Cohomology of Groups
(selected items)

§1 Abstract introduction of group cohomology (continued)

We are roughly following Chapter IV "Cohomology of Groups" by Atiyah-Wall in the Proceedings "Algebraic Number Theory" edited by Cassels-Fröhlich (1967)

G always denotes a group

Definition To $X \in \underline{Ab}$ (category of abelian groups) we associate the co induced G-module $\underline{X^*} := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G, X) \cong \{f : G \to X\}$, $\varphi \mapsto \varphi|_G$
with (left) G-action

$$(g.\varphi)(z) := \varphi(zg) \quad \forall g \in G, z \in \mathbb{Z}G, \varphi \in X^*.$$

Side-remark (getting back to a question from last time)

Different ways to turn $\mathrm{Hom}(A,B) = \mathrm{Hom}_{\mathbb{Z}}(A,B)$ into a G-module:

1) $A \in \underline{Ab}$, $B \in {}_G\underline{Mod}$ (category of left G-modules)
$(g.\varphi)(a) := g.\varphi(a) \quad \forall g \in G, a \in A, \varphi \in \mathrm{Hom}(A,B)$

2) $A \in {}_G\underline{Mod}$, $B \in \underline{Ab}$ $\quad (g.\varphi)(a) := \varphi(g^{-1}.a) \quad \forall g \in G, a \in A, \varphi \in \mathrm{Hom}(A,B)$
or $A \in \underline{Mod}_G$, $B \in \underline{Ab}$ or $(g.\varphi)(a) := \varphi(a.g)$
used to define X^* ($A = \mathbb{Z}G$ considered as right G-module; $B = X$)

3) $A, B \in {}_G\underline{Mod}$ $\rightsquigarrow$ we can also define
$(g.\varphi)(a) := g.\varphi(g^{-1}.a)$. With this G-module structure

$$\underline{\mathrm{Hom}(A,B)^G} = \{\varphi \in \mathrm{Hom}(A,B) \mid g.\varphi(g^{-1}.a) = \varphi(a) \ \forall g \in G, a \in A\}$$
$$= \{\varphi \in \mathrm{Hom}(A,B) \mid \varphi(g^{-1}.a) = g^{-1}.\varphi(a) \ \forall g \in G, a \in A\} = \underline{\mathrm{Hom}_G(A,B)}$$

Note that this G-action on $\mathrm{Hom}(A,B)$ can only be defined if A and B are both G-modules, so it's in general not available for $X^* = \mathrm{Hom}(\mathbb{Z}G, X)$.

Lemma 1 For $X \in \underline{Ab}$, $B \in {}_G\underline{Mod}$, there is a natural isomorphism between the abelian groups $\mathrm{Hom}(B, X)$ and $\mathrm{Hom}_G(B, X^*)$.

Sketch of proof: Define $\mathrm{Hom}(B, X) \longrightarrow \mathrm{Hom}_G(B, X^*)$ by $\psi'(b)(z) := \psi(z.b) \in X$

$\psi \longmapsto \psi' : B \to X^*$ for all $b \in B$, $z \in \mathbb{Z}G$

and $\mathrm{Hom}_G(B, X^*) \longrightarrow \mathrm{Hom}(B, X)$

$\varphi \longmapsto \varphi' : b \mapsto \varphi(b)(1) \in X$

Check that these are hom. of abelian groups which are inverse to each other.

Remarks (a) $^* : \underline{Ab} \to {}_G\underline{Mod}$ is the right adjoint of the "forgetful functor"

$F : {}_G\underline{Mod} \to \underline{Ab}$, and so the system of isomorphisms in Lemma 1 constitute in fact a natural equivalence between the bifunctors

$$\mathrm{Hom}(F(-), -) \longrightarrow \mathrm{Hom}_G(-, -^*)$$

(b) In connection with homology, we will later consider induced G-modules $X_* := \mathbb{Z}G \otimes_{\mathbb{Z}} X$ (with left G-module structure $g.\, z \otimes x := gz \otimes x$)

Lemma 2 If $A = X^* \in {}_G\underline{Mod}$ is coinduced ($X \in \underline{Ab}$), then $H^i(G, A) = 0$ $\forall i \geq 1$.

Proof: Pick any projective resolution $\underline{\underline{P}} \twoheadrightarrow \mathbb{Z}$ of the trivial G-module $\mathbb{Z}$, i.e. we have an exact sequence $\ldots \to P_i \to P_{i-1} \to \ldots \to P_0 \to \mathbb{Z} \to 0$ where all the P_i ($i \geq 0$) are projective G-modules.

Note that the P_i are then also projective (in fact free) $\mathbb{Z}$-modules since they are direct summands of free abelian groups. So we can also consider $\underline{\underline{P}} \twoheadrightarrow \mathbb{Z}$ as a projective resolution of $\mathbb{Z}$ in $\underline{Ab} = {}_{\mathbb{Z}}\underline{Mod}$.

Apply the contravariant functor $\mathrm{Hom}_G(-, A)$ to get the cochain complex $\underline{\underline{K}} := \mathrm{Hom}_G(\underline{\underline{P}}, A)$; By definition $H^i(G, A) = H^i(\underline{\underline{K}})$.

Lemma 1 $\Rightarrow$ ($A = X^*$)

$$\begin{array}{ccccccc}
\mathrm{Hom}_G(P_0, A) & \longrightarrow & \mathrm{Hom}_G(P_1, A) & \longrightarrow & \mathrm{Hom}_G(P_2, A) & \longrightarrow & \cdots \\
\wr\wr & \circlearrowleft & \wr\wr & \circlearrowleft & \wr\wr & & \\
\mathrm{Hom}(P_0, X) & \longrightarrow & \mathrm{Hom}(P_1, X) & \longrightarrow & \mathrm{Hom}(P_2, X) & \longrightarrow & \cdots
\end{array}$$

Naturality of the isomorphisms in Lemma 1 (which can easily be checked) implies that the squares in the above diagram are commutative.

It follows that $H^i(G, A) \cong H^i(\mathrm{Hom}(\underline{\underline{P}}, X)) \quad \forall i \geq 0$.

But now recall that by definition of Ext $H^i(\mathrm{Hom}(\underline{\underline{P}}, X)) = \mathrm{Ext}^i(\mathbb{Z}, X)$ in the category $\underline{Ab}$. Since $\mathbb{Z}$ is a projective (in fact free) $\mathbb{Z}$-module, it follows from standard properties of Ext that $\mathrm{Ext}^i(\mathbb{Z}, X) = 0 \quad \forall i \geq 1$

$$\Rightarrow \quad H^i(G, A) = 0 \quad \forall i \geq 1.$$

<u>Remark</u> Note that we can identify $\underline{Ab}$ with $\underline{Mod}_{\{1\}}$, the category of G-modules for the trivial group $G = \{1\}$. So what we showed above is

$H^i(G, X^*) = H^i(\{1\}, X)$ (and clearly $H^i(\{1\}, X) = 0 \quad \forall i \geq 1$)

This identity will be generalized later with <u>Shapiro's lemma</u>.

<u>Definition</u> (Atiyah-Wall) A <u>cohomological extension</u> of the functor A^G is a sequence of functors $H^i(G, -) : \underline{Mod}_G \rightarrow \underline{Ab}$ which satisfies

(0) $H^0(G, A) = A^G$ (more precisely: natural equivalence of the functors $H^0(G, -)$ and $-^G$)

(1) Existence of long exact sequences (functorially)

(2) $H^i(G, A) = 0 \quad \forall$ coinduced G-modules A, $\forall i \geq 1$

Recall what (1) means:

For every short exact sequence of G-modules $0 \to A \to B \to C \to 0$ $(*)$, there exists a long exact sequence of abelian groups

$$0 \to H^0(G,A) \to H^0(G,B) \to H^0(G,C) \to H^1(G,A) \to \dots \to H^i(G,A) \to H^i(G,B) \to H^i(G,C) \to H^{i+1}(G,A) \to$$

which depends functorially on the input $(*)$.

<u>Theorem 1</u> There exists one and, up to canonical equivalence, only one cohomological extension of the functor A^G.

<u>Sketch of proof</u>: <u>Existence</u> Fix a projective resolution $\underline{P} \to \mathbb{Z}$ of the trivial G-module $\mathbb{Z}$, and define $H^i(G,A) := H^i(\mathrm{Hom}_G(\underline{P}, A))$ $\forall A \in {}_G\mathrm{Mod}$, $\forall i \geq 0$.

By "standard homological algebra", this definition is (up to canonical iso.) independent of the choice of $\underline{P}$ and satisfies (0) and (1).
(2) was proved in Lemma 2.

<u>Uniqueness</u> Here we see that the role injective modules play in general homological algebra can conveniently be replaced by coinduced modules in group cohomology. We first establish that every G-module can be embedded into a coinduced one:

$A \in {}_G\underline{\mathrm{Mod}} \Rightarrow A^* = \mathrm{Hom}(\mathbb{Z}G, A) \in {}_G\underline{\mathrm{Mod}}$; $A \hookrightarrow A^*$

$a \mapsto \varphi_a : \mathbb{Z}G \to A$

$z \mapsto za$

Check that the map $a \mapsto \varphi_a$ is G-linear,
i.e. $g \cdot \varphi_a = \varphi_{g \cdot a}$ $\forall g \in G$, $a \in A$

Clearly this map is injective ($\varphi_a = 0 \Rightarrow 0 = \varphi_a(1) = a$)

Now we can see that Conditions (0) – (2) uniquely determine the sequence of functors $H^i(G, \bar{A})$ for $i \geq 0$:

$H^0(G, A) = A^G$ is determined by (0). Next consider the short exact sequence $0 \to A \to A^* \to A' \to 0$ with $A' := A^*/A$, which by (1) yields a long exact sequence

$$0 \to A^G \to (A^*)^G \to (A')^G \to H^1(G,A) \to \underset{\substack{\shortparallel \\ 0}}{H^1(G,A^*)} \to \dots \to \underset{\substack{\shortparallel \\ 0}}{H^i(G,A^*)} \to H^i(G,A') \to H^{i+1}(G,A) \to \underset{\substack{\shortparallel \\ 0}}{H^{i+1}(G,A^*)} \to$$

by (2)

Hence for all $i \geq 1$, we must have $H^{i+1}(G,A) \cong H^i(G,A')$.
So if $H^i(G,-)$ is already determined, then so is $H^{i+1}(G,-)$.
And the beginning of the long exact sequence implies

$$H^1(G,A) \cong \operatorname{coker}\left((A^*)^G \to (A')^G\right)$$

so that $H^1(G,-)$ is also uniquely determined.

<u>Remark</u> If one knows how to construct long exact sequences, the procedure described above can also be considered as an inductive definition (or construction) of the sequence of functors $H^i(G,-)$, starting with $H^0(G,-) = -^G$.

§2 The bar resolution

Question: What is a "good" projective (or even free) resolution of the trivial G-module $\mathbb{Z}$?

Obvious start: $P_0 = \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$.

$$\sum_{g\in G} n_g\, g \longmapsto \sum_{g\in G} n_g$$

Next step: Find a free G-module P_1 which projects onto $I_G = \ker \varepsilon$

$$\rightsquigarrow \quad \begin{tikzcd} P_1 \arrow[r] \arrow[dr] & P_0 \arrow[r, "\varepsilon"] & \mathbb{Z} \arrow[r] & 0 \\ & I_G \arrow[u] & & \end{tikzcd}$$

Inconvenient to follow up step-by-step.

Imitate topology (singular homology) to construct an exact sequence $\underline{\underline{P}} \overset{\varepsilon}{\twoheadrightarrow} \mathbb{Z}$ as follows ($i \geq 0$):

$\underline{P_i}$:= $\underline{\text{free } \mathbb{Z}\text{-module}}$ with $\mathbb{Z}$-basis $G^{i+1} = \{(g_0, \dots, g_i) \mid g_j \in G \ \forall 0 \leq j \leq i\}$, turned into a $\underline{G\text{-module}}$ via $g.(g_0, \dots, g_i) := (gg_0, \dots, gg_i)$

(It's not completely obvious that P_i is also a free G-module; we will return to this issue later.)

Define differentials or boundary operators

$$d_i : P_i \longrightarrow P_{i-1} \quad \text{by} \quad d_i((g_0, \dots, g_i)) := \sum_{j=0}^{i} (-1)^j (g_0, \dots, \overset{\substack{\text{omitted}\\ \downarrow}}{\hat{g}_j}, \dots, g_n)$$

and $\mathbb{Z}$-linear extension. Check: d_i is G-linear.

$$d_1 : P_1 \longrightarrow P_0 = \mathbb{Z}G \qquad\qquad d_2 : P_2 \longrightarrow P_1$$

$$(g_0, g_1) \mapsto (g_1) - (g_0) = g_1 - g_0 \qquad (g_0, g_1, g_2) \mapsto (g_1, g_2) - (g_0, g_2) + (g_0, g_1)$$

<u>Lemma 3:</u> $\cdots \to P_i \xrightarrow{d_i} P_{i-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is exact.

<u>Sketch of proof:</u> $\ker \varepsilon = I_G = \bigoplus_{g \neq 1} \mathbb{Z}(g-1) = \operatorname{im} d_1$ ✓

$(P_i)_{i \geq 0}$ is a <u>chain complex</u>, i.e. $0 = d_i \circ d_{i+1} : P_{i+1} \to P_{i-1}$ $\forall i \geq 1$

($\Leftrightarrow \operatorname{im} d_{i+1} \subseteq \ker d_i$)

This is verified by a standard computation, similarly as in topology.

<u>$\ker d_i \subseteq \operatorname{im} d_{i+1}$</u> $\forall i \geq 1$ idea: imitate the concept of a chain homotopy

$$
\begin{tikzcd}
\cdots \arrow[r] & P_{i+1} \arrow[r, "d_{i+1}"] & P_i \arrow[r, "d_i"] \arrow[dl, "s_i"'] \arrow[d, "id_{P_i}"] & P_{i-1} \arrow[r] \arrow[dl, "s_{i-1}"] & \cdots \\
\arrow[r] & P_{i+1} \arrow[r, "d_{i+1}"] & P_i \arrow[r, "d_i"] & P_{i-1} \arrow[r] & \cdots
\end{tikzcd}
$$

Find a sequence of hom. of abelian groups $(s_i : P_i \to P_{i+1})_{i \geq 0}$ s.th.
$id_{P_i} = d_{i+1} s_i + s_{i-1} d_i$ $\forall i \geq 1$. Then it follows
$x \in \ker d_i \Rightarrow x = d_{i+1} s_i(x) + s_{i-1} \underbrace{d_i(x)}_{=0} = d_{i+1}(s_i(x)) \in \operatorname{im} d_{i+1}$

Define $s_i : P_i \to P_{i+1}$ and verify $d_{i+1} s_i + s_{i-1} d_i = id_{P_i}$ $\forall i \geq 1$.
$(g_0, \dots, g_i) \mapsto (1, g_0, \dots, g_i)$

So the sequence in Lemma 3 is an exact sequence of G-modules.
Is it also a <u>free G-resolution</u> of $\mathbb{Z}$?
To answer this question, we will introduce G-modules F_i which are free G-modules by definition and compare them to the P_i.

<u>Definition</u> F_i is the free G-module with basis G^i $(i \geq 0)$
To distinguish the (G-) basis elements of F_i from the ($\mathbb{Z}$-) basis elements of P_{i-1}, we will use the "bar notation" for the elements of G^i here, i.e.
$G^i = \{ [g_1 | \ldots | g_i] \mid g_j \in G \ \forall 1 \leq j \leq i \}$. ($F_0 \cong \mathbb{Z}G$ as G-module with basis $\{[\,]\}$)

We relate F_i and P_i through hom. τ_i and σ_i $(i \geq 0)$:

$\sigma_i : F_i \longrightarrow P_i$ and $\mathbb{Z}G$-linear extension

$[g_1 | \ldots | g_i] \longmapsto (1, g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_i)$

$\tau_i : P_i \longrightarrow F_i$ and $\mathbb{Z}$-linear extension

$(g_0, \ldots, g_i) \longmapsto g_0 [g_0^{-1} g_1 | g_1^{-1} g_2 | \ldots | g_{i-1}^{-1} g_i]$

Check: τ_i is also $\mathbb{Z}G$-linear.

<u>Lemma 4</u> σ_i and τ_i are inverse to each other for all $i \geq 0$.

<u>Proof:</u> By a straightforward computation, verify that
$\tau_i \sigma_i ([g_1, \ldots, g_i]) = [g_1, \ldots, g_i] \quad \forall [g_1, \ldots, g_i] \in G^i$ and
$\sigma_i \tau_i ((g_0, g_1, \ldots, g_i)) = (g_0, g_1, \ldots, g_i) \quad \forall (g_0, g_1, \ldots, g_i) \in G^{i+1}$.

<u>Corollary</u> The P_i are free G-modules, and hence the exact sequence $\underline{P} \twoheadrightarrow \mathbb{Z}$ in Lemma 3 is a free G-resolution of the trivial G-module $\mathbb{Z}$.

Next we want to relate this resolution in terms of the F_i to get the standard "bar resolution" of $\mathbb{Z}$.

Definition Define differentials $\partial_i : F_i \to F_{i-1}$ s.th.
$$\begin{tikzcd} P_i \arrow[r, "d_i"] \arrow[d, "\tau_i"'] \arrow[dr, phantom, "\circlearrowleft"] & P_{i-1} \arrow[d, "\tau_{i-1}"] \\ F_i \arrow[r, "\partial_i"] & F_{i-1} \end{tikzcd}$$
commutes for every $i \geq 1$, i.e.
$$\partial_i := \tau_{i-1}\, d_i\, \tau_i^{-1} = \tau_{i-1}\, d_i\, \sigma_i .$$

Then Lemmas 3 and 4 immediately imply the following

Corollary:
$$\ldots \to F_i \xrightarrow{\partial_i} F_{i-1} \to \ldots \to F_1 \xrightarrow{\partial_1} F_0 = \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$
is a free G-resolution of $\mathbb{Z}$.

Side-remark: We also need that
$$\begin{tikzcd} P_0 = \mathbb{Z}G \arrow[r, "\varepsilon"] \arrow[d, "\tau_0"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{Z} \arrow[d, "id_{\mathbb{Z}}"] \\ F_0 = \mathbb{Z}G \arrow[r, "\varepsilon"] & \mathbb{Z} \end{tikzcd}$$
commutes, which follows immediately from the definition of τ_0: $\tau_0(g) = \tau_0((g)) = g[\ ] \in F_0$
$$\qquad\qquad\qquad\qquad = g \in \mathbb{Z}G .$$

In order to apply the bar resolution of the Corollary for explicit computations, we need an explicit formula for the ∂_i.

Lemma 5
$$\partial_i [g_1|\ldots|g_i] = g_1 [g_2|g_3|\ldots|g_i] + \sum_{j=1}^{i-1} (-1)^j [g_1|\ldots|g_{j-1}|g_j g_{j+1}|g_{j+2}|\ldots|g_i]$$
$$+ (-1)^i [g_1|\ldots|g_{i-1}] \quad \forall\, i \geq 1 .$$

Proof:

$\underline{i=1}$: $\partial_1([g]) = \tau_0 d_1 \sigma_1([g]) = \tau_0 d_1((1,g)) = \tau_0((g) - (1))$
$= g[\ ] - [\ ]$, i.e. $\underline{\partial_1(g) = g - 1}$

$\underline{i=2}$: $\partial_2([g_1|g_2]) = \tau_1 d_2 \sigma_2([g_1|g_2]) = \tau_1 d_2((1, g_1, g_1 g_2)) =$
$\cancel{= \tau_1((g_1, g_2) - (1, g_2) + (1, g_1)) = g_1[g_2]}$
$= \tau_1((g_1, g_1 g_2) - (1, g_1 g_2) + (1, g_1)) = g_1[g_2] - [g_1 g_2] + [g_1]$

$i \geq 2$ $\quad \partial_i([g_1|\dots|g_i]) = \tau_{i-1}\, d_i\, \sigma_i([g_1|\dots|g_i]) = \tau_{i-1}\, d_i((1, g_1, g_1g_2, \dots, g_1\cdots g_i))$

$$= \tau_{i-1}\Big((g_1, g_1g_2, \dots, g_1\cdots g_i) + \sum_{j=1}^{i-1}(-1)^j (1, g_1, \dots, g_1\cdots g_{j-1}, g_1\cdots g_{j+1}, \dots, g_1\cdots g_i) + (-1)^i (1, g_1, \dots, g_1\cdots g_{i-1})\Big)$$

$$= g_1[g_2|g_3|\dots|g_i] + \sum_{j=1}^{i-1}(-1)^j [g_1|\dots|g_{j-1}|g_j g_{j+1}|g_{j+2}|\dots|g_i] + (-1)^i [g_1|g_2|\dots|g_{i-1}]. \quad \square$$

We can now use the <u>bar resolution</u> = exact sequence of G-modules

$$(*) \qquad \dots \to F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0, \quad \text{abbreviated } \underline{\underline{F}} \twoheadrightarrow \mathbb{Z}$$

$\qquad F_0 = \mathbb{Z}G$

in order to compute $\underline{H^n(G, A) = H^n(\mathrm{Hom}_G(\underline{\underline{F}}, A))}$ for $A \in \underline{\mathrm{Mod}_G}$.

~~$\mathrm{Hom}_G(\underline{\underline{F}}, A)$ is the cochain complex~~ Applying $\mathrm{Hom}_G(-, A)$ to $(*)$ yields:

$$(**) \quad 0 \to \mathrm{Hom}_G(\mathbb{Z}, A) \xrightarrow{\varepsilon^*} \mathrm{Hom}_G(F_0, A) \xrightarrow{\delta^0} \mathrm{Hom}_G(F_1, A) \to \dots \to \mathrm{Hom}_G(F_i, A) \xrightarrow{\delta^i} \mathrm{Hom}_G(F_{i+1}, A) \to \dots$$

$\qquad \mathrm{Hom}_G(\mathbb{Z}, A) = A^G, \qquad \mathrm{Hom}_G(F_0, A) \cong A$

with $\delta^i := \partial_{i+1}^* : \mathrm{Hom}_G(F_i, A) \to \mathrm{Hom}_G(F_{i+1}, A)$

$\qquad \varphi \longmapsto \varphi \circ \partial_{i+1}$

$$\begin{tikzcd} F_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[dr] & F_i \arrow[d, "\varphi"] \\ & A \end{tikzcd}$$

Note that by the left-exactness of $\mathrm{Hom}_G(-, A)$

$(**)$ is still exact up to δ^0, i.e. ε^* is injective and

$$\ker \delta^0 = \mathrm{im}\, \varepsilon^* \cong \underline{A^G}$$

$$\ker \delta^0 = H^0(\mathrm{Hom}_G(\underline{\underline{F}}, A)) = \underline{H^0(G, A)}$$

For $\underline{i \geq 1}$, we get $\underline{H^i(G, A) = \ker \delta^i / \mathrm{im}\, \delta^{i-1}}$.

We will describe this now in more detail.

Recall that F_i is a free G-module with G-basis

$$G^i = \{ [g_1|\dots|g_i] \mid g_j \in G \ \forall\, 1 \le j \le i \} \Rightarrow$$

$$=: (g_1, \dots, g_i)$$

$$\mathrm{Hom}_G(F_i, A) \cong \{ f : G^i \to A \} =: \underline{C^i(G,A)} \quad (\text{"i-cochains"})$$

$$\varphi \longmapsto \varphi|_{G^i}$$

$$\varphi \in \mathrm{Hom}_G(F_i, A) \Rightarrow$$

$$\delta^i(\varphi) = \varphi \circ \partial_{i+1} : (g_1, \dots, g_{i+1}) \xmapsto{\partial_{i+1}}$$

$$g_1 (g_2, \dots, g_{i+1}) + \sum_{j=1}^{i} (-1)^j (g_1, \dots, g_j g_{j+1}, \dots, g_{i+1}) + (-1)^{i+1} (g_1, \dots, g_i)$$

$$\xmapsto{\varphi} g_1 \varphi((g_2, \dots, g_{i+1})) + \sum_{j=1}^{i} (-1)^j \varphi((g_1, \dots, g_j g_{j+1}, \dots, g_{i+1})) + (-1)^{i+1} \varphi((g_1, \dots, g_i))$$

$i \ge 1$

$$Z^i(G,A) := \ker \delta^i \quad (i\text{-cocycles})$$

$$B^i(G,A) := \operatorname{im} \delta^{i-1} \quad (i\text{-coboundaries})$$

$$H^i(G,A) = Z^i(G,A) / B^i(G,A)$$

$\underline{i=1}$

$$Z^1(G,A) = \{ \varphi : G \to A \mid \delta^1(\varphi) = 0 \}$$

$$= \{ \varphi : G \to A \mid g_1 \varphi(g_2) - \varphi(g_1 g_2) + \varphi(g_1) = 0 \ \forall (g_1, g_2) \in G^2 \}$$

$$\Leftrightarrow \varphi(g_1 g_2) = g_1 \varphi(g_2) + \varphi(g_1) \quad \underline{\text{"crossed hom."}}$$

If A is a trivial G-module, then $Z^1(G,A) = \mathrm{Hom}(G,A) =$ abelian group of all hom. $G \to A$ with addition $(\varphi_1 + \varphi_2)(g) := \varphi_1(g) + \varphi_2(g)$.

$$B^1(G,A) = \operatorname{im} \delta^0, \quad \delta^0 : \mathrm{Hom}_G(\mathbb{Z}G, A) \to \mathrm{Hom}_G(F_1, A)$$

If we identify $\mathrm{Hom}_G(\mathbb{Z}G, A)$ with A (as abelian group) via $\varphi \leftrightarrow \varphi(1) = a$, then $\delta^0(a)(g) = g.a - a$. Hence

$$B^1(G,A) = \{ \varphi : G \to A \mid \exists a \in A : \varphi(g) = g.a - a \ \forall g \in G \}$$

In particular, $B^1(G,A) = 0$ if A is a <u>trivial G-module</u>, in which case

$$\underline{H^1(G,A) = Z^1(G,A) = \mathrm{Hom}(G,A)} \ (\cong \mathrm{Hom}(G_{ab}, A))$$

$\underline{i=2}$ $\quad Z^2(G,A) = \{\varphi : G\times G \to A \mid \delta^2(\varphi) = 0\}$

$= \{\varphi : G\times G \to A \mid g_1\varphi((g_2,g_3)) - \varphi((g_1g_2,g_3)) + \varphi((g_1,g_2g_3)) - \varphi((g_1,g_2)) = 0 \quad \forall (g_1,g_2,g_3)\in G^3\}$

$= \{\varphi : G\times G \to A \mid g_1\varphi((g_2,g_3)) + \varphi((g_1,g_2g_3)) = \varphi((g_1g_2,g_3)) + \varphi((g_1,g_2)) \ \forall (g_1,g_2,g_3)\in G^3\}$

"factor system" with coefficients in A

$B^2(G,A) = \{\varphi : G\times G \to A \mid \exists f : G\to A \text{ with } \varphi = \delta^1(f)\}$

$= \{\varphi : G\times G\to A \mid \exists f : G\to A$ s.th.

$\varphi((g_1,g_2)) = g_1 f(g_2) - f(g_1g_2) + f(g_1) \ \forall (g_1,g_2)\in G^2\}$

Examples (a) $L|K$ finite Galois extension, $G = Gal(L|K)$, $A = L^*$ as G-module

So the group operation in A is written multiplicatively here.

Now, if you use the notation $\varphi((\sigma,\tau)) =: a_{\sigma,\tau} \in L^* \ \forall (\sigma,\tau)\in G^2$, then

$\varphi \in Z^2(G,A) \iff \rho(a_{\sigma,\tau})\, a_{\rho,\sigma\tau} = a_{\rho\sigma,\tau}\, a_{\rho,\sigma} \quad \forall (\rho,\sigma,\tau)\in G^3$

$\Updownarrow$

$\{a_{\sigma,\tau} \mid \sigma,\tau\in G\}$ is a factor set describing the central simple K-algebra

$\bigoplus_{\sigma\in G} L x_\sigma$ with $x_\sigma a x_\sigma^{-1} = \sigma(a) \ \forall a\in L$ and

$x_\sigma x_\tau = a_{\sigma,\tau} x_{\sigma\tau}$

We have seen that $\{a_{\sigma,\tau}\}$ and $\{a'_{\sigma,\tau}\}$ determine isomorphic K-algebras iff $\exists f : G \to A = L^*$ s.th. $a'_{\sigma,\tau} = \dfrac{f(\sigma)\,\sigma(f(\tau))}{f(\sigma\tau)}\, a_{\sigma,\tau} \ \forall (\sigma,\tau)\in G^2$

Note that $\delta^1(f)((\sigma,\tau)) = \sigma(f(\tau))\, f(\sigma\tau)^{-1} f(\sigma)$ so that $\{a_{\sigma,\tau}\}$ and $\{a'_{\sigma,\tau}\}$ determine isomorphic K-algebras iff the differ (multiplicatively) by a 2-coboundary.

Hence $\underline{H^2(G,L^*)} = Z^2(G,L^*)/B^2(G,L^*) = \dfrac{\{\text{factor sets}\}}{\{2\text{-coboundaries}\}} \cong \underline{Br(L|K)}$.

(b) Situation as in (a); what about $H^1(G, L^*)$?

$$Z^1(G,L^*) = \{\varphi : G \to L^* \mid \varphi(\sigma\tau) = \sigma(\varphi(\tau))\varphi(\sigma) \quad \forall (\sigma,\tau) \in G^2\}$$
$$= \{(a_\sigma)_{\sigma\in G} \subseteq (L^*)^n \mid \underline{a_{\sigma\tau} = \sigma(a_\tau)\, a_\sigma} \quad \forall (\sigma,\tau)\in G^2\} \quad (n = |G|)$$

Dedekind's lemma $\Rightarrow \{\sigma \mid \sigma \in G\}$ are lin. independent as functions $L \to L$

$$\Rightarrow \sum_{\sigma\in G} a_\sigma \sigma \neq 0$$

$$\Rightarrow \exists\, \cancel{\alpha}\, \gamma \in L^* : \quad \beta := \sum_{\tau\in G} a_\tau \tau(\gamma) \neq 0$$

*) assuming $(a_\sigma)_{\sigma\in G} \in Z^1(G,L^*)$

$$\Rightarrow \sigma(\beta) = \sum_{\tau\in G} \sigma(a_\tau)(\sigma\tau)(\gamma) \overset{*)}{=} \sum_{\tau \in G} (a_\sigma^{-1} a_{\sigma\tau})(\sigma\tau)(\gamma) =$$
$$= a_\sigma^{-1} \sum_{\tau\in G} a_{\sigma\tau}(\sigma\tau)(\gamma) = a_\sigma^{-1} \beta$$

$$\Rightarrow a_\sigma = \beta/\sigma(\beta) = \delta^0(\beta^{-1})(\sigma) \quad \forall \sigma \in G$$
$$\Rightarrow (a_\sigma)_{\sigma\in G} \in B^1(G,L^*) \Rightarrow Z^1(G,L^*) = B^1(G,L^*)$$
$$\Rightarrow H^1(G,L^*) = \{1\}$$

<u>Hilbert's Satz 90.</u>

(c) Situation as in (a) and (b); what about $H^n(G,L)$?

<u>Claim</u> L is a coinduced module, $\cong \mathrm{Hom}(\mathbb{Z}G, K) = K^*$

$$\mathrm{Hom}(\mathbb{Z}G, L) = \{f : G \to K\} \cong K^n \cong L, \quad n = [L:K]$$

as K-vector spaces

Question: Can $\{f : G \to K\} \cong L$ be made <u>G-invariant</u>?

Normal basis theorem $\exists \alpha \in L$ s.th. $\{\tau(\alpha) \mid \tau \in G\}$ is a K-basis of L

Define
$$\gamma : \{f : G \to K\} \longrightarrow L \qquad \text{additive} \checkmark$$
$$f \longmapsto \sum_{\tau \in G} f(\tau^{-1})\, \tau(\alpha)$$

$\sigma \in G \Rightarrow$

$$\gamma(\sigma . f) = \sum_{\tau \in G} (\sigma . f)(\tau^{-1})\, \tau(\alpha) = \sum_{\tau \in G} f(\tau^{-1}\sigma)\, \tau(\alpha) = \sum_{\tau' = \sigma^{-1}\tau \in G} \overset{\in K}{f(\tau'^{-1})} (\sigma\tau')(\alpha)$$
$$= \sigma\Big(\sum_{\tau' \in G} f(\tau'^{-1})\, \tau'(\alpha) \Big) = \sigma(\gamma(f))$$

Cor: $H^n(G, L) = \{0\} \quad \forall n \geq 1$

This now follows from Lemma 2.

Remark: L is also an induced module, $\cong \mathbb{Z}G \otimes_{\mathbb{Z}} K$, which implies $H_n(G, L) = \{0\} \quad \forall n \geq 1$.

§3 Cohomology of cyclic groups

$G = \langle \sigma \rangle, \quad |\sigma| = n$

$$\mathbb{Z}G = \{ \sum_{i=0}^{n-1} m_i \sigma^i \mid \text{all } m_i \in \mathbb{Z} \} \cong \mathbb{Z}[x]/(x^n - 1)$$

$I_G = \mathbb{Z}G(\sigma - 1)$: If $\sum_{i=0}^{n-1} m_i \sigma^i$ satisfies $\sum_{i=0}^{n-1} m_i = 0$, then

$$\sum_{i=0}^{n-1} m_i \sigma^i = \sum_{i=0}^{n-1} m_i \sigma^i - \sum_{i=0}^{n-1} m_i = \sum_{i=0}^{n-1} m_i (\sigma^i - 1) = (\sigma - 1) \sum_{i=0}^{n-1} m_i (1 + \sigma + \dots + \sigma^{i-1})$$

Set $\underline{N} := \sum_{i=0}^{n-1} \sigma^i \in \mathbb{Z}G \Rightarrow \underline{(\sigma-1) N = \sigma^n - 1 = 0}$

For $x \in \mathbb{Z}G$, define $\mu_x : \mathbb{Z}G \to \mathbb{Z}G$, $y \mapsto xy$ $\quad$ G-module hom.

Lemma 6 $\quad \ker \mu_{\sigma-1} = \mathbb{Z}G N, \quad \ker \mu_N = \mathbb{Z}G(\sigma - 1) = I_G$

Proof: $(\sigma-1) N = 0 = N(\sigma - 1)$ implies $\mathbb{Z}G N \subseteq \ker \mu_{\sigma-1}$ and $\mathbb{Z}G(\sigma-1) \subseteq \ker \mu_N$ ✓

$\ker \mu_{\sigma-1} \subseteq \mathbb{Z}G N$: $\left(\sum_{i=0}^{n-1} m_i \sigma^i\right)(\sigma - 1) = 0 \iff \sum_{i=0}^{n-1} m_i \sigma^{i+1} = \sum_{i=0}^{n-1} m_i \sigma^i$

$$\iff m_0 = m_1 = \dots = m_{n-1} =: m \iff \sum_{i=0}^{n-1} m_i \sigma^i = mN.$$

$\ker \mu_N \subseteq \mathbb{Z}G(\sigma - 1)$: Note first that $\sigma^i N = N$ for all i. So if

$$0 = \left(\sum_{i=0}^{n-1} m_i \sigma^i\right) N = \sum_{i=0}^{n-1} m_i (\sigma^i N) = \sum_{i=0}^{n-1} m_i N = \left(\sum_{i=0}^{n-1} m_i\right) N, \text{ then}$$

$$\sum_{i=0}^{n-1} m_i = 0 \Rightarrow \sum_{i=0}^{n-1} m_i \sigma^i \in I_G = \mathbb{Z}G(\sigma - 1). \quad \square$$

<u>Cor</u> :

$$\cdots \xrightarrow{\mu_N} \mathbb{Z}G \xrightarrow{\mu_{\sigma-1}} \mathbb{Z}G \xrightarrow{\mu_N} \mathbb{Z}G \xrightarrow{\mu_{\sigma-1}} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$\mathbb{Z}G(\sigma-1) \qquad \mathbb{Z}GN \qquad I_G = \mathbb{Z}G(\sigma-1)$$

is a free resolution of the trivial G-module $\mathbb{Z}$.

<u>Proposition</u>

$A \in {}_G\mathrm{Mod} \Rightarrow H^i(G,A) = \begin{cases} A^G & \text{if } i = 0 \\ \ker \tilde{\mu}_N / (\sigma-1).A & \text{if } i \geq 1 \text{ is odd} \\ A^G / N.A & \text{if } i \geq 2 \text{ is even} \end{cases}$

with the map $\tilde{\mu}_N : A \to A$, $a \mapsto N.a$ (which is a G-hom.)

<u>Proof</u> :

Let $\underline{\underline{P}} \twoheadrightarrow \mathbb{Z}$ be the resolution of the corollary, so $P_i = \mathbb{Z}G$ $\forall i \geq 0$
$\partial_i = \mu_{\sigma-1} : \mathbb{Z}G \to \mathbb{Z}G$ if $i \geq 1$ is odd and $\partial_i = \mu_N : \mathbb{Z}G \to \mathbb{Z}G$ if $i \geq 2$ is even.

For $x \in \mathbb{Z}G$, $\mu_x : \mathbb{Z}G \to \mathbb{Z}G$, $y \mapsto xy$ induces $\mu_x^* : \mathrm{Hom}_G(\mathbb{Z}G, A) \to \mathrm{Hom}_G(\mathbb{Z}G, A)$, $\varphi \mapsto \varphi \circ \mu_x$

If we identify $\mathrm{Hom}_G(\mathbb{Z}G, A)$ with A via $\varphi \leftrightarrow \varphi(1)$, then μ_x^* becomes $\tilde{\mu}_x : A \to A$, $a \mapsto xa$:

$$\begin{array}{ccc} \mathrm{Hom}_G(\mathbb{Z}G,A) & \xrightarrow{\mu_x^*} & \mathrm{Hom}_G(\mathbb{Z}G,A) \\ \wr\wr & & \wr\wr \\ A & \xrightarrow{\tilde{\mu}_x} & A \end{array} \qquad \begin{array}{ccc} \varphi & \longmapsto & \varphi \circ \mu_x \\ \downarrow & & \downarrow \\ \varphi(1) & \longmapsto & x\varphi(1) = \varphi(x.1) = \varphi(\mu_x(1)) \end{array}$$

It's now easy to compute the cohomology of $\mathrm{Hom}_G(\underline{\underline{P}}, A)$:

$$\begin{tikzcd}[column sep=small]
 & 0 & & 1 & & 2 & & 3 & \\
\mathrm{Hom}_G(\underline{\underline{P}}, A): \quad 0 \arrow[r] & \mathrm{Hom}_G(\mathbb{Z}G, A) \arrow[rr, "\mu^*_{\sigma-1}"] & & \mathrm{Hom}_G(\mathbb{Z}G, A) \arrow[rr, "\mu^*_N"] & & \mathrm{Hom}_G(\mathbb{Z}G, A) \arrow[rr, "\mu^*_{\sigma-1}"] & & \mathrm{Hom}_G(\mathbb{Z}G, A) \arrow[r, "\mu^*_N"] & \cdots \\
0 \arrow[r] & A \arrow[u, phantom, sloped, "\cong"] \arrow[rr, "\tilde{\mu}_{\sigma-1}"] & & A \arrow[u, phantom, sloped, "\cong"] \arrow[rr, "\tilde{\mu}_N"] & & A \arrow[u, phantom, sloped, "\cong"] \arrow[rr, "\tilde{\mu}_{\sigma-1}"] & & A \arrow[u, phantom, sloped, "\cong"] \arrow[r, "\tilde{\mu}_N"] & \cdots
\end{tikzcd}$$

Hence $\quad H^0(G, A) = \ker \tilde{\mu}_{\sigma-1} = \{a \in A \mid \sigma . a = a\} = A^G \quad (*)$

$$H^i(G, A) = \ker \tilde{\mu}_N / \operatorname{im} \tilde{\mu}_{\sigma-1} = \ker \tilde{\mu}_N / (\sigma-1)A \quad \text{if } i \geq 1 \text{ is odd.}$$

$$H^i(G, A) = \ker \tilde{\mu}_{\sigma-1} / \operatorname{im} \tilde{\mu}_N \overset{(*)}{=} A^G / N.A \quad \text{if } i \geq 2 \text{ is even.} \quad \square$$

~~(Note that ker $\tilde{\mu}_{\sigma-1} = \{a \in A$~~

<u>Remark</u> Using the same free resolution $\underline{\underline{P}} \twoheadrightarrow \mathbb{Z}$, and computing the homology of the chain complex $\underline{\underline{P}} \otimes_G A$, one can verify similarly

$$H_i(G, A) = \begin{cases} A/(\sigma-1).A = A_G & \text{if } i = 0 \\ A^G / N.A & \text{if } i \geq 1 \text{ is odd} \\ \ker \tilde{\mu}_N / (\sigma-1).A & \text{if } i \geq 2 \text{ is even} \end{cases}$$

$$\begin{tikzcd}[column sep=small]
 & 3 & & 2 & & 1 & & 0 & \\
\cdots \arrow[r] & \mathbb{Z}G \otimes_G A \arrow[rr, "\mu_{\sigma-1} \otimes 1_A"] & & \mathbb{Z}G \otimes_G A \arrow[rr, "\mu_N \otimes 1_A"] & & \mathbb{Z}G \otimes_G A \arrow[rr, "\mu_{\sigma-1} \otimes 1_A"] & & \mathbb{Z}G \otimes_G A \arrow[r] & 0 \\
\cdots \arrow[r] & A \arrow[u, phantom, sloped, "\cong"] \arrow[rr, "\tilde{\mu}_{\sigma-1}"] & & A \arrow[u, phantom, sloped, "\cong"] \arrow[rr, "\tilde{\mu}_N"] & & A \arrow[u, phantom, sloped, "\cong"] \arrow[rr, "\tilde{\mu}_{\sigma-1}"] & & A \arrow[u, phantom, sloped, "\cong"] \arrow[r] & 0
\end{tikzcd}$$

Note: $H_0(\underline{\underline{P}} \otimes_G A) = \mathbb{Z}G \otimes_G A / I_G \otimes_G A \cong A / I_G A =: A_G$.

Applications $L|K$ finite cyclic Galois extension of degree n

$G = Gal(L|K) = \langle\sigma\rangle$ finite cyclic of order n

(1) $H^2(G, L^*) = (L^*)^G / N.L^* = K^*/N_{L|K}(L^*)$

Note: Since L^* is a multiplicative group, we get for all $a \in L$

$$N.a = \left(\sum_{i=0}^{n-1} \sigma^i\right).a = \prod_{i=0}^{n-1} \sigma^i(a) = \prod_{\tau \in G} \tau(a) = N_{L|K}(a).$$

Cor: $Br(L|K) \cong K^*/N_{L|K}(L^*)$

(2) We have shown that for any finite Galois extension $L|K$ $H^1(G, L^*) = \{1\}$. Using for cyclic extensions that $H^1(G, L^*) = \ker \tilde{\mu}_N / (\sigma - 1).L^*$, we now conclude $\ker \tilde{\mu}_N = (\sigma-1).L^*$. Here is the concrete interpretation:

$$\tilde{\mu}_N : L^* \to L^* \qquad \Rightarrow \ker \tilde{\mu}_N = \{a \in L^* \mid N_{L|K}(a) = 1\}$$

$$a \mapsto N.a = N_{L|K}(a)$$

$$(\sigma-1).A = \{(\sigma-1).b \mid b \in L^*\} = \left\{\frac{\sigma(b)}{b} \mid b \in L^*\right\}$$

This yields the original version of "Hilbert's Satz 90":

Theorem If $L|K$ is a cyclic Galois extension with $Gal(L|K) = \langle\sigma\rangle$ and $a \in L^*$ satisfies $N_{L|K}(a) = 1$, then there exists $b \in L^*$ with

$$a = \frac{\sigma(b)}{b} \quad \left(\text{or } a = \frac{b}{\sigma(b)} \text{ if we replace } b \text{ with } b^{-1}\right). \quad \square$$

Cohomology of the infinite cyclic group

$G = \langle \sigma \rangle$, $|\sigma| = \infty$, $\mathbb{Z}G = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}\sigma^i \cong \mathbb{Z}[x, x^{-1}]$

$I_G = \mathbb{Z}G(\sigma - 1)$:

By an elementary fact, $I_G = \bigoplus_{g \neq 1} \mathbb{Z}(g-1)$. Here we have

$$\sigma^i - 1 = (1 + \sigma + \dots + \sigma^{i-1})(\sigma - 1) \quad \text{if } i \geq 1 \text{ and}$$
$$\sigma^{-i} - 1 = \sigma^{-i}(1 - \sigma^i) = -\sigma^{-i}(1 + \sigma + \dots + \sigma^{i-1})(\sigma - 1) \in \mathbb{Z}G(\sigma - 1).$$

Since $\mathbb{Z}G$ is an integral domain, $I_G = \mathbb{Z}G(\sigma-1)$ is a <u>free</u> $\mathbb{Z}G$-module with basis $\{\sigma - 1\}$. Hence

$$0 \to I_G \xrightarrow{\iota} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$ is a free resolution of $\mathbb{Z}$. So we have $\underline{\underline{P}} \twoheadrightarrow \mathbb{Z}$ with $P_0 = \mathbb{Z}G$, $P_1 = I_G$, $P_i = 0 \ \forall i \geq 2$

$$\Rightarrow H^i(G, A) = H_i(G, A) = \{0\} \quad \forall i \geq 2$$

The same reasoning applies to any <u>free group</u> G since then I_G is also a free $\mathbb{Z}G$-module (but this is a bit harder to prove).

We also know $H^0(G, A) = A^G$, $H_0(G, A) = A_G$. For $G = \langle \sigma \rangle$,

$$H^1(G, A) = \mathrm{Hom}_G(I_G, A) / \mathrm{im}\, \iota^* \quad \text{for } \iota^* : \mathrm{Hom}_G(\mathbb{Z}G, A) \to \mathrm{Hom}_G(I_G, A), \ \varphi \mapsto \varphi|_{I_G}$$

$$H_1(G, A) = \ker(I_G \otimes_G A \to \mathbb{Z}G \otimes_G A)$$

If A is a trivial G-module, then $H^1(G, A) \cong H_1(G, A) \cong A$.

§4 Change of Groups

We start considering subgroups of G. So let $H \leq G$.
Choose a system $\{g_i \mid i \in I\}$ of right coset representatives of $H \backslash G$, i.e.

$$G = \coprod_{i\in I} H g_i \implies \mathbb{Z}G = \bigoplus_{i\in I} (\mathbb{Z}H)\, g_i \quad \underline{\text{free (left) } H\text{-module}}.$$

If $A' \in {}_H\underline{\mathrm{Mod}}$, then we define $A = \underline{\mathrm{Coind}_H^G A'} := \mathrm{Hom}_H(\mathbb{Z}G, A')$,

A is a left G-module via $(g.\varphi)(z) := \varphi(zg) \quad \forall z \in \mathbb{Z}G, g \in G, \varphi \in A$

<u>Lemma 7</u> (Compare with Lemma 1) $A' \in {}_H\underline{\mathrm{Mod}}$, $B \in {}_G\underline{\mathrm{Mod}}$, $A = \mathrm{Coind}_H^G A' \implies$

$$\begin{array}{ccc} \mathrm{Hom}_G(B, A) & \xrightarrow{\sim} & \mathrm{Hom}_H(B, A') \\ \varphi & \longmapsto & \varphi' \end{array} \quad \text{with inverse} \quad \begin{array}{ccc} \mathrm{Hom}_H(B, A') & \xrightarrow{\sim} & \mathrm{Hom}_G(B, A) \\ \psi & \longmapsto & \psi' \end{array}$$

are isomorphisms with

$$\begin{array}{rl} \varphi' : B & \longrightarrow A' \\ b & \longmapsto \varphi(b)(1) \in A' \end{array}, \qquad \begin{array}{rl} \psi' : B & \longrightarrow A \\ \psi'(b) : \mathbb{Z}G & \longrightarrow A' \\ z & \longmapsto \psi(z.b) \end{array}$$

Generalizing the situation in Lemma 1,

$$\begin{array}{rl} \mathcal{G} : {}_H\underline{\mathrm{Mod}} & \longrightarrow {}_G\underline{\mathrm{Mod}} \\ A' & \longmapsto A = \mathrm{Coind}_H^G A' \end{array} \quad \text{is the right adjoint of} \quad \mathcal{F} : {}_G\underline{\mathrm{Mod}} \xrightarrow{\text{forget}} {}_H\underline{\mathrm{Mod}}$$

The isomorphisms above constitute in fact a natural equivalence between $\mathrm{Hom}_H(\mathcal{F}(-), -)$ and $\mathrm{Hom}_G(-, \mathcal{G}(-))$.

<u>Proposition</u> (Shapiro's Lemma; compare with Lemma 2)

$$A' \in {}_H\underline{\mathrm{Mod}}, \; A = \mathrm{Coind}_H^G A' \Rightarrow H^i(G,A) \cong H^i(H,A') \quad \forall i \geq 0.$$

<u>Proof</u> (basically the same as for Lemma 2)

Pick any projective resolution $\underline{P} \twoheadrightarrow \mathbb{Z}$ of the trivial G-module $\mathbb{Z}$, i.e. $\ldots \to P_i \to P_{i-1} \to \ldots \to P_0 \to \mathbb{Z} \to 0$ is an exact sequence of G-modules $\Rightarrow$ exact sequence of H-modules

all P_i are projective G-modules $\Rightarrow$ projective H-modules (since $\mathbb{Z}G$ is a free H-module)

Now $H^i(G,A) = H^i(\mathrm{Hom}_G(\underline{P}, A)) \quad \forall i$

But by Lemma 7, $\mathrm{Hom}_G(\underline{P}, A) \cong \mathrm{Hom}_H(\underline{P}, A')$:

$$\begin{array}{ccccccc}
\ldots \to & \mathrm{Hom}_G(P_i, A) & \longrightarrow & \mathrm{Hom}_G(P_{i+1}, A) & \to \ldots \\
& \wr\wr & \circlearrowleft & \wr\wr & \\
\ldots \to & \mathrm{Hom}_H(P_i, A') & \longrightarrow & \mathrm{Hom}_H(P_{i+1}, A') & \to \ldots
\end{array}$$

It follows that $H^i(H,A') = H^i(\mathrm{Hom}_H(\underline{P}, A'))$
$\cong H^i(\mathrm{Hom}_G(\underline{P}, A)) = H^i(G,A) \quad \forall i$. $\square$

Similarly, if $A' \in {}_H\underline{\mathrm{Mod}}$, we define $A = \underline{\mathrm{Ind}_H^G A'} := \mathbb{Z}G \otimes_H A'$.

One then also obtains a homology version of Shapiro's lemma, namely
$$H_i(G,A) \cong H_i(H,A') \quad \forall i \geq 0.$$

Another concept relating the cohomology (homology) of G to that of its subgroup $H \leq G$ is the <u>restriction</u> (corestriction).

$A, B \in {}_G\underline{\mathrm{Mod}} \Rightarrow A, B \in {}_H\underline{\mathrm{Mod}}$ and $\mathrm{Hom}_G(B, A) \hookrightarrow \mathrm{Hom}_H(B, A)$

$A \in \underline{\mathrm{Mod}}_G \Rightarrow A \in \underline{\mathrm{Mod}}_H$ and $A \otimes_H B \twoheadrightarrow A \otimes_G B$

$\underline{\underline{P}} \twoheadrightarrow \mathbb{Z}$ projective G-resolution $\Rightarrow$ projective H-resolution

$\rightsquigarrow$ hom. of cochain complexes $\mathrm{Hom}_G(\underline{\underline{P}}, A) \to \mathrm{Hom}_H(\underline{\underline{P}}, A)$

$\cancel{\longrightarrow P_i \longrightarrow P}$

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}_G(P_i, A) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Hom}_G(P_{i+1}, A) \arrow[r] \arrow[d, hook] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}_H(P_i, A) \arrow[r] & \mathrm{Hom}_H(P_{i+1}, A) &
\end{tikzcd}$$

induces hom. between the corresponding cohomology groups:

<u>Definition (restriction)</u> $\underline{\mathrm{res}^i} : \underline{H^i(G, A)} = H^i(\mathrm{Hom}_G(\underline{\underline{P}}, A)) \to H^i(\mathrm{Hom}_H(\underline{\underline{P}}, A)) = \underline{H^i(H, A)}$

Dually, the hom. of chain complexes $\underline{\underline{P}} \otimes_H A \to \underline{\underline{P}} \otimes_G A$ induces <u>corestriction</u> maps $\underline{\mathrm{cor}_i} : \underline{H_i(H, A)} \to \underline{H_i(G, A)}$ $\forall i$ of $A \in {}_G\underline{\mathrm{Mod}}$.

<u>Remarks</u> (a) $\mathrm{res}^i : H^i(G, -) \to H^i(H, -)$ and $\mathrm{cor}_i : H_i(H, -) \to H_i(G, -)$ are natural transformations, e.g. for $\varphi \in \mathrm{Hom}_G(A, A')$, the diagram

$$\begin{tikzcd}
H^i(G, A) \arrow[r] \arrow[d, "\varphi^*"'] \arrow[dr, phantom, "\circlearrowleft"] & H^i(H, A) \arrow[d, "\varphi^*"] \\
H^i(G, A') \arrow[r] & H^i(H, A')
\end{tikzcd}$$

commutes.

(6) If $K \leq H \leq G$, it follows immediately from the definitions that

$$\mathrm{res}^G_K = \mathrm{res}^H_K \circ \mathrm{res}^G_H : H^i(G,A) \to H^i(H,A) \to H^i(K,A) \text{ and}$$

$$\mathrm{cor}^K_G = \mathrm{cor}^H_G \circ \mathrm{cor}^K_H : H_i(K,A) \to H_i(H,A) \to H_i(G,A)$$

In general, there is no natural way to define maps in the opposite direction, i.e. $H^i(H,A) \to H^i(G,A)$ or $H_i(G,A) \to H_i(H,A)$ if $H \leq G$ and $A \in {}_G\underline{\mathrm{Mod}}$. However, there is a way to define such "<u>transfer</u>" maps if $\underline{n = [G:H] < \infty}$, which we will assume now.

We will concentrate on cohomology here. Recall that the def. of res^i was based on an (obvious) natural transformation between the functors

$$\mathrm{Hom}_G(-,A) \hookrightarrow \mathrm{Hom}_H(-,A) : {}_G\underline{\mathrm{Mod}} \to \underline{\mathrm{Ab}} \quad (\text{for } A \in {}_G\underline{\mathrm{Mod}})$$

If $n = [G:H] < \infty$ and $\underline{X = \{x_1, \dots, x_n\}}$ is a set of right coset representatives for $H \backslash G$, i.e. $G = \coprod_{i=1}^n Hx_i$, we will now introduce a natural transformation

$t: \mathrm{Hom}_H(-,A) \to \mathrm{Hom}_G(-,A)$. Some preliminaries first. Define

$$\hat{\ } : G \to X \quad \text{with } Hg = H\hat{g} \Leftrightarrow g\hat{g}^{-1} \in H$$
$$g \mapsto \hat{g}$$

For any $g \in G$, we get a $\underline{\text{permutation } \pi_g} : X \to X$
$$x_i \mapsto \widehat{x_i g}$$

Note: $\widehat{x_i g} = \widehat{x_j g} \Rightarrow Hx_i g = Hx_j g \Rightarrow Hx_i = Hx_j \Rightarrow i = j$

So π_g is injective $\Rightarrow$ bijective since X is finite.

Lemma 8 (a) For each $M \in \underset{G}{\underline{Mod}}$, there exists a well-defined hom.

$$t_M : Hom_H(M, A) \longrightarrow Hom_G(M, A)$$
$$\varphi \longmapsto t_M\varphi : m \longmapsto \sum_{i=1}^{n} x_i^{-1}\varphi(x_i m)$$

(b) $t : Hom_H(-, A) \to Hom_G(-, A)$ is a natural transformation

Proof: (a) It's clear that $t_M\varphi$ is additive. We have to verify that it's also a G-hom.

$$(t_M\varphi)(gm) = \sum_{i=1}^{n} x_i^{-1}\varphi(x_i g m) = \sum_{i=1}^{n} x_i^{-1}\varphi(\underbrace{(x_i g)\,\widehat{x_i g}^{-1}}_{\in H}(\widehat{x_i g})\, m) =$$

$$= \sum_{i=1}^{n} x_i^{-1}\, x_i g\, \widehat{x_i g}^{-1}\, \varphi(\widehat{x_i g}\, m) = g \sum_{i=1}^{n} \widehat{x_i g}^{-1}\varphi(\widehat{x_i g}\, m)$$

$$= g\sum_{i=1}^{n} \pi_g(x_i)^{-1}\varphi(\pi_g(x_i)\, m) = g\,(t_M\varphi)(m)$$

$t_M(\varphi + \varphi') = t_M\varphi + t_M\varphi' \quad \forall \varphi, \varphi' \in Hom_H(M, A)$ is again clear.

Remark $t_M\varphi$ is independent of the choice of X: If $X' = \{x_1', \dots, x_n'\}$ is another set of right coset representatives mod H, then there are $h_i \in H$ with $x_i' = h_i x_i \;\forall i \Rightarrow$
$x_i'^{-1}\varphi(x_i' m) = x_i^{-1}h_i^{-1}\varphi(h_i x_i m) = x_i^{-1}h_i^{-1}h_i\varphi(x_i m) = x_i^{-1}\varphi(x_i m)$
$\forall\, 1 \le i \le n,\ m \in M \Rightarrow t_{X,M}\varphi = t_{X',M}\varphi \quad \forall \varphi \in Hom_H(M, A)$.

(b) For $\psi \in Hom_G(M, N)$, we need to verify the commutativity of the diagram

$$\begin{array}{ccc} Hom_H(N,A) & \xrightarrow{t_N} & Hom_G(N,A) \\ \downarrow{\scriptstyle \psi^*} & & \downarrow{\scriptstyle \psi^*} \\ Hom_H(M,A) & \xrightarrow{t_M} & Hom_G(M,A) \end{array} \qquad \begin{array}{ccc} \varphi & \longmapsto & t_N\varphi \\ \downarrow & & \searrow \\ \varphi\psi & \longmapsto & t_M(\varphi\psi) \overset{!}{=} (t_N\varphi)\psi \end{array}$$

We verify the last equation on the previous page by applying both sides to $\underline{m \in M}$:

$$(t_N \varphi)(\psi(m)) = \sum_{i=1}^{n} x_i^{-1} \varphi(x_i \psi(m)) = \sum_{i=1}^{n} x_i^{-1} \varphi(\psi(x_i m)) = \sum_{i=1}^{n} x_i^{-1} (\varphi\psi)(x_i m)$$
$$= t_M (\varphi\psi)(m)$$

$\forall \psi \in \mathrm{Hom}_G(M,N)$, $\varphi \in \mathrm{Hom}_H(N,A) \Rightarrow \varphi\psi \in \mathrm{Hom}_H(M,A)$. $\square$

<u>Consequence</u> For any chain complex $\underline{\underline{P}}$ of G-modules, and any G-module A, t induces a hom. of cochain complexes $\mathrm{Hom}_H(\underline{\underline{P}}, A) \longrightarrow \mathrm{Hom}_G(\underline{\underline{P}}, A)$

$$\begin{array}{ccccccc}
\cdots \to & \mathrm{Hom}_H(P_i, A) & \xrightarrow{\partial_{i+1}^*} & \mathrm{Hom}_H(P_{i+1}, A) & \to \cdots \\
& t_{P_i} \downarrow & \circlearrowleft & \downarrow t_{P_{i+1}} & \\
\cdots \to & \mathrm{Hom}_G(P_i, A) & \xrightarrow{\partial_{i+1}^*} & \mathrm{Hom}_G(P_{i+1}, A) & \to \cdots
\end{array}$$

and hence homomorphisms between the cohomology of these cochain complexes.

<u>Definition</u> The <u>corestriction</u> (or transfer) map $\underline{\mathrm{cor}^i}: H^i(H,A) \longrightarrow H^i(G,A)$ $(i \geq 0)$ is the map induced by t between the cohomology of $\mathrm{Hom}_H(\underline{\underline{P}}, A)$ and $\mathrm{Hom}_G(\underline{\underline{P}}, A)$, where $\underline{\underline{P}} \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ is a projective resolution of $\mathbb{Z}$ (trivial G-module).

<u>Proposition</u> For $n = [G:H] < \infty$ and $A \in \underline{\mathrm{Mod}}_G$, $\mathrm{cor}^i \circ \mathrm{res}^i : H^i(G,A) \longrightarrow H^i(G,A)$ is given by multiplication with n.

<u>Proof</u>: Let $\underline{\underline{P}} \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ be a projective G-resolution of $\mathbb{Z}$.

$\mathrm{cor} \circ \mathrm{res}$ is induced by the composition of the cochain maps

$$\mathrm{Hom}_G(\underline{\underline{P}}, A) \hookrightarrow \mathrm{Hom}_H(\underline{\underline{P}}, A) \xrightarrow{t_{\underline{\underline{P}}}} \mathrm{Hom}_G(\underline{\underline{P}}, A)$$

On the i^{th} level, we have the maps

$$\begin{array}{ccccc} \mathrm{Hom}_G(P_i, A) & \hookrightarrow & \mathrm{Hom}_H(P_i, A) & \xrightarrow{t_{P_i}} & \mathrm{Hom}_G(P_i, A) \\ & & \varphi & \longmapsto & t_{P_i}\varphi : m \longmapsto \sum_{i=1}^{n} x_i^{-1}\varphi(x_i m) \end{array}$$

If $\varphi \in \mathrm{Hom}_G(P_i, A)$, then $\sum_{i=1}^{n} x_i^{-1}\varphi(x_i m) = \sum_{i=1}^{n} x_i^{-1}x_i\varphi(m) = n\varphi(m)$, i.e. $\underline{t_{P_i}\varphi = n\varphi}$. So the cochain map which induces $\mathrm{cor} \circ \mathrm{res}$ is

$$\begin{array}{ccccc} \cdots \longrightarrow & \mathrm{Hom}_G(P_i, A) & \xrightarrow{\partial_{i+1}^*} & \mathrm{Hom}_G(P_{i+1}, A) & \longrightarrow \cdots \\ & \downarrow \cdot n & & \downarrow \cdot n & \\ \cdots \longrightarrow & \mathrm{Hom}_G(P_i, A) & \xrightarrow{\partial_{i+1}^*} & \mathrm{Hom}_G(P_{i+1}, A) & \longrightarrow \cdots \end{array}$$

Hence also $\mathrm{cor}^i \circ \mathrm{res}^i : H^i(G, A) \to H^i(G, A)$ is given by mult. with n.

<u>Corollary</u> If $|G| = n < \infty$, then $nH^i(G, A) = 0$ $\forall i \geq 1$.

<u>Proof:</u> Choose $H = \{1\}$ and consider $H^i(G, A) \xrightarrow{\mathrm{res}^i} H^i(\{1\}, A) \xrightarrow{\mathrm{cor}^i} H^i(G, A)$

$$H^i(\{1\}, A) = 0 \quad \forall i \geq 1$$

So by the Proposition, mult. by $n : H^i(G, A) \to H^i(G, A)$ is trivial.

<u>Corollary</u> If G is finite, and A is a f.g. G-module, then $H^i(G, A)$ is finite $\forall i \geq 1$.

<u>Proof:</u> G finite, then A f.g. as G-module $\iff$ f.g. as abelian group and $\mathrm{Hom}_G(M, A) \hookrightarrow A^l$ is a f.g. abelian group for every f.g. G-module M

$$\varphi \longmapsto (\varphi(m_i))_{1 \leq i \leq l} \qquad M = \sum_{i=1}^{l} \mathbb{Z}G\, m_i$$

Since G is finite, $\mathbb{Z}$ admits a free G-resolution $\underline{\underline{F}} \to \mathbb{Z} \to 0$ in which all F_i are f.g. G-modules, e.g. the bar resolution. Then $Hom_G(\underline{\underline{F}}, A)$ is a cochain complex of f.g. abelian groups $\Rightarrow$ $H^i(G,A)$ is f.g. (as abelian group) $\forall i$. But by the previous Corollary $H^i(G,A)$ is also torsion $\forall i \geq 1$.

This implies that $\underline{H^i(G,A) \text{ is finite } \forall i \geq 1}$. $\square$

__Remark__ Similar results can be obtained for group homology.

- define the natural transformation $t : - \otimes_G A \to - \otimes_H A$ for $A \in {}_G\underline{Mod}$ by

$$t_M(m \otimes a) = \sum_{i=1}^{n} m x_i^{-1} \otimes_H t_i a$$

- get a transfer map $res_i : H_i(G,A) \to H_i(H,A)$

- verify that $cor_i \circ res_i : H_i(G,A) \to H_i(G,A)$ is multiplication with $n = [G:H]$
- deduce that $n\, H_i(G,A) = 0$ $\forall i \geq 1$ if $|G| = n$ and that $H_i(G,A)$ is finite $\forall i \geq 1$ if A is a f.g. G-module.

We now want to study more generally how group (co)homology of two groups G, G' is related if we are given a hom. $\rho : G' \to G$ and $A' \in {}_{G'}\underline{Mod}$, $A \in {}_G\underline{Mod}$ together with an additive map

$\lambda : A \to A'$ satisfying $\lambda(\rho(g')a) = g'\lambda(a)$ $\forall g' \in G', a \in A$

In this case, (ρ, λ) is called a __compatible pair__.

__Example:__ If $A \in {}_G\underline{Mod}$ is considered as a G'-module via ρ, i.e. $g'a := \rho(g')a$ $\forall g' \in G', a \in A$, then (ρ, id_A) is compatible

How can we relate $H^i(G,A)$ and $H^i(G',A')$ now?

Problem: A projective G-resolution $\underline{P} \twoheadrightarrow \mathbb{Z}$ needn't define a projective G'-resolution any longer (which was the case for $G' = H \leq G$ and $\rho : H \hookrightarrow G$).

So we go back now to our description of cohomology using the bar resolution.

Lemma 9 If (ρ, λ) is a compatible pair, then $C^i(G,A) \xrightarrow{\varphi_i} C^i(G',A')$ $(i \geq 0)$ defines a hom. of cochain complexes $f \longmapsto \lambda \circ f \circ \underbrace{(\rho \times \dots \times \rho)}_{i \text{ times}}$ and hence induces hom. $H^i(G,A) \to H^i(G',A')$.

Proof: We have to verify that the diagram commutes. For $f \in C^i(G,A)$, we get

$$\begin{array}{ccc} C^i(G,A) & \xrightarrow{\delta^i} & C^{i+1}(G,A) \\ \varphi_i \downarrow & & \downarrow \varphi_{i+1} \\ C^i(G',A') & \xrightarrow{\delta^i} & C^{i+1}(G',A') \end{array}$$

$$\delta^i(\underbrace{\varphi_i f}_{=:f'})(g_1', \dots, g_{i+1}') = g_1' f'(g_2', \dots, g_{i+1}') + \sum_{j=1}^{i} (-1)^j f'(g_1', \dots, g_j' g_{j+1}', \dots, g_{i+1}') + (-1)^{i+1} f'(g_1', \dots, g_i') =$$

$$= g_1' \lambda(f(\rho(g_2'), \dots, \rho(g_{i+1}'))) + \sum_{j=1}^{i} (-1)^j \lambda(f(\rho(g_1'), \dots, \rho(g_j' g_{j+1}'), \dots, \rho(g_{i+1}'))) + (-1)^{i+1} \lambda(f(\rho(g_1'), \dots, \rho(g_i')))$$

$$= \lambda(\rho(g_1') f(\rho(g_2'), \dots, \rho(g_{i+1}'))) + \dots$$

$$= \lambda\Big(\rho(g_1') f(\rho(g_2'), \dots, \rho(g_{i+1}')) + \sum_{j=1}^{i} (-1)^j f(\rho(g_1'), \dots, \rho(g_j')\rho(g_{j+1}'), \dots, \rho(g_{i+1}')) + (-1)^{i+1} f(\rho(g_1'), \dots, \rho(g_i'))\Big)$$

$$= \lambda(\delta^i f(\rho(g_1'), \dots, \rho(g_{i+1}'))) = \lambda \circ \delta^i f \circ \rho^{i+1}(g_1', \dots, g_{i+1}') = \varphi_{i+1}(\delta^i f)(g_1', \dots, g_{i+1}')$$

$$\Rightarrow \delta^i \varphi_i(f) = \varphi_{i+1} \delta^i(f) \quad \forall f \in C^i(G,A). \ \square$$

Remark: If $H \leq G$, $\rho : H \hookrightarrow G$ and $A' = A$, $\lambda = \mathrm{id}_A$, then $\varphi_i : C^i(G,A) \to C^i(H,A)$, $f: G^i \to A \mapsto f_| : H^i \to A$.

One can show that the so induced map $H^i(G,A) \to H^i(H,A)$ is res^i.

Here is the situation which gives rise to the <u>inflation</u> map:

$N \triangleleft G$, $\rho: G \twoheadrightarrow G/N$, $A \in \underset{G}{\underline{\mathrm{Mod}}}$, $A^N = \{a \in A \mid na = a \ \forall n \in N\} \in \underset{G/N}{\underline{\mathrm{Mod}}}$

G' $\quad G \quad$ A' $\quad A$

$\lambda: A^N \hookrightarrow A$; (ρ, λ) is <u>compatible</u>: For $a \in A^N$, $g \in G$, we get

$$\lambda(\rho(g)a) = \rho(g)a = (gN)a = ga = g\,\lambda(a)$$

So applying Lemma 9, we obtain maps, called inflation maps

$$\underline{\mathrm{inf}^i}: H^i(G/N, A^N) \longrightarrow H^i(G, A) \ \forall\, i \geq 0$$

<u>Remark</u>: $\underset{\mathrm{id}_{A^G}}{\overset{\mathrm{inf}^0}{\parallel}}: H^0(G/N, A^N) = (A^N)^{G/N} = A^G \longrightarrow A^G = H^0(G, A)$

<u>Theorem</u> (Inflation-Restriction Sequence) $N \triangleleft G$, $A \in \underset{G}{\underline{\mathrm{Mod}}}$

(a) $0 \to H^1(G/N, A^N) \xrightarrow{\mathrm{inf}^1} H^1(G, A) \xrightarrow{\mathrm{res}^1} H^1(N, A)$ is exact.

(b) If $i \geq 1$ and $H^j(N, A) = 0 \ \forall\, 1 \leq j \leq i-1$, then

$0 \to H^i(G/N, A^N) \xrightarrow{\mathrm{inf}^i} H^i(G, A) \xrightarrow{\mathrm{res}^i} H^i(N, A)$ is exact.