AZUMAYA ALGEBRAS (GG SEMINAR 2021 SPRING)

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This note is based on my $4\frac{1}{2}$ talks on March-May 2021 in the Galois-Grothendieck seminar in the University of Virginia organized by Prof. Andrei Rapinchuk. The aim of this seminar in the academic year 2020-2021 is to study Brauer groups. In this note, we are using several standard references as in the end of this file.

In this note, R will denote a commutative ring with 1.

Our goal is to generalize the notion of **Central Simple Algebras** (CSA) to similar objects over a commutative ring R. The obvious formulation "Central plus simple R-algebra" is not a good candidate. Because the center of a simple ring is necessarily a field, so this will not give us anything new. We need other formulation.

1. Definition of Azumaya Algebras

We will form a group out of the set of equivalence classes of Azumaya algebras over a fixed commutative ring R.

Let A be an R-algebra. Let $\epsilon\colon R \to A$ be the R-algebra structure map of A.

Definition 1.1. Let $A^e\colon = A \otimes_R A^o$, where A^o denotes the opposite algebra of A. The R-algebra A^e is called the **enveloping algebra** of A.

There is a natural R-algebra homomorphism $\psi\colon A^e \longrightarrow \mathrm{End}_R(A)$ defined by $a \otimes \alpha^o \mapsto (x \mapsto ax\alpha)$ and extend linearly. This map is called the **enveloping homomorphism** of A. Here $\mathrm{End}_R(A)$ stands for the algebra consisting of all R-endomorphisms as R-modules (NOT R-algebras).

Definition 1.2. An R-module A is called **faithfully projective** if it is finitely generated, projective and faithful as an R-module. An R-algebra A is called an **Azumaya Algebra** if

- A is a faithfully projective R-module.
- The map $\psi\colon A^e \longrightarrow \mathrm{End}_R(A)$ defined above is an isomorphism.

Example 1.3. If $R = k$ be a field, then a finitely generated k-algebra A is Azumaya if and only if A is central simple.

$\Leftarrow$: Suppose A is central simple over k. The algebra A is a finite dimensional vector space over k, hence free (thus projective) and faithful. Since A, A^o are both central simple, so is $A \otimes_k A^o$ (recall that we defined the group law of the Brauer group $\mathrm{Br}(k)$ in such way). Thus ψ is injective. Since $\dim_k(A \otimes A^o) = (\dim_k A)^2 = \dim_k \mathrm{End}_k(A)$, ψ is also surjective. Therefore ψ is an isomorphism.

$\Rightarrow$: Let A be an Azumaya k-algebra. In this case, a projective module is always free. Thus $A \simeq k^n$ as a k-module for some $n \in \mathbb{N}$. So

$$A^e \simeq \mathrm{End}_k(A) \simeq M_n(k),$$

which is a central simple algebra. Recall that $A_1 \otimes_k A_2$ is a Central Simple Algebra if and only if both A_1 and A_2 are central simple. We conclude the argument.

As an equivalent description, Auslander and Goldman defined "Azumaya algebra" to be both central and separable in their seminal paper in 1960 [1]. See also the book by F. DeMeyer and E. Ingraham [2]. Being technical itself (*separable* means A is a projective A^e-module), separable algebras will not discussed in detail here, but we want to emphasize one important property.

Proposition 1.4. *Azumaya algebras are central, i.e. $Z(A) = R$. Here we identify R with the image $R \cdot 1$ of the algebra-structure map $\epsilon\colon R \to A$. (Since A is faithful, this ϵ is injective.)*

Proof. Consider the **trace ideal** of A in R (which is a two-sided ideal)

$$\mathcal{T}_R(A) = \langle f(a); f \in \mathrm{Hom}_R(A, R), a \in A\rangle.$$

Since A is faithfully projective, there exists a **dual basis** $\{(f_i, a_i); 1 \le i \le n\}$, i.e. for any $a \in A$,

$$a = \sum_{i=1}^{n} f_i(a)a_i.$$

This implies that $\mathcal{T}_R(A)A = A$, by Nakayama's lemma, $R = \mathcal{T}_R(A) + \mathrm{Ann}_R(A) = \mathcal{T}_R(A)$. In other words, the trace ideal is all of R (this means A is a **generator** (**progenerator**) over R in the context of Morita's theorem).

(Remark: In fact, the trace ideal here is always principal.)

Claim. $R\colon = R \cdot 1$ is an R-module direct summand of A.

In order to verify the existence of a left inverse for ε, it suffices to show that

$$\mathrm{Hom}_R(A, R) \to \mathrm{Hom}_R(R, R), \ \ g \mapsto g \circ \varepsilon$$

is onto. Let $\mathfrak{m}$ be any maximal ideal of R, then $R/\mathfrak{m} \otimes_R A = A/\mathfrak{m}A$ is a progenerator of the field $R/\mathfrak{m}$. In other words, $A/\mathfrak{m}A$ is a **non-zero** finite dimensional vector space over $R/\mathfrak{m}$.

Since $0 \to R/\mathfrak{m} \to A/\mathfrak{m}A$ is split exact over $R/\mathfrak{m}$, the map

$$\mathrm{Hom}_{R/\mathfrak{m}}(A/\mathfrak{m}A, R/\mathfrak{m}) \to \mathrm{Hom}_{R/\mathfrak{m}}(R/\mathfrak{m}, R/\mathfrak{m})$$

is onto. Therefore

$$R/\mathfrak{m} \otimes_R \mathrm{Hom}_R(A, R) \to R/\mathfrak{m} \otimes_R \mathrm{Hom}_R(R, R)$$

is also onto. It remains to apply Lemma 1.5 below to get a section.

Let $A = R \oplus P$ be such a direct sum decomposition. Then for any $p_0 \in P \backslash \{0\}$, $p_0 \otimes 1$ and $1 \otimes p_0$ are different elements in $A^e \simeq \mathrm{End}_R(A)$. In fact, by the universal property of the tensor product,

$$\mathrm{Hom}_R(A^e, A) \simeq \mathrm{Bil}_R(A \times A^o, A).$$

So it suffices to find a bilinear map $\Phi\colon A \times A^o \to A$ such that $\Phi(1, p_0) \neq \Phi(p_0, 1)$. Recall that any element in A has a unique expression $r + p$ where $r \in R, p \in P$. Since $x \mapsto xp_0$ and $x \mapsto p_0x$ are two different maps, the bilinear function

$$\Phi\colon (r + p, r' + p') \mapsto p(r' + p')$$

works.

□

Lemma 1.5. *Let M, N be R-modules, with N finitely generated. Then for $f \in \mathrm{Hom}_R(M,N)$, f is onto if and only if for any maximal ideal $\mathfrak{m}$ of R, the induced map $\bar{f}\colon M/\mathfrak{m}M \to N/\mathfrak{m}N$ is onto.*

Proof. Let C be the cokernel of f. Then for any $\mathfrak{m} \in \mathrm{Spm}(R)$, we have $C = \mathfrak{m}C$. Using Nakayama's lemma for a maximal ideal $\mathfrak{m}$ which contains $\mathrm{Ann}_R(C)$, we have $C = 0$. □

2. Constructing Brauer group over a commutative ring

Theorems about endomorphisms algebras of projective R-modules can often be reduced to similar and much simpler questions about endomorphism algebras of free R-modules. Concerning the latter, the following result is standard whose proof needs nothing but linear algebra.

Proposition 2.1. *For a commutative ring R, we have*

(1) $M_m(R) \otimes M_{m'}(R) \simeq M_{mm'}(R)$.
(2) *The map $w : \mathrm{End}_R(R^m) \otimes \mathrm{End}_R(R^{m'}) \to \mathrm{End}_R(R^m \otimes R^{m'})$ defined by $f \otimes g \mapsto (x \otimes y \mapsto f(x) \otimes g(y))$ is an isomorphism. Notation: we will not distinguish $f \otimes g$ and its image under w.*

The result also holds if we replace free modules by projective modules.

Proposition 2.2. *Let P, Q be finitely generated projective R-modules, then the map $w : \mathrm{End}_R(P) \otimes \mathrm{End}_R(Q) \to \mathrm{End}_R(P \otimes Q)$ defined by $f \otimes g \mapsto (x \otimes y \mapsto f(x) \otimes g(y))$ is an isomorphism.*

Before giving the proof, recall the following construction which generalizes matrix representation of a linear transformation.

Lemma 2.3. *Let $E_1, E_2, ..., E_n; F_1, F_2, ..., F_m$ be R modules and*

$$\varphi : \bigoplus_{i=1}^{n} E_i \to \bigoplus_{j=1}^{m} F_j$$

be an R-module homomorphism. Then φ can be represented by a unique matrix

$$M(\varphi) = \begin{pmatrix} \varphi_{11} & \varphi_{12} & \cdots & \varphi_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{m1} & \varphi_{12} & \cdots & \varphi_{mn} \end{pmatrix}$$

where $\varphi_{ij} \in \mathrm{Hom}_R(E_i, F_j)$.

If E_i, F_j are both free of rank 1, this is linear algebra since φ_{ij} are giving by $x \mapsto a_{ij}x, a_{ij} \in R$.

Proof. In fact, φ_{ij} is equal to the composition

$$E_i \to \bigoplus_{k=1}^{n} E_k \to \bigoplus_{l=1}^{m} F_l \to F_j.$$

□

In particular, for R-modules M and N, we have

$$i : \mathrm{End}(M) \to \mathrm{End}(M \oplus N),\ f \mapsto (x \mapsto (f(\pi_M(x)),$$

and

$$j : \mathrm{End}(M \oplus N) \to \mathrm{End}(M),\ g \mapsto (y \mapsto \pi_M(g(y))).$$

The composition $j \circ i$ is the identity, so i is injective and j is surjective.

Proof of Proposition 2.2. Let P', Q' be finitely generated projective modules such that $P \oplus P' \simeq R^m$ and $Q \oplus Q' \simeq R^n$. The following diagram commutes

$$\begin{array}{ccc} \mathrm{End}_R(P) \otimes \mathrm{End}_R(Q) & \longrightarrow & \mathrm{End}_R(P \otimes_R Q) \\ \downarrow\ \ \uparrow & & \downarrow\ \ \uparrow \\ \mathrm{End}_R(R^m) \otimes \mathrm{End}_R(R^n) & \longrightarrow & \mathrm{End}_R(R^m \otimes R^n) \end{array}$$

The bottom arrow is bijective, so is the top arrow. □

Just as matrix algebras $M_l(k)$ represents the neutral element $[1] \in \mathrm{Br}(k)$, we expect $\mathrm{End}_R(P)$ for a faithfully projective R-module plays a similar role. First, we have to verify that it is an Azumaya algebra itself.

Proposition 2.4. *If P is a faithful projective R-module then $\mathrm{End}_R(P)$ is an Azumaya algebra.*

Proof. Since P is finitely generated projective, there is an R-module Q such that $P \oplus Q \simeq R^n$. Consider homomorphisms

$$\mathrm{End}_R(P) \to \mathrm{End}_R(P \oplus Q) \to \mathrm{End}_R(P)$$

whose composition is the identity, we see that $\mathrm{End}_R(P)$ is finitely generated projective.

If $r \in \mathrm{Ann}(\mathrm{End}_R(P))$, then it annihilates the identity map. So $r = 0$, which implies the faithfulness $\mathrm{End}_R(P)$.

To verify condition (2) in the definition of Azumaya algebra, consider the following commutative diagram.

$$\begin{array}{ccc} \mathrm{End}_R(P) \otimes \mathrm{End}_R(P)^o & \xrightarrow{\Psi_P} & \mathrm{End}_R(\mathrm{End}_R(P)) \\ \downarrow\ \ \uparrow & & \downarrow\ \ \uparrow \\ \mathrm{End}_R(P \oplus Q) \otimes \mathrm{End}_R(P \oplus Q)^o & \xrightarrow{\Psi_{P \oplus Q}} & \mathrm{End}_R(\mathrm{End}_R(P \oplus Q)) \end{array}$$

Thus it suffices to show that $\Psi_{P \oplus Q}$ is an isomorphism. Let $e_1, \dots, e_n$ be the natural R-basis for $P \oplus Q \simeq R^n$, and let $E_{ij} \in \mathrm{End}_R(R^n)$ such that $E_{ij}(e_k) = \delta_{ik} e_j$. Then $\{E_{ij}; 1 \le i, j \le n\}$ is an R-basis for $\mathrm{End}_R(R^n)$, and $\{E_{ij} \otimes E^o_{kl}; 1 \le i, j, k, l \le n\}$ is an R-basis for $\mathrm{End}_R(R^n) \otimes \mathrm{End}_R(R^n)^o$. We also have, by definition of $\Psi_{P \oplus Q}$,

$$\Psi_{P \oplus Q}(E_{ij} \otimes E^o_{kl})(E_{st}) = E_{ij} E_{st} E_{kl} = \delta_{js} \delta_{tk} E_{il}.$$

Thus $\Psi_{P \oplus Q}$ maps basis to basis, so it is an isomorphism. □

The next result tells us that $\otimes_R$ gives the operation of the Brauer group.

Proposition 2.5. *If A and B are Azumaya algebras, so is $A \otimes_R B$.*

Proof. The following diagram is commutative

$$\begin{array}{ccc} (A\otimes B)\otimes(A\otimes B)^o & \xrightarrow{\Psi_{A\otimes B}} & \mathrm{End}(A\otimes B) \\ \uparrow & & \uparrow w \\ (A\otimes A^o)\otimes(B\otimes B^o) & \xrightarrow{\Psi_A\otimes\Psi_B} & \mathrm{End}(A)\otimes\mathrm{End}(B) \end{array}$$

Here ψ_A (resp. ψ_B) denotes the isomorphism coming from the fact A (resp. B) is an azumaya algebra, w is the isomorphism given by Proposition 2.2. The left side vertical isomorphism comes from the commutativity of the tensor product and the fact that $(A\otimes B)^o = A^o\otimes B^o$. This shows that $\psi_{A\otimes B}$ is an isomorphism. □

Definition 2.6. Let A and B be Azumaya algebras over R. We write $A\sim B$ if there exists faithfully projective R-modules P and Q such that

$$A\otimes\mathrm{End}(P)\simeq B\otimes\mathrm{End}(Q).$$

This is an equivalence relation. The only thing to check is transitivity, so suppose that $A\sim B$ and $B\sim C$ for Azumaya algebras A, B and C, then there exist faithfully projective R-modules P, P', Q and Q' such that

$$A\otimes\mathrm{End}(P)\simeq B\otimes\mathrm{End}(Q),\ B\otimes\mathrm{End}(P')\simeq C\otimes\mathrm{End}(Q').$$

Then

$$A\otimes\mathrm{End}(P\otimes P')\simeq C\otimes\mathrm{End}(Q\otimes Q').$$

Remark 2.7. *When $R = k$ is a field, and A, B are Central Simple Algebras, then the $\sim$ is exactly the equivalence relation to define* $\mathrm{Br}(k)$*. In fact, if $[A] = [B]\in$ $\mathrm{Br}(k)$, then there is a central division algebra D over k such that $A\simeq M_n(D)\simeq D\otimes_k M_n(k)$ and $B\simeq M_m(D)\simeq D\otimes_k M_m(k)$. We see that*

$$A\otimes_k\mathrm{End}_k(k^m)\simeq A\otimes_k M_m(k)\simeq B\otimes_k M_n(k)\simeq B\otimes_k\mathrm{End}_k(k^n).$$

For another direction, let D, D' be central division algebras such that $A\simeq M_n(D)\simeq D\otimes_k M_n(k)$ and $B\simeq M_m(D')\simeq D'\otimes_k M_m(k)$. If

$$A\otimes_k\mathrm{End}_k(k^s)\simeq B\otimes_k\mathrm{End}_k(k^t)$$

then $sn = tm$ and $D\simeq D'$.

Definition 2.8. Define the Brauer set

$$\mathrm{Br}(R) := \{\text{Azumaya algebras over } R\}/\sim.$$

We denote by $[A]$ the equivalence class of A under this $\sim$.

Now let us give an abelian group structure of $\mathrm{Br}(R)$.

(1) Group operation is the tensor product. This is well-defined. Suppose $A_1\sim A_2$ and $B_1\sim B_2$. Suppose $A_i\otimes\mathrm{End}(P_i)\simeq B_i\otimes\mathrm{End}(Q_i)$ for $i = 1, 2$, then $(A_1\otimes B_1)\otimes\mathrm{End}(P_1\otimes Q_1)\simeq(A_2\otimes B_2)\otimes\mathrm{End}(P_2\otimes Q_2)$.
(2) Associativity and commutativity come from those properties of $\otimes$.
(3) Identity element is $[R]$. For any Azumaya algebra, $[A][R] = [A\otimes R] = [A]$. (More generally, for any faithfully projective R-module P, we have $\mathrm{End}(P)$ corresponds to the identity element.)
(4) Inverse element. For any Azumaya algebra A, we have $[A][A^o] = [A\otimes A^o] = [\mathrm{End}(A)] = [1]$.

The next proposition gives a description of equivalence relation which is more similar to the one in defining Brauer groups over a field.

Proposition 2.9. *For $[A], [B] \in \mathrm{Br}(R)$, we have*

(1) *$[A] = [1]$ if and only if $A \simeq \mathrm{End}_R(P)$ for some faithfully projective R-module P.*
(2) *$[A] = [B]$ if and only if $A \otimes_R B^o \simeq \mathrm{End}_R(Q)$ for some faithfully projective R-module Q.*

The proof needs Morita's theory.

3. Functoriality of $\mathrm{Br}(\cdot)$

Our goal is to show that

$$\mathrm{Br}(\cdot) : \mathrm{CommRing} \longrightarrow \mathrm{AbGp}$$

is a covariant functor. The key ingredient of the demonstration is to understand how Azumaya algebras behave under Base Change. Suppose that $f : R \to S$ is a commutative ring homomorphism which makes S to be an R-algebra. Then if A is an R-algebra, $A \otimes_R S$ becomes an S-algebra. An obvious candidate for $\mathrm{Br}(f)$ is $[A] \mapsto [A \otimes_R S]$. We need to show that this is well defined. Most efforts will be spent to verify that $(A \otimes_R S) \otimes_S (A \otimes_R S)^o \to \mathrm{End}_S(A \otimes_R S)$ is an isomorphism.

Lemma 3.1. *If A is a faithfully projective R-algebra and S is a commutative R-algebra, then $\mathrm{End}_R(A) \otimes S \simeq \mathrm{End}_S(A \otimes_R S)$.*

Proof. We define an S-algebra homomorphism

$$\varphi_A \colon \mathrm{End}_R(A) \otimes S \to \mathrm{End}_S(A \otimes_R S),\ \ f \otimes s \mapsto (a \otimes s' \mapsto f(a) \otimes ss').$$

As A is faithfully projective, there is an R-module B such that $A \oplus B \simeq R^n$. We then have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{End}_R(A) \otimes_R S & \xrightarrow{\quad\varphi_A\quad} & \mathrm{End}_S(A \otimes_R S) \\
\downarrow \ \ \uparrow & & \downarrow \ \ \uparrow \\
\mathrm{End}_R(R^n) \otimes_R S & \xrightarrow{\quad\varphi_{R^n}\quad} & \mathrm{End}_S(R^n \otimes_R S)
\end{array}$$

It suffices to show that φ_{R^n} is an isomorphism, i.e. free case. Let $e_1, ..., e_n$ be the natural basis for R^n, then $e_i \otimes_R 1$ is a natural basis for $R^n \otimes_R S$. Take $E_{ij} \in \mathrm{End}_R(R^n)$ such that $E_{ij}(e_k) = \delta_{ik} e_j$ so that E_{ij} is a basis for $\mathrm{End}_R(R^n)$. Thus $E_{ij} \otimes 1 (1 \leq i, j \leq n)$ form a basis for $\mathrm{End}_R(R^n) \otimes_R S$. Notice that

$$\varphi_{R^n}(E_{ij} \otimes_R 1) = (e_k \otimes_R 1 \mapsto \delta_{ik} e_j \otimes_R 1),$$

i.e. φ_{R^n} maps a basis to a basis, thus it is an isomorphism. □

Now we can show that $\mathrm{Br}(f)$ is well-defined.

Proposition 3.2. *If A is an Azumaya R-algebra and S is a commutative R-algebra, then $A \otimes_R S$ is an Azumaya S-algebra.*

Proof. In fact, "being faithfully projective" is preserved under base change. Let B be an R-module such that $A \oplus B \simeq R^n$, then $(A \otimes_R S) \oplus (B \otimes_R S) \simeq S^n$, thus $A \otimes_R S$ is projective. Recall that $\mathrm{Ann}(M) \cap \mathrm{Ann}(N) = \mathrm{Ann}(M \oplus N)$, we obtain that $A \oplus_R S$ is faithful.

(It is important to notice that if A is faithful, then $A \otimes_R S$ is not necessarily a faithful S-module. For example, $R = \mathbb{Z}, A = 2\mathbb{Z}$ and $S = \mathbb{Z}/2\mathbb{Z}$.)

Next it suffices to observe the following commutative diagram.

$$\begin{array}{ccc} (A \otimes_R S) \otimes_S (A \otimes_R S)^o & \xrightarrow{\Psi_{A\otimes_R S}} & \mathrm{End}_S(A \otimes_R S) \\ \downarrow & & \uparrow \\ (A \otimes_R A^o) \otimes_R S & \longrightarrow & \mathrm{End}_R(A) \otimes_R S \end{array}$$

□

Now, let us prove the functoriality.

Theorem 3.3. *We have* $\mathrm{Br}(\cdot)$ *is a functor.*

Proof. Suppose $A \sim A'$ as R-algebras. Then there exists faithfully projective R-modules P, P' such that

$$A \otimes_R \mathrm{End}_R(P)) \simeq A' \otimes_R \mathrm{End}_R(P').$$

Thus

$$(A \otimes_R S) \otimes S(\mathrm{End}_R(P) \otimes S) \simeq (A' \otimes_R S) \otimes S(\mathrm{End}_R(P') \otimes S).$$

This implies that $A \otimes_R S \sim A' \otimes S$, i.e. $\mathrm{Br}(f)\colon \mathrm{Br}(R) \to \mathrm{Br}(S)$ is well-defined. Now we verify that $\mathrm{Br}(f)$ is a group homomorphism. Suppose $[A], [B] \in \mathrm{Br}(R)$, then $[A][B] = [A \otimes_R B]$. It remains to observe that

$$(A \otimes_R B) \otimes_R S = (A \otimes_R S) \otimes_S (B \otimes_R S).$$

□

4. Quaternion algebras: first examples of Azumaya algebras

Example 4.1. Let $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ be the usual real quaternion algebra. Then there is no subring A of $\mathbb{H}$ such that (1) A is free of rank four over $\mathbb{Z}$, (2) $A \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{H}$, (3) A is an Azumaya algebra over $\mathbb{Z}$. Well, suppose the contrary, then $A \otimes_{\mathbb{Z}} \mathbb{F}_2 \simeq M_2(\mathbb{F}_2)$, LHS is commutative, but not RHS, a contradiction. In particular, the following algebras are NOT Azumaya over $\mathbb{Z}$.

- $A = \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus k = \mathbb{Z}[i, j, k]$, i.e. the $\mathbb{Z}$-algebra of "integer quaternions";
- $A = \mathbb{Z}[i, j, k, \frac{1+i+j+k}{2}]$ (as an maximal order, a more natural object).

In fact, $\mathrm{Br}(\mathbb{Z}) \simeq \{0\}$.

Example 4.2. Let R be a commutative ring in which 2 is invertible, then the algebra of R-quaternions is an Azumaya algebra. That is $Q\colon = R \oplus Ri \oplus Rj \oplus Rk$ with $i^2 = j^2 = k^2 = -1, ij = k = -ji$. The isomorphism can be obtained explicitly. For example, $\frac{1}{4}(1 \otimes 1 - i \otimes i - j \otimes j - k \otimes k)$ maps to E_{11} matrix (under the enveloping homomorphism). Thus the map is surjective, hence bijective due to the easy lemma below.

Lemma 4.3. *Let M be a finitely generated R-module, then any surjective endomorphism of M is an automorphism.*

Proof. Let θ be such a surjection. View M as an $R[X]$-module via $X \cdot m = \theta(m)$. Then $(X)M = M$. By Nakayama's lemma, there exists $p(X)X = Xp(X) \in XR[X]$ such that $(1 - p(X)X)M = (1 - Xp(X))M = 0$. So $p(\theta)$ is an inverse of θ.

□

5. Azumaya algebras are separable

Definition 5.1. An R-algebra A can be viewed an an A^e-module via $(a_1 \otimes a_2^o a) = a_1 a a_2$. The algebra A is called **separable** if A is a projective A^e-module.

Theorem 5.2. *If A is Azumaya over R, then A is separable.*

Proof. It suffices to show that A is a projective $\mathrm{End}_R(A)$-module. We have already seen that $A = R \oplus P$ as R-modules. Take R-module homomorphism $g\colon A \to R$ such that $g(r+p) = r$. Then the surjection

$$\mathrm{End}_R(A) \to A,\ f \mapsto f(1)$$

has an $\mathrm{End}_R(A)$-module section, namely

$$a \mapsto (a' \mapsto g(a')a).$$

Thus A is projective $\mathrm{End}_R(A)$-module. □

Remark 5.3. *Up to now we have verified that all Azumaya algebras are central and separable. The converse is also true but the proof is technical, so will be skipped.*

6. Brauer group of a local ring

In this section, let $(R, \mathfrak{m}, k = \bar{R})$ be a commutative Noetherian local ring.

In this situation, the definition of Azumaya Algebras is simpler, i.e. an R-algebra A is Azumaya if it is free of finite rank as an R-module and the enveloping homomorphism is an ismomorphism.

When defining the Brauer group, the equivalence relation is just the same as the one of Central Simple Algebras.

Remark 6.1. *Azumaya himself only defined the Azumaya algebras over local rings, and Auslander and Goldman generalized this to arbitrary commutative rings.*

Proposition 6.2 (Skolem-Noether)**.** *Let A be an Azumaya R-algebra, then any automorphism of A is inner, that is, of the form $a \mapsto u^{-1}au$ for some unit u in A.*

Proof. Let $\varphi\colon A \to A$ be such an automorphism. We can realize A as an $A^e = A \otimes_R A^o$-module in two different ways

$$(a_1 \otimes a_2^o)a = a_1 a a_2 \text{ and } (a_1 \otimes_2^o)a = \varphi(a_1) a a_2.$$

We denote the resulting A^e-modules to be A and A' respectively. Both $\overline{A'}\colon = A' \otimes_R R/\mathfrak{m}$ and $\overline{A}$ are simple $\overline{A^e}$-modules. Since $\overline{A^e} = \overline{A} \otimes_{R/\mathfrak{m}} \overline{A^o}$ is a finite dimensional simple algebra over $k = R/\mathfrak{m}$, there is only one simple $\overline{A^e}$-module up to isomorphism.

To see this, let $\overline{A^e} = M_m(P)$ where P is a division algebra over k. Notice that the category of $\overline{A^e}$-modules is equivalent to the category of P-modules via

$$M \mapsto M^{\oplus n} \text{ and } N \mapsto E_{11}N.$$

Also observe that every module over a division algebra is free.

So we have an $\overline{A^e}$-module isomorphism

$$\overline{\eta}\colon \overline{A} \to \overline{A'}.$$

By separability, A is a projective A^e-module. Now the map

$$A \to \overline{A} \to \overline{A'}$$

lifts to an A^e-module homomorphism $\eta\colon A \to A'$. The surjectivity of $\overline{\eta}$ implies that $\eta(A)+\mathfrak{m}A' = A'$, now Nakayama's lemma tells us η is also surjective. Let $u = \eta(1)$. Then for any $a \in A$, we have

$$\eta(a) = \eta(a1) = \eta((a \otimes 1)1) = \varphi(a)u,$$

and also

$$\eta(a) = \eta(1a) = \eta((1 \otimes a)1) = ua$$

Thus $\varphi(a)u = ua$. It remains to check that u is a unit, in fact, by the surjectivity of η, there is $b \in A$ such that $\eta(b) = 1$, then $\varphi(b) = u^{-1}$. □

Corollary 6.3. *The automorphism group of $M_n(R)$ (as an R-algebra) is $\mathrm{PGL}_n(R) = \mathrm{GL}_n(R)/R^\times$.*

Next we talk about the existence of a "good splitting" of the Azumaya algebra.

Theorem 6.4 (Hensel's lemma)**.** *Let $(R, \mathfrak{m}) = (\mathcal{O}_v, \varpi\mathcal{O}_v)$ or $(F[[X]], XF[[X]])$. Let $f(X) \in R[X]$, if $\bar{f} \in k[X]$ factors as $\bar{f} = g_0h_0$ as a product of two monic coprime polynomials. Then there exist $g, h \in R[X]$, both monic such that $f = gh, \bar{g} = g_0, \bar{h} = h_0$*

A local ring $(R, \mathfrak{m})$ satisfying the above lemma is called **Henselian ring** or **Hensel ring**. It can be shown that if R is a complete local ring, i.e.

$$R \simeq \hat{R}_{\mathfrak{m}} := \varprojlim_i R/\mathfrak{m}^i,$$

then R is Henselian.

Lemma 6.5. *Let R be a Henselian ring, then any finite local R-algebra B is also Henselian. The same for every non-zero quotient ring R/J.*

Proposition 6.6. *Let R be a Henselian ring. Then the map $\mathrm{Br}(R) \to \mathrm{Br}(k), [A] \mapsto [A \otimes_R k]$ is injective.*

Proof. Let $[A]$ be in the kernel of the above map. Then A is an Azumaya algebra over R such that there is an isomorphism $\bar{A} \to M_n(k)$. Let $\varepsilon \in \bar{A}$ be the unique element which maps to $E_{11} \in M_n(k)$. Then ε is idempotent, i.e. $\varepsilon^2 = \varepsilon$. Let $\varepsilon = \sum_{i=1}^l a_i \otimes y_i = \sum_{i=1}^l a_i y_i \otimes 1$. Then $a := \sum_{i=1}^l a_i y_i \in A$ maps to ε. Since A, as an R-algebra, is finitely generated. We have a is integral over R. Let $B = R[a]$.

Fact. Let A be a faithful R-algebra. Then $a \in A$ is integral if and only if there is an R-subalgebra B of A containing a such that B is a finitely generated R-module (standard in "Algebraic Number Theory", the proof uses Cayley-Hamilton's theorem).

(Remark 1: each B_i is in fact isomorphic to $B_{\mathfrak{m}_i}$ for some maximal ideal $\mathfrak{m}_i$ of B.)

(Remark 2: an alternative definition of Hensel ring is that every finite R-algebra is a direct product of local rings.)

Now we want to show that ε lifts to an idempotent element $e \in R[a] = B$. Notice that $a^2 - a \in \mathfrak{m}B$, then there is a monic polynomial

$$\eta(X) = X^d + \sum a_j X^j \text{ with } a_j \in \mathfrak{m},$$

such that $\eta(a^2 - a) = 0$. Then $f(X) = \eta(X^2 - X) \in R[X]$ is a monic polynomial such that $f(X) \equiv X^d(X-1)^d \pmod{\mathfrak{m}}$ and $f(a) = 0$. Thus by Hensel property, there exist monic $g, h \in R[X]$ such that

$$g(X) \equiv X^d \pmod{\mathfrak{m}} \text{ and } h(X) \equiv (X-1)^d \pmod{\mathfrak{m}}$$

Then $b_1 = g(a) \in B$ is a lift of $\varepsilon^d = \varepsilon$ and $b_2 = h(a) \in B$ is a lift of $(\varepsilon - 1)^d = (-1)^d(1-\varepsilon)$ and moreover $b_1 b_2 = 0$. Thus $(b_1, b_2)B/\mathfrak{m}B = B/\mathfrak{m}B$ and $V(b_1, b_2) \subset \operatorname{Spec}(B)$ is disjoint from $V(\mathfrak{m}B)$. Since $\operatorname{Spec}(B) \to \operatorname{Spec}(R)$ is closed (going-up property), we can find $r \in R$ which maps to invertible elements in $R/\mathfrak{m}$ whose image in B lies in (b_1, b_2).

Easy lemma. Let S/R be a ring extension, I, J be ideal of R, S respectively. If the closure of the image of $V(J)$ is disjoint from $V(I)$, then $\exists t \in R$ which maps to $1 \in R/I$ and to an element in J in S.

Let I' be an ideal in R such that $V(I') = \overline{\text{image of } V(J)}$. Then $V(I) \cap V(I') = \emptyset$ so $I + I' = R$. Write $1 = t + s$ with $t \in I$ and $s \in I'$. We have $V(J) \subset V(t')$ where t' is the image of t in S. Hence $t'^n \in J$ for some n. Replacing t by t^n, we win.

After replacing R by the localization R_r, we get $(b_1, b_2) = B$. Then $\operatorname{Spec}(B) = D(b_1) \coprod D(b_2)$; disjoint because $b_1 b_2 = 0$, covers $\operatorname{Spec}(B)$ because $(b_1, b_2) = B$. Let $e \in B$ correspond to the open and closed subset $D(b_1)$. Since b_1 is a lift of ε and b_2 is a lift of $(-1)^d(1-\varepsilon)$, by the uniqueness property (one-to-one correspondence between open and closed subsets and idempotents), e is a lifting of ε.

Then $A = Ae \oplus A(1-e)$, in fact, if $a_1 e = a_2(1-e)$, then $a_1 e = a_1 e^2 = a_2(1-e)e = a_2 0 = 0$. So, the R-module Ae and $A(1-e)$ are finitely generated and free, now we consider the former one. Let

$$\varphi\colon A \to \operatorname{End}_R(Ae);\ a \mapsto (xe \mapsto axe).$$

Then $\ker(\varphi) \cap R = \{0\}$ since Ae is free. Let $A = Ra_1 \oplus \cdots \oplus Ra_l$, consider the $\chi_i \in \operatorname{End}(A)$ such that $\chi_i(a_j) = \delta_{ij}$, then χ_i is given by $y \mapsto \sum \theta_i^{(k)} y \tilde{\theta_i}^{(k)}$ due to the enveloping isomorphism. For any $a \in \ker(\varphi)$, write $a = \sum r_i a_i$, then $r_i = \chi_i(a) = \sum \theta_i^{(k)} a \tilde{\theta_i}^{(k)} \in \ker(\varphi)$. Thus $r_i = 0 \Rightarrow \varphi$ is injective.

Now consider the induced map $\bar{\varphi}\colon \bar{A} \to \operatorname{End}_k(\bar{A}\varepsilon)$. Similar argument shows that φ is also injective. Since $\bar{A}$ and $\operatorname{End}_k(\bar{A}\varepsilon)$ have the same dimension$= n^2$, $\bar{\varphi}$ is an isomorphism, thus surjective. Then Nakayama's lemma shows that φ is also surjective, cf. Lemma 1.5. □

Corollary 6.7. *If R is strictly local, i.e. the residue field k of R is separably closed, then* $\operatorname{Br}(R) = \{0\}$.

Remark 6.8. *Using étale cohomology, we can show that if R is local Henselian ring, then the map* $\operatorname{Br}(R) \to \operatorname{Br}(k)$ *is an isomorphism.*

Let us recall étale extension. An R-algebra S is called **étale** if it is (commutative,) separable, flat and finitely presented as an R-module. Here finitely presented means $S = R[X_1, X_2, \cdots, X_l]/I$ for a finitely generated ideal I. If R is Noetherian, then "finitely presented" above can be replace by "finitely generated".

When R is a field, the description is simpler.

Recall. A finite commutative algebra L over a field k is called **étale** if one of the following equivalent condition holds.

(1) $L = \prod_{i=1}^{r} L_i$ where L_i/k is a finite separable field extension.
(2) $[L\colon K] < \infty$ and $L \times L \to k, (x, y) \mapsto \mathrm{Tr}(xy)$ is non-degenerate.
(3) $L \otimes_{\overline{K}} \overline{K} \simeq \overline{K}^n$ for some $n \in \mathbb{N}$.

Lemma 6.9. *If $(R, \mathfrak{m}, k)$ is Henselian local. Then $S \mapsto S \otimes_R k$ induces an equivalence of the category of finite étale R-algebras and the category of finite étale k-algebras.*

In fact, for any étale k-algebra k', write $k' = k[X]/(\bar{h}(X))$. Then $S = R[X]/(h(X))$ satisfies $B \otimes_R k = k'$.

Proposition 6.10. *If A is an Azumaya algebra over a Henselian local ring R, then there is a finite étale faithfully flat ring homomorphism $R \to S$ such that $A \otimes_R S \simeq M_n(S)$ as S-algebras.*

This is because of Lemma 6.9 and that the proposition is true for fields. In fact, there is a finite separable extension k'/k such that $(A \otimes_R k) \otimes_k k' \simeq M_n(k')$, i.e. $[(A \otimes_R k) \otimes_k k'] = [0] \in \mathrm{Br}(k')]$. Now by Lemma 6.9, there exists S as a finite étale R-algebra (being local itself) such that $S \otimes_R k = k'$. Since $[A \otimes_R S]$ maps to $[M_n(k')] = [0]$ under $\mathrm{Br}(S) \to \mathrm{Br}(k')$. By the injectivity of this map, $A \otimes_R S$ splits.

Thus, every Azumaya algebra over a Henselian local ring has rank n^2 for some n.

Before we continue, we want to review the **faithfully flat descent.** For a systematic introduction to descent theory, see Knus-Ojanguren' book [5]. In particular, we can find in [5] why the classical Galois descent is a special situation of faithfully flat descent.

Definition 6.11. An R-module M is faithfully flat if any complex of R-modules

$$M_1 \to M_2 \to M_3$$

it is exact if and only if

$$M \otimes_R M_1 \to M \otimes_R M_2 \to M \otimes_R M_3$$

is exact. Another equivalent description is A is flat and for every nonzero R-module N, $M \otimes_R N$ is nonzero as well.

Theorem 6.12. *If $f\colon R \to S$ faithfully flat ring homomorphism, then the* **Amitsur Complex**, *i.e. the sequence $\mathcal{C}^\bullet(S/R)$ defined by*

$$\text{(AC)} \qquad 0 \to R \xrightarrow{f} S \xrightarrow{d^0} S^{\otimes 2} \to \cdots \to S^{\otimes r} \xrightarrow{d^{r-1}} S^{\otimes (r+1)} \to \cdots$$

is exact (as R-modules). Here $d^r = \sum_{i=0}^{r+1} (-1)^i e_i$ and

$$e_i(b_0 \otimes \cdots \otimes b_{r-1}) = b_0 \otimes \cdots b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_{r-1}.$$

If we tensor the above sequence by any R-module M, the we still get an exact sequence.
(AC1)

$$0 \to M \xrightarrow{1\otimes f} M\otimes_R S \xrightarrow{1\otimes d^0} M\otimes_R S^{\otimes 2} \to \cdots \to M\otimes_R S^{\otimes r} \xrightarrow{1\otimes d^{r-1}} M\otimes_R S^{\otimes(r+1)} \to \cdots$$

Proof. Since $\forall r \otimes s \in \ker(f) \otimes_R S$, it is equal to $1 \otimes f(r)s = 1 \otimes 0 = 0$, thus $\ker(f) \otimes_R S = 0$. Since S/R is faithfully flat, $\ker(f) = 0$. Thus R can be viewed as a subring of S.

Now the exactness at S, i.e. $1 \otimes s = s \otimes 1$ if and only if $s \in R$. We proceed in three steps.

Step 1. Suppose there is a section, i.e. $g\colon S \to R$ such that $g \circ f = g|_R = \mathrm{Id}_R$. Consider the map $h\colon = g \otimes \mathrm{Id}_S\colon S \otimes_R S \to S$. Let $s \in \ker(d^0)$, then

$$0 = h(0) = h(1 \otimes s - s \otimes 1) = s - g(s),$$

which implies that $s = g(s) \in R$.

In general, define a (contracting homotopy) operator $k^r\colon S^{\otimes(r+2)} \to S^{\otimes(r+1)}$ by

$$k^r(x_0 \otimes \cdots \otimes x_{r+1}) = g(x_0)x_1 \otimes \cdots \otimes x_{r+1}.$$

We can verify by direct computation that $d^{r-1}k^{r-1} + k^r d^r$ is equal to the identity map on $S^{\otimes(r+1)}$. Now for any $y \in \ker(d^r)$, we have $y = \mathrm{Id}(y) = d^{r-1}k^{r-1}(y) \in \mathrm{Im}(d^{r-1})$.

Step 2. Suppose $R \to R'$ is a faithfully flat extension. Then $S \otimes_R R'$ is also faithfully flat over R'. Tensoring (AC) by $\otimes_R R'$ and notice that

$$(S \otimes_R S) \otimes_R R' = (S \otimes_R R') \otimes_{R'} (S \otimes_R R'),$$

we get

$$0 \to R' \to S \otimes_R R' \to (S \otimes_R R') \otimes_{R'} (S \otimes_R R') \to \cdots$$

Since R' is faithfully flat, we can always replace the pair (R, S) with $(R', S \otimes_R R')$.

Step 3. Consider arbitrary $f\colon S \to R$. We use the previous reduction to $R' = S$. So we get faithfully flat extension $S \to S \otimes_R S, s \mapsto s \otimes 1$. We construct a section $g\colon S \otimes S \to S, s \otimes s' \mapsto ss'$. But this puts us in Case 1, we win.

□

Now we will have a closer look at splitting rings of an Azumaya algebra. Recall that a Central Division Algebra D over a field k is always split by a maximal subfield P of D, the map is given by

$$D \otimes_k P \simeq \mathrm{End}_P(D) = M_n(P),\ t \otimes p \mapsto (y \mapsto typ).$$

Conversely, and field $P \supset F$ which splits D with $[P\colon k] = \sqrt{\dim_k D} = l$ is isomorphic to a maximal subfield of D. For a Central simple algebra A which is not necessary division, we have the following generalization.

Proposition 6.13. *Let $\overline{A}$ be a Central Simple Algebra over a field k of dimension n^2, then there exists a commutative k-subalgebra $\overline{S} \subset \overline{A}$ such that*

1. *$\overline{S}$ is a maximal commutative k-subalgebra of $\overline{A}$.*
2. *$\overline{S}$ is separable over k of dimension n.*
3. *$\overline{S} = k(\alpha)$ for some $\alpha \in \overline{A}$.*

(4) $\overline{A}$ *is a free* $\overline{S}$*-module of rank* n.
(5) $\overline{S}$ *is a splitting ring for* $\overline{A}$.

Proof. If k is finite, $\overline{A} \simeq M_n(k)$. Let $P = k[a_0]$ be a Galois extension of degree n (In fact, if $k = \mathbb{F}_q$, then let $P = \mathbb{F}_{q^n}$). Let $P \to \mathrm{End}_k(P) = \overline{A}, p \mapsto (x \mapsto px)$ be the left regular representation. Then it maps to a maximal commutative subalgebra $\overline{S}$ of $\overline{A}$.

Let k be infinite. By Artin-Wedderburn theorem, $\overline{A} = M_r(D)$ for a unique Central Divison Algebra D over k. Let $P = k(u)$ be a maximal subfield of D where u is separable over k. Then $[P\colon D] = l = \sqrt{\dim_k D}$. Since k is infinite, the set of minimal polynomials of $au, a \in k^\times$ is an infinite subset of $k[X]$ (they all have degree n since $P = k(au)$, i.e. au is a primitive element). So we can find r irreducible polynomials $f_i(X) \colon = \text{min.poly.}_k(a_i u)$ such that $f \colon = f_1 \cdots f_r$ has no repeated roots and $\deg(f) = rl = n$. Take $\alpha = \mathrm{diag}\{a_1 u, \cdots, a_r u\} \in M_r(D)$. Then $\text{min.poly.}_k(\alpha) = f$. So $\overline{S} = k(\alpha)$ is a subalgebra of $\overline{A}$ with $\dim_k \overline{S} = n$.

Since f has no multiple roots, $S = k[X]/(f)$ is separable over k. Thus $\overline{A}$ is projective as an $\overline{S}$-module. Notice that $P^r \simeq \overline{S}$ via diagonal embedding. Let $\overline{S}' \supset \overline{S}$ be a commutative k-subalgebra of $\overline{A}$. Then any $y = (y_{ij}) \in \overline{S'}$ commutes with elements in $\overline{S}$. In particular, take $E_{ii} \in \overline{S}$.

$$yE_{ii} = E_{ii}y \Rightarrow y_{ij} = 0 \text{ if } i \neq j.$$

We see that $y = \mathrm{diag}\{y_{11}, \cdots, y_{rr}\} \subset D^r$. Since P is a maximal subfield of D, we have $y_{ii} \in P$. So $\overline{S}$ is a maximal commutative subalgebra of $\overline{A}$.

□

We have the following local ring analogue of Proposition 6.13.

Theorem 6.14. *Let A be an Azumaya R-algebra of rank n^2.*

(1) *Let $a \in A$. Let S be a faithfully flat R-algebra which splits A, and let $\varphi_S \colon A \otimes_R S \simeq M_n(S)$ be an isomorphism. Then the characteristic polynomial $ch_a(X)$ of $\varphi_S(a \otimes 1)$ belongs to $R[X]$, is independent of S and $ch_a(a) = 0$. This $ch_a(X)$ is called the* ***Cayley-Hamilton polynomial*** *of a.*
(2) *There is a maximal commutative étale subalgebra S of A of rank n that is a direct summand of A. Moreover, A is a free module over S. (Such S is called a* ***maximal étale subalgebra*** *of A.)*
(3) *The subalgebra S as above splits A.*

Proof. (1) We first remark that for any two isomorphisms $\varphi_1, \varphi_2 \colon A \otimes_R S \simeq M_n(S)$ of S-algebras, $\varphi_1(a \otimes 1)$ and $\varphi_2(a \otimes 1)$ have the same characteristic polynomial. In fact, for any maximal ideal $\mathfrak{n}$ of S(S is not necessarily local), the Skolem-Noether theorem implies that there is $u \in \mathrm{GL}_n(S_\mathfrak{n})$ such that $\varphi_2(a \otimes 1) = u^{-1}\varphi_1(a \otimes 1)u$. So two characteristic polynomials have the same image in $S_\mathfrak{n}[X]$ for all $\mathfrak{n} \in \mathrm{Spm}(S)$. This proves the remark.

Now let T/R be another faithfully flat ring extension such that there is a $\varphi_T\colon A\otimes_R T \xrightarrow{\sim} M_n(T)$. Consider the commutative diagram of $(S\otimes_R T)$-algebras

$$\begin{array}{ccc}(A\otimes_R S)\otimes_R T & \xrightarrow{\varphi_S\otimes \mathrm{Id}_T} & M_n(S)\otimes_R T = M_n(S\otimes_R T)\\ \downarrow & & \downarrow \\ S\otimes_R (A\otimes_R T) & \xrightarrow{\mathrm{Id}_S\otimes\varphi_T} & S\otimes_R M_n(T) = M_n(S\otimes_R T)\end{array}$$

where the left vertical map is $(a\otimes s)\otimes t\mapsto s\otimes(a\otimes t)$, and the right vertical map is the unique isomorphism defined by the rest of the diagram.

Let $S = T$ then by the above remark, $\varphi_S(a\otimes 1_S)\otimes 1_S, 1_S\otimes\varphi_T(a\otimes 1_S)\in M_n(S\otimes_R T)$ have the same characteristic polynomial. In other words,

$$1_{S[X]}\otimes ch_{\varphi_S(a\otimes 1_S)}(X) - ch_{\varphi_S(a\otimes 1_S)}(X)\otimes 1_{S[X]} = 0.$$

Take $M = R[X]$ in the Amitsur complex (AC1), we get the exact sequence

$$0\to R[X]\to S[X]\to S[X]\otimes_{R[X]} S[X]\to\cdots$$

So $ch_{\varphi_S(a\otimes 1_S)}(X)\in\ker(1\otimes d^0) = R[X]$, i.e. the characteristic polynomial is defined over R.

Now let S, T be any two faithfully flat splitting extension of R. Then $S\otimes_R T$ is also a faithfully flat algebra over R. We have the following Amitsur complex

$$0\to R[X]\to (S\otimes_R T)[X]\to (S\otimes_R T)[X]\otimes_{R[X]}(S\otimes_R T)[X]\to\cdots$$

Since $(\varphi_S(a\otimes 1_S))\otimes 1_T$ and $1_S\otimes(\varphi_T(a\otimes 1_T))$ have the same characteristic polynomial, due to the exactness at $(S\otimes_R T)[X]$, they come from the same element in $R[X]$.

Finally, let S/R be a faithfully flat splitting extension, then $A\to A\otimes_R S$ is injective. Thus $ch_a(a)\mapsto ch_a(a\otimes 1) = 0$, which implies that $ch_a(a) = 0$.

(2) We can choose $a\in A$ such that $\alpha = \bar{a}$, so $k(\bar{a}) = k(\alpha)$ is a splitting ring of $\overline{A}$ as Proposition 6.13.

Let $S = R[X]/(ch_a(X))$, this is an étale algebra over R of rank n there is a canonical map $S\to A; \bar{X}\mapsto a$. As $S\otimes_R k\xrightarrow{\sim} k[\bar{a}]\hookrightarrow\bar{A}$ is injective, it follows from a standard lemma below that $S\to A$ is injective and S is a direct summand of A.

Easy lemma. Let $\varphi\colon M\to N$ be a homomorphism of two finitely generated R-modules with N-free. If $\bar{\varphi} = \varphi\otimes 1$ is injective, then φ has a section, in particular it is injective. Moreover, if $\bar{\varphi}$ is an isomorphism, so is φ.

In fact, let $\varphi'\colon N\to M$ such that $\bar{\varphi'}\bar{\varphi} = Id_{\bar{M}}$. Let $\psi = \varphi'\varphi$, then Nakayama's lemma implies that ψ is surjective. Regard M as an $R[X]$-module via $X\cdot m = \psi(m)$. Then by Nakayama's lemma, there is $f(X)\in R[X]$ such that $(1-\psi f(\psi))M = 0$. So $f(\psi)\varphi'$ is a left inverse of φ.

In fact, we can show that A is a free module over S. We need the following

Lemma. Let S be a semilocal ring, i.e. a commutative ring with finitely many maximal ideals, then any finitely generated projective S-module M of constant rank is free.

We just need to mimic the proof of the fact "projective modules over local rings are free" to prove this Lemma (use Nakayama's lemma). Since A/R is projective, S/R separable (modulo $\mathfrak{m}$ and use the Proposition 6.13), so A/S is projective. In our situation, $S \otimes_R k = S/\mathfrak{m}S = k(\alpha)$ is isomorphic to P^r, thus S has at most finitely many maximal ideals. For very $\mathfrak{n} \in \mathrm{Spm}(S)$, $A/\mathfrak{n}A$ is free over $S/\mathfrak{n}$ of rank n, i.e. A is of constant rank.

(3) Let S as in (2), view A as a right S-module. Consider the map

$$\alpha\colon A \otimes_R S \to \mathrm{End}_S(A);\ a_0 \otimes s \mapsto (x \mapsto a_0 x s).$$

Modulo $\mathfrak{m}$, we get

$$\alpha_{\mathfrak{m}}\colon A \otimes_k S/\mathfrak{m} \to \mathrm{End}_{S/\mathfrak{m}}(A \otimes_k S/\mathfrak{m}),$$

this is an isomorphism by theory of Central Simple Algebras over fields (since $S \otimes_R K$ is a splitting ring of $\overline{A}$). Since both $A \otimes_R S$ and $\mathrm{End}_S(A)$ are finitely generated over S and since A is free over S, so is $\mathrm{End}_S(A)$. Thus the previous lemma tells us α is an isomorphism. □

7. The Brauer group over a scheme, a brief introduction

Let X be a local Notherian scheme. An $\mathcal{O}_X$-algebra $\mathcal{A}$ is called an Azumaya algebra over X if it is coherent as an $\mathcal{O}_X$-module and if, for every closed point x of X, $\mathcal{A}_x$ is an Azumaya algebra over the local ring $\mathcal{O}_{X,x}$. The condition imply that $\mathcal{A}$ is locally free of finite rank as an $\mathcal{O}_X$-module. We also have for *every* point x of X, $\mathcal{A}_x$ is an Azumaya algebra over $\mathcal{O}_{X,x}$. To see this, consider the affine case $X = \mathrm{Spec}(R)$, take any prime ideal $\mathfrak{p} \in X$ and take a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$. There is a natural map $R_{\mathfrak{m}} \to R_{\mathfrak{p}}, a/t \mapsto a/t$. Then using standard commutative algebra, we get

$$A_{\mathfrak{p}} = A_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} R_{\mathfrak{p}}.$$

By the functoriality of $\mathrm{Br}(\cdot)$, we see that $A_{\mathfrak{p}}$ is an Azumaya algebra over $\mathcal{O}_{X,\mathfrak{p}} = R_{\mathfrak{p}}$.

There are several equivalent descriptions of Azumaya algebra over X.

Theorem 7.1. *Let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra which is of finite type as an $\mathcal{O}_X$-module. Then the following are equivalent.*

1. *$\mathcal{A}$ is an Azumaya algebra over X.*
2. *$\mathcal{A}$ is locally free as an $\mathcal{O}_X$-module and for all $x \in X$, $\mathcal{A}(x) \colon = \mathcal{A}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ is a Central Simple Algebra over $\kappa(x)$.*
3. *$\mathcal{A}$ is locally free as an $\mathcal{O}_X$-module and the enveloping homomorphism $\psi\colon \mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^o \to \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{A})$ is an isomorphism.*
4. *There is a covering $(U_i \to X)$ for the étale topology on X such that for each i, there exists an $r_i \in \mathbb{N}$, $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \simeq \mathcal{M}_{r_i}(\mathcal{O}_{U_i})$.*
5. *Same thing holds as above when replacing "étale topology" by "flat topology".*

In particular, if $X = \mathrm{Spec}(R)$, then any Azumaya algebra over X has shape $\widetilde{A}$ for some Azumaya algebra A over R (as defined in Section 1).

Artin's Question. Suppose X is proper over $\mathrm{Spec}\mathbb{Z}$, is $\mathrm{Br}(X)$ finite?

If $\dim X = 1$, the class field theory gives the positive response. But this question is open even for a surface over a finite field. Known situation: for all $K3$ surfaces, the answer is yes.

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