one to obtain the following corollary.

PROPOSITION 3.6 (SGA 4, IX, 5.7). *Let k be an algebraically closed field, X an algebraic curve over k, and $\mathcal{F}$ a torsion sheaf on X. Then*

(i) $H^q(X,\mathcal{F}) = 0$ for $q > 2$.

(ii) If X is affine, then we even have $H^q(X,\mathcal{F}) = 0$ for $q > 1$.

For the proof, as well as for an exposition of the "trace method", we refer to SGA 4, IX, 5.

4 Unwindings

To calculate the cohomology of varieties of dimension > 1 we fibre by curves, which reduces the problem to the study of morphisms with fibres of dimension $\leqslant 1$. This principle has several variants. We mention some of them.

4.1. Let A be a k-algebra of finite type with generators $a_1,\dots,a_n$. If we put

$$X_0 = \mathrm{Spec}(k),\dots,X_i = \mathrm{Spec}(k[a_1,\dots,a_i]),\dots,X_n = \mathrm{Spec}(A),$$

then the canonical inclusions $k[a_1,\dots,a_i] \to k[a_1,\dots,a_i,a_{i+1}]$ define morphisms

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$$

whose fibres are of dimension $\leqslant 1$.

4.2. In the case of a smooth morphism, we can be more precise. We define an *elementary fibration* to be a morphism of schemes $f\colon X \to S$ that can be embedded in a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] & Y \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

satisfying the following conditions:

(i) j is an open immersion dense in each fibre and $X = \bar{X} \smallsetminus Y$;

(ii) $\bar{f}$ is smooth and projective with irreducible geometric fibres of dimension 1;

(iii) g is a finite étale covering with no empty fibres.

We define a *good neighbourhood* relative to S to be an S-scheme X such that there exist S-schemes $X = X_n, \dots, X_0 = S$ and elementary fibrations $f_i : X_i \to X_{i-1}$, $i = 1, \dots, n$. One can show SGA 4, XI, 3.3 that, if X is a smooth scheme over an algebraically closed field k, then every rational point of X admits an open neighbourhood that is a good neighbourhood (relative to $\operatorname{Spec}(k)$).

4.3. We can unwind a proper morphism $f : X \to S$ as follows. According to Chow's lemma, there exists a commutative diagram

$$\begin{tikzcd} X \arrow[dr, "f"'] & & \bar{X} \arrow[ll, "\pi"'] \arrow[dl, "\bar{f}"] \\ & S & \end{tikzcd}$$

where π and $\bar{f}$ are projective morphisms, π being moreover an isomorphism over a dense open of X. Locally on S, $\bar{X}$ is a closed subscheme of a projective space of type $\mathbb{P}^n_S$.

We unwind this last statement by considering the projection $\varphi : \mathbb{P}^n_S \to \mathbb{P}^1_S$ that sends the point with homogeneous coordinates $(x_0, x_1, \dots, x_n)$ to (x_0, x_1). It is a rational map defined outside the closed subset $Y \simeq \mathbb{P}^{n-2}_S$ of $\mathbb{P}^n_S$ with homogeneous equations $x_0 = x_1 = 0$. Let $u : P \to \mathbb{P}^n_S$ be the blow up with centre Y; the fibres of u have dimension $\leqslant 1$. Moreover, there exists a natural morphism $v : P \to \mathbb{P}^1_S$ extending the rational map φ, and v makes P a $\mathbb{P}^1_S$-scheme locally isomorphic to the projective space of type $\mathbb{P}^{n-1}$, which can in turn be projected onto $\mathbb{P}^1$, etc.

4.4. A smooth projective variety X can be fibred by a *Lefschetz pencil*. The blow up $\widetilde{X}$ of the intersection of the axis of the pencil with X projects onto $\mathbb{P}^1$, and the fibres of this projection are the hyperplane sections of X by the hyperplanes of the pencil.

IV The proper base change theorem

1 Introduction

This chapter is devoted to the proof and applications of

THEOREM 1.1. *Let $f : X \to S$ be a proper morphism of schemes and $\mathcal{F}$ a torsion abelian sheaf on X. For every $q \geqslant 0$, the fibre of $R^q f_* \mathcal{F}$ at a geometric point s of S is isomorphic to the cohomology $H^q(X_s, \mathcal{F})$ of the fibre $X_s = X \otimes_S \operatorname{Spec} k(s)$ of f at s.*

For $f : X \to S$ a separated proper continuous map (separated means that the diagonal of $X \times_S X$ is closed) between topological spaces and $\mathcal{F}$ an abelian sheaf on X, the analogous result is well known and elementary: as f is closed, the sets

$f^{-1}(V)$ for V a neighbourhood of s form a fundamental system of neighbourhoods of X_s, and it can be shown that $H^\bullet(X_s,\mathcal{F}) = \varinjlim_U H^\bullet(U,\mathcal{F})$ with U running over the neighbourhoods of X_s. In practice, X_s even has a fundamental system $\mathcal{U}$ of neighbourhoods U for which it is a deformation retract, and, for $\mathcal{F}$ constant, we therefore have $H^\bullet(X_s,\mathcal{F}) = H^\bullet(U,\mathcal{F})$. In pictorial terms: the special fibre swallows the general fibre.

In the case of schemes the proof is more delicate and it is essential to assume that $\mathcal{F}$ is torsion SGA 4, XII, 2. Taking account of the description of the fibres of $R^q f_*\mathcal{F}$ in II.3.3, we see that the Theorem 1.1 is essentially equivalent to

THEOREM 1.2. *Let A be a strictly henselian local ring and $S = \mathrm{Spec}(A)$. Let $f\colon X \to S$ be a proper morphism with closed fibre X_0. Then, for every torsion abelian sheaf $\mathcal{F}$ on X and $q \geqslant 0$,*

$$H^q(X,\mathcal{F}) \xrightarrow{\sim} H^q(X_0,\mathcal{F}).$$

By passage to the limit, we see that it suffices to prove the theorem when A is the strict henselization of a $\mathbb{Z}$-algebra of finite type at a prime ideal. We first treat the case $q = 0$ or 1 and $\mathcal{F} = \mathbb{Z}/n$ (§2). An argument based on the notion of a constructible sheaf (§3) shows that it suffices to consider the case that $\mathcal{F}$ is constant. On the other hand, the unwinding (III.4.3) allows us to assume that X_0 is a curve. In this case it only remains to prove the theorem for $q = 2$ (§4).

Among other applications (§6), the theorem makes it possible to define the notion of cohomology with proper support (§5).

2 *Proof for $q = 0$ or 1 and $\mathcal{F} = \mathbb{Z}/n$*

The result for $q = 0$ and $\mathcal{F}$ constant is equivalent to the following proposition [Zariski connectedness theorem].

PROPOSITION 2.1. *Let A be a noetherian henselian local ring and $S = \mathrm{Spec}(A)$. Let $f\colon X \to S$ be a proper morphism and X_0 the closed fibre of f. Then the sets of connected components $\pi_0(X)$ and $\pi_0(X_0)$ are in natural bijection.*

Proving this amounts to showing that the sets of subsets of X and X_0 that are both open and closed, $\mathrm{Of}(X)$ and $\mathrm{Of}(X_0)$, are in natural bijection. We know that the set $\mathrm{Of}(X)$ is in natural bijection with the set of idempotents of $\Gamma(X,\mathcal{O}_X)$, and similarly $\mathrm{Of}(X_0)$ is in natural bijection with the set of idempotents of $\Gamma(X_0,\mathcal{O}_{X_0})$. It is therefore a question of showing that the canonical map

$$\mathrm{Idem}\,\Gamma(X,\mathcal{O}_X) \to \mathrm{Idem}\,\Gamma(X_0,\mathcal{O}_{X_0})$$

is bijective.

Let $\mathfrak{m}$ denote the maximal ideal of A, $\Gamma(X,\mathcal{O}_X)^\wedge$ the completion of $\Gamma(X,\mathcal{O}_X)$ for the $\mathfrak{m}$-adic topology, and, for every integer $n \geqslant 0$, $X_n = X \otimes_A A/\mathfrak{m}^{n+1}$. According to the finiteness theorem for proper morphisms (EGA, III, 3.2), $\Gamma(X,\mathcal{O}_X)$ is a finite A-algebra; as A is henselian, it follows that the canonical map

$$\mathrm{Idem}\,\Gamma(X,\mathcal{O}_X) \to \mathrm{Idem}\,\Gamma(X,\mathcal{O}_X)^\wedge$$

is bijective.

According to the comparison theorem for proper morphisms (EGA, III, 4.1), the canonical map

$$\Gamma(X, \mathcal{O}_X)^\wedge \to \varprojlim \Gamma(X_n, \mathcal{O}_{X_n})$$

is bijective. In particular, the canonical map

$$\mathrm{Idem}\,\Gamma(X, \mathcal{O}_X)^\wedge \to \varprojlim \mathrm{Idem}\,\Gamma(X_n, \mathcal{O}_{X_n})$$

is bijective. But, since X_n and X_0 have the same underlying topological space, the canonical map

$$\mathrm{Idem}\,\Gamma(X_n, \mathcal{O}_{X_n}) \to \mathrm{Idem}\,\Gamma(X_0, \mathcal{O}_{X_0})$$

is bijective for all n, which completes the proof.

As $H^1(X, \mathbb{Z}/n)$ is in natural bijection with the set of isomorphism classes of Galois finite étale coverings of X with group $\mathbb{Z}/n$, the theorem for $q = 1$ and $\mathcal{F} = \mathbb{Z}/n$ follows from the next proposition.

PROPOSITION 2.2. *Let A be a noetherian henselian local ring and $S = \mathrm{Spec}(A)$. Let $f: X \to S$ be a proper morphism and X_0 the closed fibre of f. Then the restriction functor*[11]

$$\mathrm{Rev.et}(X) \to \mathrm{Rev.et}(X_0)$$

is an equivalence of categories.

[If X_0 is connected and we have chosen a geometric point of X_0 as base point, this amounts to saying that the canonical map $\pi_1(X_0) \to \pi_1(X)$ on the (profinite) fundamental groups is bijective.]

Proposition 2.1 shows that this functor is fully faithful. Indeed, if X' and X'' are two finite étale coverings of X, then an X-morphism from X' to X'' is determined by its graph, which is an open and closed subset of $X' \times_X X''$.

It remains to show that every finite étale covering X_0' of X_0 extends to X. The finite étale coverings do not depend on nilpotent elements (SGA 1, Chapt. 1), and so X_0' lifts uniquely to a finite étale coverng X_n' of X_n for all $n \geqslant 0$, i.e., to a finite étale covering $\mathfrak{X}'$ of the formal scheme $\mathfrak{X}$ obtained by completing X along X_0. According to Grothendieck's theorem on the algebraization of formal coherent sheaves [existence theorem, EGA, III, 5], $\mathfrak{X}'$ is the formal completion of a finite étale covering $\bar{X}'$ of $\bar{X} = X \otimes_A \hat{A}$.

By passage to the limit, it suffices to prove the proposition in the case that A is the henselization of a $\mathbb{Z}$-algebra of finite type. We can then apply Artin's approximation theorem to the functor $\mathcal{F}: (A-\text{algebras}) \to (\text{sets})$ which, to an A-algebra B, makes correspond the set of isomorphism classes of finite étale coverings of $X \otimes_A B$. Indeed, this functor is locally of finite presentation: if B_i is an filtered inductive system of A-algebras and $B = \varinjlim B_i$, then $\mathcal{F}(B) = \varinjlim \mathcal{F}(B_i)$. According to Artin's theorem, given an element $\xi \in \mathcal{F}(\hat{A})$, there exists a $\xi \in \mathcal{F}(A)$ having the same image as $\bar{\xi}$ in $\mathcal{F}(A/\mathfrak{m})$. When we take ξ to be the isomorphism class of $\bar{X}'$, this gives us a finite étale covering X' of X whose restriction to X_0 is isomorphic to X_0'.

[11]A "revêtement étale" is a "finite étale covering", and so these are the categories of finite étale coverings.

3 Constructible sheaves

In this paragraph, X is a *noetherian* scheme, and "sheaf on X" means an *abelian* sheaf on X_{et}.

DEFINITION 3.1. We say that a sheaf $\mathcal{F}$ on X is *locally constant constructible* (abbreviated l.c.c) if it is represented by a finite étale covering of X.

DEFINITION 3.2. We say that a $\mathcal{F}$ sheaf on X is *constructible* if it satisfies the following equivalent conditions:

(i) There exists a finite surjective family of subschemes X_i of X such that the restriction of $\mathcal{F}$ to X_i is l.c.c..

(ii) There exists a finite family of finite morphisms $p_i: X_i' \to X$ and a monomorphism $\mathcal{F} \to \prod p_{i*}C_i$, where, for each i, C_i is a constant constructible sheaf (= defined by a finite abelian group) over X_i'.

It is easily checked that the category of constructible sheaves on X is abelian. Moreover, if $u: \mathcal{F} \to \mathcal{G}$ is a homomorphism of sheaves and $\mathcal{F}$ is constructible, then the sheaf $\operatorname{im}(u)$ is constructible.

LEMMA 3.3. *Every torsion sheaf $\mathcal{F}$ is a filtered inductive limit of constructible sheaves.*

Indeed, if $j: U \to X$ is étale of finite type, then an element $\xi \in \mathcal{F}(U)$ such that $n\xi = 0$ defines a homomorphism of sheaves $j_!\underline{\mathbb{Z}/n}_U \to \mathcal{F}$ whose image (the smallest subsheaf of $\mathcal{F}$ having ξ as a local section) is a constructible subsheaf of $\mathcal{F}$. It is clear that $\mathcal{F}$ is the inductive limit of such subsheaves.

DEFINITION 3.4. Let $\mathcal{C}$ be an abelian category and T a functor from $\mathcal{C}$ to the category of abelian groups. We say that T is *effaceable* in $\mathcal{C}$ if, for every object A of $\mathcal{C}$ and every $\alpha \in T(A)$, there exists a monomorphism $u: A \to M$ in $\mathcal{C}$ such that $T(u)\alpha = 0$.

LEMMA 3.5. *The functors $H^q(X, -)$ for $q > 0$ are effaceable in the category of constructible sheaves on X.*

It suffices to remark that, if $\mathcal{F}$ is a constructible sheaf, then there exists an integer $n > 0$ such that $\mathcal{F}$ is a sheaf of $\mathbb{Z}/n$-modules. Then there exists a monomorphism $\mathcal{F} \hookrightarrow \mathcal{G}$, where $\mathcal{G}$ is a sheaf of $\mathbb{Z}/n$-modules such that $H^q(X, \mathcal{G}) = 0$ for all $q > 0$. We can, for example, take $\mathcal{G}$ to be the Godement resolution $\prod_{x \in X} i_{x*}\mathcal{F}_{\bar{x}}$, where x runs through the points of X and $i_x: \bar{x} \to X$ is a geometric point centred at x. According to 3.3, $\mathcal{G}$ is an inductive limit of constructible sheaves, from which the lemma follows, because the functors $H^q(X, -)$ commute with inductive limits.

LEMMA 3.6. *Let $\varphi^\bullet: T^\bullet \to T'^\bullet$ be a morphism of cohomological functors defined on an abelian category $\mathcal{C}$ with values in the category of abelian groups. Suppose that T^q is effaceable for $q > 0$ and let $\mathcal{E}$ be a subset of objects of $\mathcal{C}$ such that every object of $\mathcal{C}$ is contained in an object belonging to $\mathcal{E}$. Then the following conditions are equivalent:*

(i) $\varphi^q(A)$ is bijective for all $q \geqslant 0$ and all $A \in \operatorname{Ob}\mathcal{C}$.

(ii) $\varphi^0(M)$ *is bijective and* $\varphi^q(M)$ *is surjective for all* $q > 0$ *and all* $M \in \mathcal{E}$.
(iii) $\varphi^0(A)$ *is bijective for all* $A \in \mathrm{Ob}\,\mathcal{C}$ *and* T'^q *is effaceable for all* $q > 0$.

The proof is by induction on q and does not present difficulties.

PROPOSITION 3.7. *Let* X_0 *be a subscheme of* X. *Suppose that, for all* $n \geqslant 0$ *and every* X' *scheme finite over* X, *the canonical map*

$$H^q(X', \mathbb{Z}/n) \to H^q(X'_0, \mathbb{Z}/n), \qquad X'_0 = X' \times_X X_0,$$

is bijective for $q = 0$ *and surjective for* $q > 0$. *Then, for every torsion sheaf* $\mathcal{F}$ *on* X *and all* $q \geqslant 0$, *the canonical map*

$$H^q(X, \mathcal{F}) \to H^q(X_0, \mathcal{F})$$

is bijective.

By passage to the limit, it suffices to prove the assertion for a constructible $\mathcal{F}$. We apply Lemma 3.6 taking $\mathcal{C}$ to be the category of constructible sheaves on X, T^q to be $H^q(X, -)$, T'^q to be $H^q(X_0, -)$, and $\mathcal{E}$ to be the set of constructible sheaves of the form $\prod p_{i*}C_i$, where $p_i : X'_i \to X$ is a finite morphism and C_i is a finite constant sheaf on X'_i.

4 *End of the proof*

By the method of fibring by curves (III.4.3), it suffices to prove the theorem in relative dimension $\leqslant 1$. According to the preceding paragraph, it suffices to show that, if S is the spectrum of a strictly henselian local noetherian ring, $f : X \to S$ a proper morphism whose closed fibre X_0 is of dimension $\leqslant 1$, and n an integer $\geqslant 0$, then the canonical homomorphism

$$H^q(X, \mathbb{Z}/n) \to H^q(X_0, \mathbb{Z}/n)$$

is bijective for $q = 0$ and surjective for $q > 0$.

We saw the cases $q = 0$ and 1 earlier, and we know that $H^q(X_0, \mathbb{Z}/n) = 0$ for $q \geqslant 3$; it therefore suffices to treat the case $q = 2$. We can obviously assume that n is a power of a prime number. If $n = p^r$, where p is the residue field characteristic of S, then Artin-Schreier theory shows that we have $H^2(X_0, \mathbb{Z}/p^r) = 0$. If $n = \ell^r$, $\ell \neq p$, then we deduce from Kummer theory a commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}(X) & \xrightarrow{\alpha} & H^2(X, \mathbb{Z}/\ell^r) \\ \downarrow & & \downarrow \\ \mathrm{Pic}(X_0) & \xrightarrow{\beta} & H^2(X_0, \mathbb{Z}/\ell^r) \end{array}$$

where the map β is surjective. [We saw this in Chapter III for a smooth curve over an algebraically closed field, but similar arguments apply to any curve over a separably closed field.]

To conclude, it suffices to show

PROPOSITION 4.1. *Let S be the spectrum of a henselian noetherian local ring and $f: X \to S$ a proper morphism whose closed fibre X_0 is of dimension $\leqslant 1$. Then the canonical restriction map*

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X_0)$$

is surjective. [In fact, this holds with f any separated *morphism of finite type.]*

To simplify the proof, we will assume that X is integral, although this is not necessary. As every invertible sheaf on X_0 is associated with a Cartier divisor (because X_0 is a curve, and so quasi-projective), it suffices to show that the canonical map $\mathrm{Div}(X) \to \mathrm{Div}(X_0)$ is surjective.

Every divisor on X_0 is a linear combination of divisors whose support is concentrated in a single non-isolated closed point of X_0. Let x be such a point, $t_0 \in \mathcal{O}_{X_0,x}$ a non-invertible regular element of $\mathcal{O}_{X_0,x}$, and D_0 the divisor with support in x and local equation t_0. Let U be an open neighbourhood of x in X such that there exists a section $t \in \Gamma(U, \mathcal{O}_U)$ lifting t_0. Let Y be the closed subscheme of U with equation $t = 0$; by taking U sufficiently small, we can assume that x is the only point of $Y \cap X_0$. Then Y is quasi-finite above S at x. As S is the spectrum of a local henselian ring, we deduce that $Y = Y_1 \coprod Y_2$, where Y_1 is finite over $S$and Y_2 does not meet X_0. In addition, as X is separated over S, Y_1 is closed in X.

After replacing U with a smaller open neighbourhood of x, we can assume that $Y = Y_1$, that is, that Y is closed in X. We then define a divisor D on X lifting D_0 by putting $D|X \smallsetminus Y = 0$ and $D|U = \mathrm{div}(t)$, which makes sense because t is invertible on $U \smallsetminus Y$.

4.2 Remark

In the case that f is proper, we could also make a proof in the same style as that of the Proposition 2.2. Indeed, as X_0 is a curve, there is no obstruction to lifting an invertible sheaf on X_0 to the infinitesimal neighbourhoods X_n of X_0, and so to the formal completion $\mathfrak{X}$ of X along X_0. We can then conclude by applying successively Grothendieck's existence theorem and Artin's approximation theorem.

5 *Cohomology with proper support*

DEFINITION 5.1. Let X be a separated scheme of finite type over a field k. According to a theorem of Nagata (1962, 1963), there exists a scheme $\overline{X}$ over k and an open immersion $j: X \to \overline{X}$. For a torsion sheaf $\mathcal{F}$ on X, we let $j_!\mathcal{F}$ denote the extension by 0 of $\mathcal{F}$ to $\overline{X}$, and we define the *cohomology groups with proper support* $H_c^q(X, \mathcal{F})$ by setting

$$H_c^q(X, \mathcal{F}) = H^q(\overline{X}, j_!\mathcal{F}).$$

We show that this definition is independent of compactification $j: X \to \overline{X}$ chosen. Let $j_1: X \to \overline{X}_1$ and $j_2: X \to \overline{X}_2$ be two compactifications. Then X maps into $\overline{X}_1 \times \overline{X}_2$ by $x \mapsto (j_1(x), j_2(x))$, and the closed image $\overline{X}_3$ of X by this map is a compactification of X. We have a commutative diagram

$$\begin{tikzcd} & \overline{X}_1 & \\ X \arrow[ur, "j_1"] \arrow[rr, "(j_1,j_2)"] \arrow[dr, "j_2"'] & & \overline{X}_3, \arrow[ul, "p_1"'] \arrow[dl, "p_2"] \\ & \overline{X}_2 & \end{tikzcd}$$

where p_1 and p_2, the restrictions of the natural projections to $\overline{X}_3$, are proper morphisms.

It suffices therefore to treat the case where we have a commutative diagram

$$\begin{tikzcd} & \overline{X}_2 \arrow[d, "p"] \\ X \arrow[ur, "j_2"] \arrow[dr, "j_1"'] & \\ & \overline{X}_1 \end{tikzcd}$$

with p a proper morphism.

LEMMA 5.2. *We have $p_*(j_{2!}\mathcal{F}) = j_{1!}\mathcal{F}$ and $R^q p_*(j_{2!}\mathcal{F}) = 0$, for $q > 0$.*

Note immediately that the lemma completes the proof because, from the Leray spectral sequence of the morphism p, we can deduce that, for all $q \geqslant 0$,

$$H^q(\overline{X}_2, j_{2!}\mathcal{F}) = H^q(\overline{X}_1, j_{1!}\mathcal{F}).$$

To prove the lemma, we argue fibre by fibre using the base change theorem (1.1) for p. The result is immediate because, over a point of X, p is an isomorphism, and over a point of $\overline{X}_1 \smallsetminus X$, the sheaf $j_{2!}\mathcal{F}$ is zero on the fibre of p.

5.3. Similarly, for a separated morphism of finite type of noetherian schemes $f: X \to S$, there exists a proper morphism $\overline{f}: \overline{X} \to S$ and an open immersion $j: X \to \overline{X}$ over S. We then define the higher direct images with proper support of a torsion sheaf $\mathcal{F}$ on X by setting

$$R^q f_! \mathcal{F} = R^q f_*(j_!\mathcal{F}).$$

One can check as before that this definition is independent of the chosen compactification.

THEOREM 5.4. *Let $f: X \to S$ be a separated morphism of finite type of noetherian schemes and $\mathcal{F}$ a torsion sheaf on X. Then the fibre of $R^q f_!\mathcal{F}$ at a geometric point s of S is canonically isomorphic to the cohomology with proper support $H^q_c(X_s, \mathcal{F})$ of the fibre X_s of f at s.*

This is a simple variant of the base change theorem for a proper morphism (1.1). More generally, if

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "g'"'] \arrow[d, "f'"] \\ S & S' \arrow[l, "g"] \end{tikzcd}$$

is a cartesian diagram, we have a canonical isomorphism

$$g^*(R^q f_! \mathcal{F}) \simeq R^q f'_!(g'^* \mathcal{F}). \tag{5.4.1}$$

6 Applications

VANISHING THEOREM 6.1. *Let $f: X \to S$ be a separated morphism of finite type whose fibres are of dimension $\leqslant n$ and $\mathcal{F}$ a torsion sheaf on X. Then $R^q f_! \mathcal{F} = 0$ for $q > 2n$.*

Thanks to the base change theorem, we can assume that S is the spectrum of a separably closed field. If $\dim X = n$, then there is an open affine U of X such that $\dim(X \smallsetminus U) < n$. We then have an exact sequence $0 \to \mathcal{F}_U \to \mathcal{F} \to \mathcal{F}_{X \smallsetminus U} \to 0$ and, by induction on n, it suffices to prove the theorem for $X = U$ affine. Then the method of fibring by curves (III.4.1) and the base change theorem reduce the problem to the case of a curve over a separably closed field for which we can deduce the desired result from Tsen's theorem (III.3.6).

FINITENESS THEOREM 6.2. *Let $f: X \to S$ be a separated morphism of finite type and $\mathcal{F}$ a constructible sheaf on X. Then the sheaves $R^q f_! \mathcal{F}$ are constructible.*

We consider only the case where $\mathcal{F}$ is killed by an integer invertible on X.

Proving the theorem comes down to the case where $\mathcal{F}$ is a constant sheaf $\underline{\mathbb{Z}/n}$ and $f: X \to S$ is a smooth proper morphism whose fibres are geometrically connected curves of genus g. For n invertible on X, the sheaves $R^q f_* \mathcal{F}$ are then locally free of finite rank and zero for $q > 2$ (6.1). Replacing $\mathbb{Z}/n$ with the locally isomorphic sheaf (on S) μ_n, we have canonically

$$\begin{aligned} R^0 f_* \mu_n &= \mu_n \\ R^1 f_* \mu_n &= \underline{\mathrm{Pic}}(X/S)_n \\ R^2 f_* \mu_n &= \mathbb{Z}/n. \end{aligned} \tag{6.2.1}$$

THEOREM 6.3 (COMPARISON WITH THE CLASSICAL COHOMOLOGY). *Let $f: X \to S$ be a separated morphism of schemes of finite type over $\mathbb{C}$ and $\mathcal{F}$ a torsion sheaf on X. We use the exponent $^{\mathrm{an}}$ to denote the functor of passing to the usual topological spaces, and $R^q f_!^{\mathrm{an}}$ the derived functors of the direct-image functor with proper support f^{an}. Then*

$$(R^q f_! \mathcal{F})^{\mathrm{an}} \simeq R^q f_!^{\mathrm{an}} \mathcal{F}^{\mathrm{an}}.$$

In particular, for $S =$ a point and $\mathcal{F}$ the constant sheaf $\mathbb{Z}/n$,

$$H_c^q(X, \mathbb{Z}/n) \simeq H_c^q(X^{\mathrm{an}}, \mathbb{Z}/n).$$

Unwinding using the base change theorem brings us back to case that X is a smooth proper curve with $S =$ a point and $\mathcal{F} = \mathbb{Z}/n$. The relevant cohomology groups are then zero for $q \neq 0,1,2$, and we can invoke GAGA (Serre 1956): indeed, if X is proper over $\mathbb{C}$, we have $\pi_0(X) = \pi_0(X^{\mathrm{an}})$ and $\pi_1(X) =$ profinite completion of $\pi_1(X^{\mathrm{an}})$, whence the assertion for $q = 0,1$. For $q = 2$, we use the Kummer exact sequence and the fact that, by GAGA again, $\mathrm{Pic}(X) = \mathrm{Pic}(X^{\mathrm{an}})$.

THEOREM 6.4 (COHOMOLOGICAL DIMENSION OF AFFINE SCHEMES). *Let X be an affine scheme of finite type over a separably closed field and $\mathcal{F}$ a torsion sheaf on X. Then $H^q(X,\mathcal{F}) = 0$ for $q > \dim(X)$.*

For the very pretty proof, we refer to SGA 4, XIV, §2 and 3.

6.5 Remark

This theorem is in a way a substitute for Morse theory. Indeed, consider the classical case, where X is affine and smooth over $\mathbb{C}$ and is embedded in an affine space of type $\mathbb{C}^N$. Then, for almost all points $p \in \mathbb{C}^N$, the function "distance to p" on X is a Morse function and the indices of its critical points are less than $\dim(X)$. Thus X is obtained by gluing handles of index smaller than $\dim(X)$, whence the classical analog of (6.4).

V Local acyclicity of smooth morphisms

Let X be a complex analytic variety and $f : X \to D$ a morphism from X to the disk. We denote by $[0,t]$ the closed line segment with endpoints 0 and t in D and by $]0,t]$ the semi-open segment. If f is *smooth*, then the inclusion

$$j : f^{-1}(]0,t]) \hookrightarrow f^{-1}([0,t])$$

is a homotopy equivalence: we can push the special fibre $X_0 = f^{-1}(0)$ into $f^{-1}([0,t])$.

In practice, for t small enough, $f^{-1}(]0,t])$ will be a fibre bundle on $]0,t]$ so that the inclusion

$$X_t = f^{-1}(t) \hookrightarrow f^{-1}(]0,t])$$

will also be a homotopy equivalence. We then define the *cospecialization morphism* to be the homotopy class of maps

$$\mathrm{cosp} : X_0 \hookrightarrow f^{-1}([0,t]) \overset{\approx}{\longleftarrow} f^{-1}(]0,t]) \overset{\approx}{\longleftarrow} X_t.$$

This construction can be expressed in pictorial terms by saying that for a smooth morphism, the general fibre swallows the special fibre.

Let us assume no longer that f is necessarily smooth (but assume that $f^{-1}(]0,t])$ is a fibre bundle over $]0,t]$). We can still define a morphism $\mathrm{cosp}^\bullet$ on cohomology provided $j_*\mathbb{Z} = \mathbb{Z}$ and $R^q j_*\mathbb{Z} = 0$ for $q > 0$. Under these assumptions, the Leray spectral sequence for j shows that we have

$$H^\bullet\left(f^{-1}([0,t]),\mathbb{Z}\right) \xrightarrow{\sim} H^\bullet\left(f^{-1}(]0,t]),\mathbb{Z}\right)$$

and $\operatorname{cosp}^\bullet$ is the composite morphism

$$\operatorname{cosp}^\bullet\colon H^\bullet(X_t,\mathbb{Z}) \xleftarrow{\sim} H^\bullet\left(f^{-1}(]0,t]),\mathbb{Z}\right) \xleftarrow{\sim} H^\bullet\left(f^{-1}([0,t]),\mathbb{Z}\right) \to H^\bullet(X_0,\mathbb{Z}).$$

The fibre of $R^q j_*\mathbb{Z}$ at a point $x \in X_0$ is computed as follows. We take in the ambient space a ball B_ε with centre x and sufficiently small radius ε, and for η sufficiently small, we put $E = X \cap B_\varepsilon \cap f^{-1}(\eta t)$; it is the *variety of vanishing cycles* at x. We have

$$(R^q j_*\mathbb{Z})_x \xleftarrow{\sim} H^q\left(X \cap B_\varepsilon \cap f^{-1}(]0,\eta t]),\mathbb{Z}\right) \xrightarrow{\sim} H^q(E,\mathbb{Z}),$$

and the cospecialization morphism is defined in cohomology when the varieties of vanishing cycles are acyclic [$H^0(E,\mathbb{Z}) = \mathbb{Z}$ and $H^q(E,\mathbb{Z}) = 0$ for $q > 0$], which can be expressed by saying that f is *locally acyclic*.

This chapter is devoted to the analogue of this situation for a smooth morphism of schemes and the étale cohomology. However it is essential in this context to restrict the coefficients to be torsion of order *prime to the residue characteristics*. Paragraph 1 is devoted to generalities on locally acyclic morphisms and cospecialization maps. In paragraph 2, we show that a smooth morphism is locally acyclic. In paragraph 3, we combine this result with those of the preceding chapter to deduce two applications: a specialization theorem for cohomology groups (the cohomology of the geometric fibres of a proper smooth morphism is locally constant) and a base change theorem for a smooth morphism.

In the following, we fix an integer n, and "scheme" means "scheme on which n is invertible". "Geometric point" will always mean "algebraic geometric point" $x : \operatorname{Spec}(k) \to X$ with k algebraically closed (II.3.1).

1 Locally acyclic morphisms

1.1. NOTATION. Given a S scheme and a geometric point s of S, we let $\widetilde{S}^s$ denote the spectrum of the strict localization of S at s.

DEFINITION 1.2. We say that a geometric point t of S is a *generization* of s if it is defined by an algebraic closure of the residue field of a point of $\widetilde{S}^s$. We also say that s is a *specialization* of t, and we call the S-morphism $t \to \widetilde{S}^s$ the *specialization arrow*.

DEFINITION 1.3. Let $f\colon X \to S$ be a morphism of schemes. Let s be a geometric point of S, t a generization of s, x a geometric point of X above s, and $\widetilde{X}_t^x = \widetilde{X}^x \times_{\widetilde{S}^s} t$. Then we say that $\widetilde{X}_t^x$ is a *variety of vanishing cycles* of f at the point x.

We say that f is *locally acyclic* if the reduced cohomology of every variety of vanishing cycles $\widetilde{X}_t^x$ is zero:

$$\widehat{H}^\bullet(\widetilde{X}_t^x,\mathbb{Z}/n) = 0, \tag{1.3.1}$$

i.e., $H^0(\widetilde{X}_t^x,\mathbb{Z}/n) = \mathbb{Z}/n$ and $H^q(\widetilde{X}_t^x,\mathbb{Z}/n) = 0$ for $q > 0$.

LEMMA 1.4. *Let $f: X \to S$ be a locally acyclic morphism and $g: S' \to S$ a quasi-finite morphism (or projective limit of quasi-finite morphisms). Then the morphism $f': X' \to S'$ deduced from f by base change is locally acyclic.*

One can show that, in fact, every variety of vanishing cycles of f' is a variety of vanishing cycles of f.

LEMMA 1.5. *Let $f: X \to S$ be a locally acyclic morphism. For every geometric point t of S and corresponding cartesian diagram*

$$\begin{array}{ccc} X_t & \xrightarrow{\varepsilon'} & X \\ \downarrow & & \downarrow f \\ t & \xrightarrow{\varepsilon} & S, \end{array}$$

we have $\varepsilon'_*\mathbb{Z}/n = f^*\varepsilon_*\mathbb{Z}/n$ and $R^q\varepsilon'_*\mathbb{Z}/n = 0$ for $q > 0$.

Let $\overline{S}$ be the closure $\varepsilon(t)$ and S' the normalization of $\overline{S}$ in $k(t)$. Consider the cartesian diagram

$$\begin{array}{ccccc} X_t & \xrightarrow{i'} & X' & \xrightarrow{\alpha'} & X \\ \downarrow & & \downarrow f' & & \downarrow f \\ t & \xrightarrow{i} & S' & \xrightarrow{\alpha} & S. \end{array}$$

The local rings of S' are normal with separably closed fields of fractions. They are therefore strictly henselian, and the local acyclicity of f' (1.4) gives $i'_*\mathbb{Z}/n = \mathbb{Z}/n$, $R^q i'_*\mathbb{Z}/n = 0$ for $q > 0$. As α is integral, we then have

$$R^q\varepsilon'_*\mathbb{Z}/n = \alpha'_* R^q i'_*\mathbb{Z}/n = \alpha'_* f'^* R^q i_*\mathbb{Z}/n = f^*\alpha_* R^q i_*\mathbb{Z}/n = f^* R^q\varepsilon_*\mathbb{Z}/n,$$

and the lemma follows.

1.6. Given a locally acyclic morphism $f: X \to S$ and a specialization arrow $t \to \widetilde{S}^s$, we will define canonical homomorphisms, called cospecialization maps

$$\mathrm{cosp}^\bullet: H^\bullet(X_t, \mathbb{Z}/n) \to H^\bullet(X_s, \mathbb{Z}/n)$$

relating the cohomology of the general fibre $X_t = X \times_S t$ to that of the special fibre $X_s = X \times_X s$.

Consider the cartesian diagram

$$\begin{array}{ccccc} X_t & \xrightarrow{\varepsilon'} & \widetilde{X} & \longleftarrow & X_s \\ \downarrow & & \downarrow f' & & \downarrow \\ t & \xrightarrow{\varepsilon} & \widetilde{S}^s & \longleftarrow & s. \end{array}$$

deduced from f by base change. According to 1.4, f' is still locally acyclic. From the definition of local acyclicity, we deduce immediately that the restriction to X_s of the sheaf $R^q\varepsilon'_*\mathbb{Z}/n$ is $\mathbb{Z}/n$ for $q = 0$, and 0 for $q > 0$. By 1.5, we even know that $R^q\varepsilon'_*\mathbb{Z}/n = 0$ for $q > 0$. We define $\mathrm{cosp}^\bullet$ to be the composite

$$H^\bullet(X_t, \mathbb{Z}/n) \simeq H^\bullet(X, \varepsilon'_*\mathbb{Z}/n) \to H^\bullet(X_s, \mathbb{Z}/n)\,. \tag{1.6.1}$$

Variant: Let $\overline{S}$ be the closure of $\varepsilon(t)$ in $\widetilde{S}^s$, let S' be the normalization of $\overline{S}$ in $k(t)$, and let X'/S' be deduced from X/S by base change. The diagram (1.6.1) can also be written

$$H^\bullet(X_t, \mathbb{Z}/n) \simeq H^\bullet(X', \mathbb{Z}/n) \to H^\bullet(X_s, \mathbb{Z}/n).$$

THEOREM 1.7. *Let S be a locally noetherian scheme, s a geometric point of S, and $f: X \to S$ a morphism. We assume that*

(a) *the morphism f is locally acyclic,*

(b) *for every specialization arrow $t \to \widetilde{S}^s$ and for all $q \geqslant 0$, the cospecialization maps $H^q(X_t, \mathbb{Z}/n) \to H^q(X_s, \mathbb{Z}/n)$ are bijective.*

Then the canonical homomorphism $(R^q f_* \mathbb{Z}/n)_s \to H^q(X_s, \mathbb{Z}/n)$ is bijective for all $q \geqslant 0$.

In proving the theorem, we can clearly assume that $S = \widetilde{S}^s$. We will in fact show that, for every sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules $\mathcal{F}$ on S, the canonical homomorphism $\varphi^q(\mathcal{F}): (R^q f_*(f^*\mathcal{F}))_s \to H^q(X_s, f^*\mathcal{F})$ is bijective.

Every sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules is a filtered inductive limit of constructible sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules (IV.3.3). Moreover, every constructible sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules embeds into a sheaf of the form $\prod i_{\lambda *} C_\lambda$, where $(i_\lambda: t_\lambda \to S)$ is a finite family of generizations of s and C_λ is a free $\mathbb{Z}/n\mathbb{Z}$-module of finite rank over t_λ. After the definition of the cospecialization maps, condition b) means that the homomorphisms $\varphi^q(\mathcal{F})$ are bijective if $\mathcal{F}$ is of this form.

We conclude with the help of a variant of Lemma 3.6 from Chapter IV:

LEMMA 1.8. *Let $\mathcal{C}$ be an abelian category in which filtered inductive limits exist. Let $\varphi^\bullet: T^\bullet \to T'^\bullet$ be a morphism of cohomological functors, commuting with filtered inductive limits, defined on $\mathcal{C}$ and with values in the category of abelian groups. Suppose that there exist two subsets $\mathcal{D}$ and $\mathcal{E}$ of objects of $\mathcal{C}$ such that*

(a) *every object of $\mathcal{C}$ is a filtered inductive limit of objects belonging at $\mathcal{D}$,*

(b) *every object belonging to $\mathcal{D}$ is contained in an object belonging to $\mathcal{E}$.*

Then the following conditions are equivalent:

(i) *$\varphi^q(A)$ is bijective for all $q \geqslant 0$ and all $A \in \operatorname{Ob}\mathcal{C}$.*

(ii) *$\varphi^q(M)$ is bijective for all $q \geqslant 0$ and all $M \in \mathcal{E}$.*

The lemma is proved by passing to the inductive limit, using induction on q, and by repeated application of the five-lemma to the diagram of exact cohomology sequences deduced from an exact sequence $0 \to A \to M \to A' \to 0$, with $A \in \mathcal{D}$, $M \in \mathcal{E}$, $A' \in \operatorname{Ob}\mathcal{C}$.

COROLLARY 1.9. *Let S be the spectrum of a strictly henselian local noetherian ring and $f: X \to S$ a locally acyclic morphism. Suppose that, for every geometric point t of S we have $H^0(X_t, \mathbb{Z}/n) = \mathbb{Z}/n$ and $H^q(X_t, \mathbb{Z}/n) = 0$ for $q > 0$ (i.e., the geometric fibres of f are acyclic). Then $f_*\mathbb{Z}/n = \mathbb{Z}/n$ and $R^q f_* \mathbb{Z}/n = 0$ for $q > 0$.*

COROLLARY 1.10. *Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of locally noetherian schemes. If f and g are locally acyclic, then so also is $g \circ f$.*

We may suppose that X, Y, and Z are strictly local and that f and g are local morphisms. We have to show that, if z is an algebraic geometric point of Z, then $H^0(X_z, \mathbb{Z}/n) = \mathbb{Z}/n$ and $H^q(X_z, \mathbb{Z}/n) = 0$ for $q > 0$.

As g is locally acyclic, we have $H^0(Y_z, \mathbb{Z}/n) = \mathbb{Z}/n$ and $H^q(Y_z, \mathbb{Z}/n) = 0$ for $q > 0$. Moreover, the morphism $f_z: X_z \to Y_z$ is locally acyclic (1.4) and its geometric fibres are acyclic because they are varieties of vanishing cycles of f. From 1.9, we have $R^q f_{z*}\mathbb{Z}/n = 0$ for $q > 0$. Moreover, $f_{z*}\mathbb{Z}/n$ is constant with fibre $\mathbb{Z}/n$ over Y_z. We conclude with the help of the Leray spectral sequence of f_z.

2 Local acyclicity of a smooth morphism

THEOREM 2.1. *Smooth morphisms are locally acyclic.*

Let $f: X \to S$ be a smooth morphism. The assertion is local for the étale topology on X and S, so we can assume that X is affine space of dimension d over S. By passage to the limit, we can assume that S is noetherian, and the transitivity of local acyclicity 1.10 shows that it suffices to treat the case $d = 1$.

Let s be a geometric point of S and x a geometric point of X centred at a closed point of X_s. We have to show that the geometric fibres of the morphism $\widetilde{X}^x \to \widetilde{S}^s$ are acyclic. We now put $S = \widetilde{S}^s = \operatorname{Spec}(A)$ and $X = \widetilde{X}^x$. We have $X \simeq \operatorname{Spec} A\{T\}$, where $A\{T\}$ is the henselization of $A[T]$ at the point $T = 0$ above s.

If t is a geometric point of S, then the fibre X_t is a projective limit of smooth affine curves over t. Therefore $H^q(X_t, \mathbb{Z}/n) = 0$ for $q \geqslant 2$ and it suffices to show that $H^0(X_t, \mathbb{Z}/n) = \mathbb{Z}/n$ and $H^1(X_t, \mathbb{Z}/n) = 0$ for n prime to the residue characteristic of S. This follows from the next two propositions.

PROPOSITION 2.2. *Let A be a strictly henselian local ring, $S = \operatorname{Spec}(A)$, and $X = \operatorname{Spec} A\{T\}$. Then the geometric fibres of $X \to S$ are connected.*

By passage to the limit, we need only consider the case that A is a strict henselization of a $\mathbb{Z}$-algebra of finite type.

Let $\overline{t}$ be a geometric point of S, localized at t, and k' a finite separable extension of $k(t)$ in $k(\overline{t})$. We put $t' = \operatorname{Spec}(k')$ and $X_{t'} = X \times_S \operatorname{Spec}(k')$. We have to show that, for all $\overline{t}$ and t', $X_{t'}$ is connected (by which we mean connected and nonempty). Let A' be the normalization of A in k', i.e., the ring of elements of k' integral over the image of A in $k(t)$. We have $A\{T\} \otimes_A A' \xrightarrow{\sim} A'\{T\}$: the ring on the left is indeed henselian local (because A' is finite over A and local) and a limit of étale local algebras over $A'[T] = A[T] \otimes_A A'$. The scheme $X_{t'}$ is again the fibre at t' of $X' = \operatorname{Spec}(A'\{T\})$ on $S' = \operatorname{Spec}(A')$. The local scheme X' is normal, therefore integral; its localization $X_{t'}$ is still integral, a fortiori, connected.

PROPOSITION 2.3. *Let A be a strictly henselian local ring, $S = \operatorname{Spec}(A)$, and $X = \operatorname{Spec}(A\{T\})$. Let $\overline{t}$ be a geometric point of S and $X_{\overline{t}}$ the corresponding geometric fibre. Then every finite étale Galois covering of $X_{\overline{t}}$ of order prime to the characteristic of the residue field of A is trivial.*

LEMMA 2.3.1 (ZARISKI-NAGATA PURITY THEOREM IN DIMENSION 2). *Let C be a regular local ring of dimension 2 and C' a finite normal C-algebra, étale over the open complement of a closed point of* $\mathrm{Spec}(C)$. *Then C' is étale over C.*

Indeed C' is normal of dimension 2, so $\mathrm{prof}(C') = 2$. As $\mathrm{prof}(C') + \dim\mathrm{proj}(C') = \dim(C) = 2$, we conclude that C' is free over C. Then the set of points of C where C' is ramified is defined by a single equation, namely, the discriminant. As it does not contain a point of height 1, it is empty.

LEMMA 2.3.2 (SPECIAL CASE OF ABHYANKAR'S LEMMA). *Let V be a discrete valuation ring, $S = \mathrm{Spec}(V)$, π a uniformizer, η the generic point of S, X irreducible and smooth over S of relative dimension 1, $\widetilde{X}_1$ a finite étale Galois covering X_η of degree n invertible on S, and $S_1 = \mathrm{Spec}\left(V[\pi^{1/n}]\right)$. Denote by a subscript $_1$ base change from S to S_1. Then, $\widetilde{X}_{1\eta}$ extends to a finite étale covering of X_1.*

Let $\widetilde{X}_1$ be the normalization of X_1 in $\widetilde{X}_{1\eta}$. In view of the structure of the tame inertia groups of the discrete valuation rings that are localizations of X at generic points of the special fibre X_s, we see that $\widetilde{X}_1$ is étale over X_1 over the generic fibre and at generic points of the special fibre. By 2.3.1, it is étale everywhere.

2.3.3. We now prove Proposition 2.3. Let t denote the point at which $\bar{t}$ is localized. We are free to replace A with the normalization of A in a finite separable extension $k(t')$ of $k(t)$ in $k(\bar{t})$ (cf. 2.2). This, and a preliminary passage to the limit, allow us to assume that

(a) A is noetherian and normal, and t is the generic point of S.

(b) The finite étale covering of $X_{\bar{t}}$ in question comes from a finite étale covering of X_t.

(c) In fact, it comes from a finite étale covering $\widetilde{X}_U$ of the inverse image X_U of a nonempty open U of S (consequence of (b): t is the limit of the U).

(d) The complement of U is of codimension $\geqslant 2$ (this at the cost of enlarging $k(t)$; apply 2.3.2 to the discrete valuation rings obtained by localizing S at points of $S \smallsetminus U$ of codimension 1 in S; there are only finitely many such points).

(e) The finite étale covering $\widetilde{X}_U$ is trivial above the subscheme $T = 0$ (this requires that $k(t)$ be further enlarged).

When these conditions are met, we will see that the finite étale covering $\widetilde{X}_U$ is trivial.

LEMMA 2.3.4. *Let A be a noetherian normal strictly local henselian ring, U an open of* $\mathrm{Spec}(A)$ *whose complement has codimension $\geqslant 2$, V its inverse image in $X = \mathrm{Spec}(A\{T\})$, and V' a finite étale covering of V. If V' is trivial over $T = 0$, then V' is trivial.*

Let $B = \Gamma(V', \mathcal{O})$. As V' is the inverse image of V in $\mathrm{Spec}(B)$, it suffices to show that B is finite and étale over $A\{T\}$ (hence decomposed as $A\{T\}$ is strictly henselian). Let $\widehat{X} = A[[T]]$, and denote by $(-)^\wedge$ base change from X to $\widehat{X}$. The scheme $\widehat{X}$ is faithfully flat over X. Therefore $\Gamma(\widehat{V}', \mathcal{O}) = B \otimes_{A\{T\}} A[[T]]$, and it suffices to show that this ring $\widehat{B}$ is finite and étale over $A[[T]]$.

Let V_m (resp. V'_m) be the subscheme of $\widehat{V}$ (resp. $\widehat{V}'$) defined by the equation $T^{m+1} = 0$. By hypothesis, V'_0 is a trivial finite étale covering of V_0, i.e., a sum of n copies V_0. Likewise V'_m / V_m is trivial because finite étale coverings are insensitive to nilpotents. We deduce a map

$$\varphi: \Gamma(\widehat{V}', \mathcal{O}) \to \varprojlim_m \Gamma(V'_m, \mathcal{O}) = (\varprojlim_m \Gamma(V, \mathcal{O}))^n.$$

By hypothesis, the complement of U is of depth $\geqslant 2$: we have

$$\Gamma(V_m, \mathcal{O}) = A[T]/(T^{m+1}),$$

and φ is a homomorphism from $\widehat{B}$ to $A[[T]]^n$. Over U, it provides n distinct sections of $\widehat{V}'/\widehat{V}$; it follows that $\widehat{V}'$ is trivial, i.e., a sum of n copies of $\widehat{V}$. The complement of $\widehat{V}$ in $\widehat{X}$ still being of codimension $\geqslant 2$ (therefore of depth $\geqslant 2$), we deduce that $\widehat{B} = A[[T]]$, whence the lemma.

3 Applications

THEOREM 3.1 (SPECIALIZATION OF COHOMOLOGY GROUPS). *Let $f: X \to S$ be a proper locally acyclic morphism, for example, a proper smooth morphism. Then the sheaves $R^q f_* \mathbb{Z}/n$ are constant and locally constructible and for every specialization arrow $t \to \widetilde{S}^s$, the cospecialization arrows $H^q(X_t, \mathbb{Z}/n) \to H^q(X_s, \mathbb{Z}/n)$ are bijective.*

This follows immediately from the definition of the cospecialization morphisms and the finiteness and base change theorems for proper morphisms.

THEOREM 3.2 (SMOOTH BASE CHANGE). *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \Big\downarrow{\scriptstyle f'} & & \Big\downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

be a cartesian diagram with g smooth. For every torsion sheaf $\mathcal{F}$ on X whose torsion is prime to the residue characteristics of S,

$$g^* R^q f_* \mathcal{F} \xrightarrow{\sim} R^q f'_*(g'^* \mathcal{F})$$

By passing to an open covering of X, we may suppose that X is affine, and then by passing to the limit, that X is of finite type on S. Then f factors into an open immersion $j: X \to \overline{X}$ and a proper morphism $\overline{f}: \overline{X} \to S$. From the Leray spectral sequence for $\overline{f} \circ j$ and the base change theorem for proper morphism, we deduce that it suffices to prove the theorem in the case where $X \to S$ is an open immersion.

In this case, if $\mathcal{F}$ is of the form $\varepsilon_* C$, where $\varepsilon: t \to X$ is a geometric point of X, then the theorem is a corollary of 1.5. The general case follows from Lemma 1.8.

COROLLARY 3.3. *Let K/k be an extension of separably closed fields, X a k-scheme, and n an integer prime to the characteristic of k. Then the canonical map $H^q(X,\mathbb{Z}/n) \to H^q(X_K,\mathbb{Z}/n)$ is bijective for all $q \geqslant 0$.*

It suffices to remark that $\bar{K}$ is an inductive limit of smooth $\bar{k}$-algebras.

THEOREM 3.4 (RELATIVE PURITY). *Consider a diagram*

$$\begin{tikzcd} U \arrow[r, hook, "j"] \arrow[dr] & X \arrow[d, "f"] & Y \arrow[l, hook', "i"'] \arrow[dl, "h"] \\ & S & \end{tikzcd} \tag{3.4.1}$$

with f smooth of pure relative dimension N, h smooth of pure relative dimension $N-1$, i a closed immersion, and $U = X \smallsetminus Y$. For n prime to the residue characteristics of S, we have

$$\begin{aligned} j_*\mathbb{Z}/n &= \mathbb{Z}/n \\ R^1 j_*\mathbb{Z}/n &= \mathbb{Z}/n(-1)_Y \\ R^q j_*\mathbb{Z}/n &= 0 \qquad \text{for } q \geqslant 2. \end{aligned}$$

In these formulas, $\mathbb{Z}/n(-1)$ denotes the $\mathbb{Z}/n$-dual of μ_n. If t is a local equation for Y, then the isomorphism $R^1 j_*\mathbb{Z}/n \simeq \mathbb{Z}/n(-1)_Y$ is defined by the map

$$a\colon \mathbb{Z}/n \to R^1 j_*\mu_n$$

sending 1 to the class of the μ_n-torsor of nth roots of t.

The question is local. This allows us to replace (X,Y) with a locally isomorphic pair, for example,

$$\begin{tikzcd} \mathbf{A}^1_T \arrow[r, hook, "j"] \arrow[dr, "g"'] & \mathbb{P}^1_T \arrow[d, "f"] & T \arrow[l, hook', "i"'] \arrow[dl] \\ & T & \end{tikzcd}$$

with $T = \mathbf{A}^{n-1}_S$ and i the section at infinity. Corollary 1.9 applies to g, and shows that $R^q g_*\mathbb{Z}/n = \mathbb{Z}/n$ for $q = 0$ and 0 for $q > 0$. For f, we have moreover (6.2.1)

$$R^q f_*\mathbb{Z}/n = \mathbb{Z}/n,\, 0,\, \mathbb{Z}/n(-1),\, 0 \text{ for } q = 0,\, 1,\, 2,\, > 2.$$

It is easily checked that $j_*\mathbb{Z}/n = \mathbb{Z}/n$, and that the $R^q j_*\mathbb{Z}/n$ are concentrated on $i(T)$ for $q > 0$. The Leray spectral sequence

$$E_2^{pq} = R^p f_* R^q j_*\mathbb{Z}/n \implies R^{p+q} g_*\mathbb{Z}/n$$

therefore simplifies to

$$\begin{tikzcd} i^*R^q j_*\mathbb{Z}/n & 0 & \cdots & & \\ i^*R^1 j_*\mathbb{Z}/n \arrow[drr, "d_2"] & 0 & \cdots & & \\ \mathbb{Z}/n & 0 & \mathbb{Z}/n(-1) & 0 & \cdots \end{tikzcd} .$$

$R^q j_*\mathbb{Z}/n = 0$ for $q \geqslant 2$, and $R^1 j_*\mathbb{Z}/n$ is the extension by zero of a locally free sheaf of rank one over T (isomorphic, via d_2, to $\mathbb{Z}/n(-1)$). The map a defined above, being injective (as can be checked fibre by fibre), is an isomorphism, which completes the proof of Theorem 3.4.

3.5. We refer to SGA 4, XVI, §4, §5 for the proofs of the following applications of the acyclicity theorem (2.1).

3.5.1. *Let $f: X \to S$ be a morphism of schemes finite type over $\mathbb{C}$ and $\mathcal{F}$ a constructible sheaf on X. Then*

$$(R^q f_*\mathcal{F})^{\mathrm{an}} \sim R^q f_*^{\mathrm{an}}(\mathcal{F}^{\mathrm{an}})$$

(cf. IV.6.3; in ordinary cohomology, it is necessary to assume that $\mathcal{F}$ is constructible and not just torsion).

3.5.2. *Let $f: X \to S$ be a morphism of schemes of finite type over a field k of characteristic* 0 *and $\mathcal{F}$ a sheaf on X. If $\mathcal{F}$ is constructible, then so also are the sheaves $R^q f_*\mathcal{F}$.*

The proof uses resolution of singularities and Theorem 3.4. It is generalized to the case of a morphism of finite type of excellent schemes of characteristic 0 in SGA 4, XIX, 5. Another proof, independent of resolution, is given in SGA $4\frac{1}{2}$ Th. Finitude, 1.1. It applies to a morphism of schemes of finite type over a field or over a Dedekind ring.

VI Poincaré duality

1 Introduction

Let X be an oriented topological manifold of pure dimension N, and assume that X admits a finite open covering $\mathcal{U} = (U_i)_{1 \leqslant i \leqslant K}$ such that all nonempty intersections of the opens U_i are homeomorphic to balls. For such a manifold, the Poincaré duality theorem can be described as follows.

A. The cohomology of X is the Čech cohomology of the covering $\mathcal{U}$. This is the cohomology of the complex

$$0 \to \mathbb{Z}^{A_0} \to \mathbb{Z}^{A_1} \to \cdots \tag{1}$$

where

$$A_k = \{(i_0, \ldots, i_k) : i_0 < \cdots < i_k \text{ and } U_{i_0} \cap \cdots \cap U_{i_k} \neq \varnothing\}.$$

B. For $a = (i_0, \ldots, i_k) \in A_k$, let $U_a = U_{i_0} \cap \cdots \cap U_{i_k}$ and let j_a be the inclusion of U_a into X. The constant sheaf $\mathbb{Z}$ on X admits the (left) resolution

$$\begin{array}{ccccccc} \cdots \longrightarrow & \bigoplus_{a \in A_1} j_{a!}\mathbb{Z} & \longrightarrow & \bigoplus_{a \in A_0} j_{a!}\mathbb{Z} & \longrightarrow & 0 \\ & & & \big\downarrow & & \\ & & & \mathbb{Z}. & & \end{array} \tag{2}$$

The cohomology with compact support $H_c^\bullet(X, j_{a!}\mathbb{Z})$ is nothing but the cohomology with proper support of the (oriented) ball U_a:

$$H_c^i(X, j_{a!}\mathbb{Z}) = \begin{cases} 0 & \text{if } i \neq N \\ \mathbb{Z} & \text{if } i = N. \end{cases}$$

The spectral sequence of hypercohomology for the complex (2) and cohomology with proper support, show therefore that $H_c^i(X)$ is the $(i-N)$th cohomology group of the complex

$$\cdots \to \mathbb{Z}^{A_1} \to \mathbb{Z}^{A_0} \to 0. \tag{3}$$

This complex is the dual of the complex (1), whence Poincaré duality.

The essential points of this construction are

(a) the existence of a cohomology theory with proper support:

(b) the fact that every point x of a manifold X of pure dimension N has a fundamental system of open neighbourhoods U for which

$$H_c^i(U) = \begin{cases} 0 & \text{for } i \neq N, \\ \mathbb{Z} & \text{for } i = N. \end{cases} \tag{4}$$

Poincaré duality in étale cohomology can be constructed on this model. For X smooth of pure dimension N over an algebraically closed field k, n invertible on X, and x a closed point of X, the key point is to calculate the projective limit over the étale neighbourhoods U of x,

$$\varprojlim H_c^i(U, \mathbb{Z}/n) = \begin{cases} 0 & \text{if } i \neq 2N \\ \mathbb{Z}/n & \text{if } i = 2N. \end{cases} \tag{5}$$

Just as when working with topological manifolds we have to directly treat first the case of an open ball (or simply the interval $(0,1]$), here we must directly treat first the case of the curves (§2). The local acyclicity theorem for smooth morphisms then allows us to reduce the general case (5) to this particular case (§3).

The isomorphisms (4) and (5) are not canonical: they depend on the choice of an *orientation* of X. For n invertible on a scheme X, μ_n is a sheaf of free $\mathbb{Z}/n$-modules of rank one. We denote by $\mathbb{Z}/n(N)$ its Nth tensor power ($N \in \mathbb{Z}$). The intrinsic form of the second line of (5) is

$$\varprojlim H_c^{2N}(U, \mathbb{Z}/n(N)) = \mathbb{Z}/n, \tag{5$'$}$$

and $\mathbb{Z}/n(N)$ is called the *orientation sheaf* of X. The sheaf $\mathbb{Z}/n(N)$ is constant and isomorphic to $\mathbb{Z}/n$, and so we can move the sign N and write instead

$$\varprojlim H_c^{2N}(U, \mathbb{Z}/n) = \mathbb{Z}/n(-N). \tag{5$''$}$$

Now Poincaré duality takes the form of a perfect duality, with values in $\mathbb{Z}/n(-N)$, between $H^i(X, \mathbb{Z}/n)$ and $H_c^{2N-i}(X, \mathbb{Z}/n)$.

2 The case of curves

2.1. Let $\bar{X}$ be a smooth projective curve over an algebraically closed field k, and let n be invertible on $\bar{X}$. When $\bar{X}$ is connected, the proof of (III.3.5) gives us a canonical isomorphism

$$H^2(\bar{X}, \mu_n) = \operatorname{Pic}(\bar{X})/n\operatorname{Pic}(\bar{X}) \xrightarrow[\sim]{\deg} \mathbb{Z}/n.$$

Let D be a reduced divisor on $\bar{X}$ and $X = \bar{X} \smallsetminus D$,

$$X \xhookrightarrow{j} \bar{X} \xhookleftarrow{i} D.$$

The exact sequence

$$0 \to j_!\mu_n \to \mu_n \to i_*\mu_n \to 0$$

gives us an isomorphism

$$H_c^2(X, \mu_n) = H^2(\bar{X}, j_!\mu_n) \xrightarrow{\sim} H^2(\bar{X}, \mu_n) \xrightarrow{\sim} \mathbb{Z}/n.$$

because $H^i(\bar{X}, i_*\mu_n) = H^i(D, \mu_n) = 0$ for $i > 0$. When $\bar{X}$ is disconnected, we have similarly

$$H_c^2(X, \mu_n) \simeq (\mathbb{Z}/n)^{\pi_0(X)},$$

and we define the *trace morphism* to be the sum

$$\operatorname{Tr}: H_c^2(X, \mu_n) \simeq (\mathbb{Z}/n)^{\pi_0(X)} \xrightarrow{\Sigma} \mathbb{Z}/n.$$

THEOREM 2.2. *The form* $\operatorname{Tr}(a \cup b)$ *identifies each of the two groups* $H^i(X, \mathbb{Z}/n)$ *and* $H_c^{2-i}(X, \mu_n)$ *with the dual (with values in* $\mathbb{Z}/n$*) of the other.*

Transcendental proof. If $\bar{X}$ is a smooth projective curve over the spectrum S of a discrete valuation ring and $j: X \hookrightarrow \bar{X}$ is the inclusion of the complement of a divisor D étale over S the cohomologies (resp. the cohomologies with proper support) of the special and generic geometric fibres of X/S are "the same," i.e., the fibres of locally constant sheaves on S. This can be deduced from the similar facts for $\bar{X}$ and D using the exact sequence $0 \to j_!\mathbb{Z}/n \to \mathbb{Z}/n \to (\mathbb{Z}/n)_D \to 0$ (for cohomology with proper support) and the formulas $j_*\mathbb{Z}/n = \mathbb{Z}/n$, $R^1 j_*\mathbb{Z}/n \simeq (\mathbb{Z}/n)_D(-1)$, $R^i j_*\mathbb{Z}/n = 0$ ($i \geqslant 2$) (for ordinary cohomology) (V.3.4).

This principle of specialization reduces the general case of 2.2 to the case where k has characteristic 0. By V.3.3, we can then take $k = \mathbb{C}$. Finally, for $k = \mathbb{C}$, the groups $H^\bullet(X, \mathbb{Z}/n)$ and $H_c^\bullet(X, \mu_n)$ coincide with the groups of the same name, calculated for the classical topological space X_{cl} and, via the isomorphism $\mathbb{Z}/n \to \mu_n: x \mapsto \exp\left(\frac{2\pi i x}{n}\right)$, the trace morphism becomes identified with "integration over the fundamental class," so that 2.2 results from Poincaré duality for X_{cl}.

2.3. *Algebraic proof*

For a very economical proof, see SGA $4\frac{1}{2}$, Dualité §2. Here is another, tied to the autoduality of the Jacobian.

We return to the notation of 2.1. We may suppose — and we do suppose — that X is connected. The cases $i = 0$ and $i = 2$ being trivial, we suppose also that $i = 1$. Define ${}_D\mathbb{G}_m$ by the exact sequence

$$0 \to {}_D\mathbb{G}_m \to \mathbb{G}_m \to i_*\mathbb{G}_m \to 0$$

(sections of $\mathbb{G}_m$ congruent to $1 \mod D$). The group $H^1(\bar{X}, {}_D\mathbb{G}_m)$ classifies the invertible sheaves on $\bar{X}$ trivialized over D. It is the group of points of $\mathrm{Pic}_D(\bar{X})$, which is an extension of $\mathbb{Z}$ (the degree) by the group of points of Rosenlicht's generalized Jacobian $\mathrm{Pic}^0_D(\bar{X})$ (corresponding to the conductor 1 at each point of D). This last is itself an extension of the abelian variety $\mathrm{Pic}^0(\bar{X})$ by the torus $\mathbb{G}_m^D/(\mathbb{G}_m$ diagonal).

(a) The exact sequence $0 \to j_!\mu_n \to {}_D\mathbb{G}_m \xrightarrow{x\mapsto x^n} {}_D\mathbb{G}_m \to 0$ provides an isomorphism
$$H^1_c(X,\mu_n) = \mathrm{Pic}^0_D(\bar{X})_n. \tag{2.3.1}$$

(b) The map sending $x \in X(k)$ to the class of the sheaf invertible $\mathcal{O}(x)$ on $\bar{X}$ trivialized by 1 over D comes from a morphism
$$f: X \to \mathrm{Pic}_D(\bar{X}).$$

For the rest, we fix a base point 0 and we put $f_0(x) = f(x) - f(0)$. For a homomorphism $v{:}\,\mathrm{Pic}^0_D(\bar{X})_n \to \mathbb{Z}/n$, let $\bar{v} \in H^1\left(\mathrm{Pic}^0_D(\bar{X}), \mathbb{Z}/n\right)$ denote the image by v of the class in $H^1\left(\mathrm{Pic}^0_D(\bar{X}), \mathrm{Pic}^0_D(\bar{X})_n\right)$ of the torsor defined by the extension

$$0 \longrightarrow \mathrm{Pic}^0_D(\bar{X})_n \longrightarrow \mathrm{Pic}^0_D(\bar{X}) \xrightarrow{\ n\ } \mathrm{Pic}^0_D(\bar{X}) \longrightarrow 0$$

Geometric class field theory (as explained in Serre 1959) shows that the map $v \mapsto f_0^*(\bar{v})$:
$$\mathrm{Hom}\left(\mathrm{Pic}^0_D(\bar{X})_n, \mathbb{Z}/n\right) \to H^1(X, \mathbb{Z}/n) \tag{2.3.2}$$
is an isomorphism. To deduce (2.2) from (2.3.1) and (2.3.2), it remains to show that

$$\mathrm{Tr}\left(u \cup f_0^*(\bar{v})\right) = -v(u). \tag{2.3.3}$$

This compatibility is proved in SGA $4\frac{1}{2}$, Dualité, 3.2.4.

3 The general case

Let X be a smooth algebraic variety of pure dimension N over an algebraically closed field k. To state the Poincaré duality theorem, we must first define the trace morphism

$$\mathrm{Tr}{:}\ H^{2N}_c(X, \mathbb{Z}/n(N)) \to \mathbb{Z}/n.$$

The definition is a painful unwinding starting from the case of curves SGA 4. XVIII, §2. We then have

THEOREM 3.1. *The form* $\mathrm{Tr}(a \cup b)$ *identifies each of the groups* $H^i_c(X, \mathbb{Z}/n(N))$ *and* $H^{2N-i}(X, \mathbb{Z}/n)$ *with the dual (with values in* $\mathbb{Z}/n$*) of the other.*

Let $x \in X$ be a closed point and X_x the strict localization of X at x. We suppose that, for U running over the étale neighbourhoods of x,

$$H_c^\bullet(X_x, \mathbb{Z}/n) = \varprojlim H_c^\bullet(U, \mathbb{Z}/n). \tag{1}$$

It would be better to consider rather the pro-object "$\varprojlim$"$H_c^\bullet(U, \mathbb{Z}/n)$ but, the groups in play being finite, the difference is inessential. As we endeavored to explain in the introduction, (3.1) follows from

$$\begin{gathered} H_c^i(X_x, \mathbb{Z}/n) = 0 \text{ for } i \neq 2N \text{ and} \\ \mathrm{Tr}\colon H_c^{2N}(X_x, \mathbb{Z}/n(N)) \xrightarrow{\sim} \mathbb{Z}/n \text{ is an isomorphism.} \end{gathered} \tag{2}$$

The case $N = 0$ is trivial. When $N > 0$, let Y_y denote the strict localization at a closed point of a smooth scheme Y of pure dimension $N-1$, and let $f\colon X_x \to Y_y$ be an essentially smooth morphism (of relative dimension one). The proof uses the Leray spectral sequence for cohomology with proper support for f to reduce the question to the case of curves. The "cohomology with proper support" considered being defined by limits (1), the existence of such a spectral sequence poses various problems of passage to the limit, treated with too much detail in SGA 4, XVIII. Here we are content to calculate. For every geometric point z of Y_y, we have

$$(R^i f_! \mathbb{Z}/n)_z = H_c^i(f^{-1}(z), \mathbb{Z}/n).$$

The geometric fibre $f^{-1}(z)$ is a projective limit of smooth curves over an algebraically closed field. It satisfies Poincaré duality. Its ordinary cohomology is given by the local acyclicity theorem for smooth morphisms,

$$H^i(f^{-1}(z), \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n & \text{for } i = 0 \\ 0 & \text{for } i > 0. \end{cases}$$

By duality, we have

$$H_c^i(f^{-1}(z), \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n(-1) & \text{for } i = 2 \\ 0 & \text{for } i \neq 2, \end{cases}$$

and the Leray spectral sequence becomes

$$H_c^i(X_x, \mathbb{Z}/n(N)) = H_c^{i-2}\big(Y_y, \mathbb{Z}/n(N-1)\big).$$

We conclude by induction on N.

4 *Variants and applications*

It is possible to construct, in étale cohomology, a "duality formalism" (= functors Rf_*, $Rf_!$, f^*, $Rf^!$, satisfying various compatibilities and adjunction formulas) parallel to that existing in coherent cohomology. In this language, the results of the preceding paragraph can be rewritten as follows: if $f\colon X \to S$ is smooth of pure relative dimension N and $S = \mathrm{Spec}(k)$ with k algebraically closed, then

$$Rf^!\mathbb{Z}/n = \mathbb{Z}/n[2N](N).$$

This statement is valid without hypothesis on S. It admits the

COROLLARY 4.1. *If f is smooth of pure relative dimension N, and the sheaves $R^i f_! \mathbb{Z}/n$ are locally constant, then the sheaves $R^i f_* \mathbb{Z}/n$ are also locally constant, and*

$$R^i f_* \mathbb{Z}/n = \underline{\mathrm{Hom}}\left(R^{2N-i} f_! \mathbb{Z}/n(N), \mathbb{Z}/n\right).$$

In particular, under the hypotheses of the corollary, the sheaves $R^i f_* \mathbb{Z}/n$ are constructible. Starting from that, we can show that, if S is of finite type over the spectrum of a field or Dedekind ring, then, for every morphism of finite type $f: X \to S$ and every constructible sheaf $\mathcal{F}$ on X, the sheaves $R^i f_* \mathcal{F}$ are constructible SGA $4\frac{1}{2}$, Th. finitude, 1.1.

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